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SOME ALMOST HERMITIAN QUATERNION MANIFOLDS

1. Introduction. A differentiable manifold with an almost quaternion structure has been studied by various mathematicians including C.Ehresmann, P.Liebermann, K.Yano, M.Obata, S.Ishihara and H.Wakakuwa. A.Gray [1] has given some classifications of almost Hermitian manifolds, and obtained inclusion relations between them. In the present paper we classify almost quaternion manifolds and later prove inclusion relations between them. In the last section we study a conformal diffeomorphism between almost Hermitian quaternion manifolds and use this diffeomorphism to compare classifications of these two manifolds.

2. Preliminaries. Let M be an n-dimensional $(n=4m)$, C^∞ real differentiable manifold, F(M) the ring of real valued differentiable functions over M, and $\mathfrak{X}(M)$ the module of derivatives of F(M). Then $\mathfrak{X}(M)$ is a Lie algebra over the real numbers and the elements of $\mathfrak{X}(M)$ are called vector fields.

If M is equipped with (1,1) tensor fields F and G satisfying

$$(2.1) \qquad F^2 = -I, \quad G^2 = -I, \quad FG = -GF,$$

then $H \overset{\text{def}}{=\!=} FG$ satisfies the identities

$$H^2 = -I, \; GH = -HG = F, \; HF = -FH = G .$$

Such a manifold M is said to be a differentiable manifold with an almost quaternion structure [3].

Each differentiable manifold with an almost quaternion structure (F, G, H) admits a positive definite Riemannian metric defined by a scalar product $<\cdot,\cdot>$ such that

$$(2.2)\qquad \begin{cases} <FX,FY> = <X,Y>, \\ <GX,GY> = <X,Y>, \\ <HX,HY> = <X,Y>, \end{cases}$$

for all $X,Y \in \mathcal{X}(M)$ and the manifold is then said to have an almost Hermitian quaternion structure [3].

Let $'F$, $'G$, $'H$ be defined as follows

$$'F(X,Y) = <FX,Y>,$$
$$'G(X,Y) = <GX,Y>,$$
$$'H(X,Y) = <HX,Y>; \quad X,Y \in \mathcal{X}(M).$$

It is easy to check that $'F$, $'G$, $'H$ are skew symmetric.

The torsion tensor $S(X,Y)$ associated with tensors F and G is a tensor of type (1,2) defined [4] by

$$(2.3)\qquad \begin{aligned} 2S(X,Y) = {} & [FX,GY] - F[X,GY] - G[FX,Y] + GF[X,Y] + \\ & + [GX,FY] - G[X,FY] - F[GX,Y] + FG[X,Y], \end{aligned}$$

where $X,Y \in \mathcal{X}(M)$. The Nijenhuis tensor corresponding to the (1,1) tensor F is given [4] by,

$$^F N(X,Y) = [FX,FY] - F[X,FY] - F[FX,Y] - [X,Y].$$

Let ∇_X be the Riemannian connection on M. Then

$$(2.4)\qquad \nabla_X(F)(Y) = \nabla_X(FY) - F\,\nabla_X Y,$$

and

$$(2.5)\qquad \nabla_X('F)(Y,Z) = <\nabla_X(F)(Y),Z>,$$

where X, Y, $Z \in \mathcal{X}(M)$. Using the above two identities, it is easy to verify the following

T h e o r e m 2.1. The following equalities hold for arbitrary $X,Y,Z \in \mathcal{X}(M)$

(i) $$\nabla_X(F)(GY) = F\,\nabla_X(F)(HY).$$

(ii) $$\nabla_X(G)(FY) = -G\,\nabla_X(G)(HY).$$

(iii) $$\nabla_X(F)(Y) = G\,\nabla_X(H)(Y) + \nabla_X(G)(HY).$$

(iv) $$\nabla_X('F)(GY,Z) = -\nabla_X('F)(HY,FZ).$$

3. Special almost Hermitian quaternion manifolds and their inclusion relations

In this section we shall require the following explicit formulas for the exterior and co-derivatives of the 2-form 'F,

(3.1) $$d\,'F(X,Y,Z) = \underset{X,Y,Z}{C}\ \nabla_X('F)(Y,Z),$$

(3.2) $$\delta\,'F(X) = -\sum_{i=1}^{m}\Big\{\nabla_{E_i}('F)(E_i,X) + \nabla_{FE_i}('F)(FE_i,X) + \nabla_{GE_i}('F)(GE_i,X) + \nabla_{HE_i}('F)(HE_i,X)\Big\},$$

where C denotes the cyclic permutation over (X,Y,Z) and $\{E_1, \dots, E_m, FE_1,\dots,FE_m, GE_1,\dots,GE_m, HE_1,\dots,HE_m\}$ form the frame field on an open subset of M [3].

T h e o r e m 3.1. Let $X,Y,Z \in \mathcal{X}(M)$. Then

(3.3) $${}^{F}N(X,Y) = \nabla_{FX}(F)(Y) + \nabla_X(F)(FY) - \nabla_{FY}(F)(X) - \nabla_Y(F)(FX).$$

(3.4) $$2S(X,Y) = \nabla_{FX}(G)(Y) + \nabla_{GX}(F)(Y) - F\,\nabla_X(G)(Y) - G\,\nabla_X(F)(Y) - \nabla_{FY}(G)(X) - \nabla_{GY}(F)(X) + F\,\nabla_Y(G)(X) + G\,\nabla_Y(F)(X).$$

$$(3.5)\quad 2\nabla_X('F)(Y,Z) = d'F(X,Y,Z) - d'F(X,FY,FZ) + \langle X, N(Y,FZ)\rangle .$$

$$(3.6)\quad \begin{aligned} &2\nabla_X('F)(Y,Z) + 2\nabla_{FX}('F)(FY,Z) = \\ &= d'F(X,Y,Z) - d'F(X,FY,FZ) + d'F(Z,FX,FY) + d'F(Y,FZ,FX). \end{aligned}$$

$$(3.7)\quad \begin{aligned} &\langle S(X,Y),Z\rangle - \langle S(X,Z),Y\rangle - \langle S(Y,Z),X\rangle = \\ &= \nabla_{FX}('G)(Y,Z) - F\,\nabla_X('G)(Y,Z) + \nabla_{GX}('F)(Y,Z) - G\,\nabla_X('F)(Y,Z). \end{aligned}$$

P r o o f. The proof of (3.3) and (3.4) follows from the fact, that

$$\nabla_X Y - \nabla_Y X = [X,Y] \; ;$$

(3.5) and (3.6) are consequences of (3.3), (3.1) and the formula

$$\nabla_X('F)(FY,Z) = \nabla_X('F)(Y,FZ).$$

The relation (3.7) follows from (3.4).

We shall call an almost Hermitian quaternion manifold

1° K.-quaternion manifold or (K.Q) iff

$$\nabla_X F = 0, \quad \nabla_X G = 0 \, ;$$

2° A.K.-quaternion manifold or (A.K.Q) iff

$$d'F = 0, \quad d'G = 0 \, ;$$

3° ^{G}(N.K.)-quaternion manifold or ^{G}(N.K.Q.) iff

$$\nabla_X(G)(Y) + \nabla_Y(G)(X) = 0 \, , \quad \nabla_X(G)(FY) + \nabla_Y(G)(FX) = 0;$$

4° ^{F}(N.K.)-quaternion manifold or ^{F}(N.K.Q) iff

$$\nabla_X(F)(Y) + \nabla_Y(F)(X) = 0, \quad \nabla_X(F)(GY) + \nabla_Y(F)(GX) = 0 \, ;$$

5^0 $^G(Q.K)$-quaternion manifold or $^G(Q.K.Q)$ iff

$$\nabla_X(G)(Y) + \nabla_{FX}(G)(FY) = 0 ;$$

6^0 $^F(Q.K)$-quaternion manifold or $^F(Q.K.Q)$ iff

$$\nabla_X(F)(Y) + \nabla_{GX}(F)(GY) = 0 ;$$

7^0 $^G(S.K.)$-quaternion manifold or $^G(S.K.Q.)$ iff

$$\delta' G = 0 ;$$

8^0 $^F(S.K.)$-quaternion manifold or $^F(S.K.Q.)$ iff

$$\delta' F = 0 ;$$

9^0 H.-quaternion manifold or (H.Q) iff

$$^F N(X,Y) = 0, \quad ^G N(X,Y) = 0, \quad S(X,Y) = 0.$$

T h e o r e m 3.2. The special almost Hermitian quaternion manifolds 1^0 - 9^0 satisfy the following inclusion relations:

(i) $$K.Q. \subseteq A.K.Q.$$

(ii) $$K.Q. \subseteq H.K.Q.$$

(iii) $$K.Q. \subseteq {}^G(N.K.Q) \subseteq {}^G(Q.K.Q) \subseteq {}^G(S.K.Q).$$

(iv) $$K.Q. \subseteq {}^F(N.K.Q) \subseteq {}^F(Q.K.Q) \subseteq {}^F(S.K.Q).$$

P r o o f. The relation $K.Q. \subseteq A.K.Q.$ follows from (2.5) and (3.1), the relation $K.Q. \subseteq H.K.Q.$ follows from (3.1) and (3.4). The relations $K.Q. \subseteq {}^G(N.K.Q)$ and $K.Q. \subseteq {}^F(N.K.Q.)$ are obvious. The inclusion ${}^G(N.K.Q) \leq {}^G(Q.K.Q)$ is an immediate consequence of the definitions 3^0 and 5^0 and the inclusion ${}^G(Q.K.Q) \leq {}^G(S.K.Q)$ is a consequence of (3.2) and (2.5).

In a similar manner we can prove that

$${}^F(N.K.Q.) \subseteq {}^F(Q.K.Q) \subseteq {}^F(S.K.Q).$$

T h e o r e m 3.3. If M is a ${}^{G}(N.K)$ quaternion manifold and also an A.K.-and ${}^{F}(N.K.)$-quaternion manifold, then it is K.-quaternion manifold.

P r o o f. If $M \in {}^{G}(N.K.Q)$, then $d'G = 3\nabla_X('G)(Y,Z)$, and $d'F = 3\nabla_X('F)(Y,Z)$, if $M \in {}^{F}(N.K.Q)$. Now if $M \in A.K.Q$, it means that $d'G = 0$, $d'F = 0$, then $\nabla_X F = 0$, $\nabla_X G = 0$. Hence, M is K.-quaternion.

Using the definitions of special spaces given earlier in this section, we readily deduce the following theorems.

T h e o r e m 3.4. A ${}^{F}(N.K)$-quaternion manifold is ${}^{G}(N.K)$-quaternion iff

$$\nabla_X(H)(FY) + \nabla_Y(H)(FX) = 0 .$$

P r o o f . Using (iii) of Theorem (2.1) we get

$$\nabla_X(G)(Y) + \nabla_Y(G)(X) =$$

$$= H\left\{\nabla_X(F)(Y) + \nabla_Y(F)(X)\right\} + \nabla_X(H)(FY) + \nabla_Y(H)(FX),$$

and

$$\nabla_X(F)(GY) + \nabla_Y(F)(GX) =$$

$$= F\left\{\nabla_X(H)(FY) + \nabla_Y(H)(FX)\right\} - \left\{\nabla_X(G)(FY) + \nabla_Y(G)(FX)\right\} .$$

The proof of the theorem follows immediately from the above two expressions.

T h e o r e m 3.5. A ${}^{G}(S.K)$-quaternion manifold is ${}^{F}(S.K)$-quaternion manifold iff

$$\nabla_X(H)(FY) - \nabla_{FX}(H)(Y) + \nabla_{GX}(H)(HY) - \nabla_{HX}(H)(GY) = 0 .$$

P r o o f . Using theorem (2.1) and the formula (3.2) we get

$$\delta' G = H\delta' F - \sum_{i=1}^{m}\left\{\nabla_{E_i}('H)(FE_i,X) - \nabla_{FE_i}('H)(E_i,X) + \right.$$

$$\left. + \nabla_{GE_i}('H)(HE_i,X) - \nabla_{HE_i}('H)(GE_i,X)\right\} ,$$

and hence the proof follows.

As a consequence of the above theorem we have

Corollary 3.5. A ^{G}(S.K)-quaternion manifold is ^{F}(S.K.)-quaternion provided

$$\nabla_X(H)(FY) - \nabla_{FX}(H)(Y) = 0 .$$

4. Conformal diffeomorphism of almost Hermitian quaternion manifolds

We consider two manifolds $(M, <\cdot,\cdot>)$ and $(M^o, <\cdot,\cdot>^o)$. Let $\Phi: M \longrightarrow M^o$ be a diffeomorphism. For $X \in \mathcal{X}(M)$ let $X^o = \Phi_* X$, where Φ_* is the Jacobian or the differential of Φ. Then Φ is called [1] a conformal diffeomorphism iff there exists $\sigma \in F(M)$ such that

$$(4.1) \qquad <X^o, Y^o>^o \cdot \Phi = e^{2\sigma} <X,Y>; \quad X,Y \in \mathcal{X}(M).$$

For $f \in F(M)$, grad $f \in \mathcal{X}(M)$ is defined by

$$<\operatorname{grad} f, X> = X(f); \quad X \in \mathcal{X}(M).$$

If $\Phi : M \longrightarrow M^o$ is a conformal diffeomorphism then [1]

$$(4.2) \quad \nabla^o_{X^o} Y^o = \left\{ \nabla_X Y + X(\sigma)Y + Y(\sigma)X - <X,Y> \operatorname{grad} \sigma \right\}^o .$$

L e m m a. The forms and structures of M and M^o are related by the following equalities:

$$(4.3) \qquad {}'F^o(X^o, Y^o) \cdot \Phi = e^{2\sigma}\, {}'F(X,Y),$$

$$(4.4) \qquad \Phi^* ({}'F^o) = e^{2\sigma}\, {}'F ,$$

$$(4.5) \qquad \Phi^*(d\,{}'F^o) = e^{2\sigma} \left\{ 2d\sigma \wedge {}'F + d\,{}'F \right\} ,$$

$$(4.6) \qquad \nabla^o_{X^o}(F^o)(Y^o) =$$

$$= \left\{ \nabla_X(F)(Y) + FY(\sigma)X - Y(\sigma)FX + <FX,Y> \operatorname{grad}\sigma + <X,Y> F \operatorname{grad}\sigma \right\}^o ,$$

$$\nabla^o_{X^o}('F^o)(Y^o,Z^o)\cdot\Phi =$$

(4.7) $$= e^{2\sigma}\{\nabla_X('F)(Y,Z) + FY(\sigma)<X,Z> - Y(\sigma)'F(X,Z) + $$

$$+ 'F(X,Y)Z(\sigma) - <X,Y> FZ(\sigma)\},$$

(4.8) $$\{F_N(X,Y)\}^o = F_{N^o}(X^o,Y^o),$$

for $X,Y,Z\in\mathfrak{X}(M)$.

Theorem 4.1. If $X,Y\in\mathfrak{X}(M)$ and $\dim M=n=4m$, then

(4.9) $$\delta^o\,'G^o(X^o)\cdot\Phi = \delta'G(X) + (n-2)GX(\sigma).$$

and

(4.10) $$\{S(X,Y)\}^o = S^o(X^o,Y^o).$$

Proof. We first observe that if

$$\{E_1,\ldots,E_m,\ FE_1,\ldots,FE_m,\ GE_1,\ldots,GE_m,\ HE_1,\ldots,HE_m\}$$

is a frame field on an open subset of M, then

$$\{(e^{-\sigma}E_1)^o,\ \ldots,\ (e^{-\sigma}E_m)^o,\ (e^{\sigma}FE_1)^o,\ \ldots,\ (e^{-\sigma}HE_m)^o\}$$

is a frame field on an open subset of M^o.
Now, we have

$$\delta^o\,'G^o(X^o)\cdot\Phi = -\ e^{-2\sigma}\sum_{i=1}^{m}\{\nabla^o_{E_i^o}('G^o)(E_i^o,X^o) +$$

$$+\nabla^o_{F^oE_i^o}('G^o)(F^oE_i^o,X^o) + \nabla^o_{G^oE_i^o}('G^o)(G^oE_i^o,X^o) +$$

$$+\nabla^o_{H^oE_i^o}('G^o)(H^oE_i^o,X^o)\}\cdot\Phi\ .$$

Hence by (4.7) - we get

$$\delta^{o}\,'G^{o}(X^{o})\cdot\Phi = \sum_{i=1}^{m}\Big\{\nabla_{E_i}('G)(E_i,X) + \nabla_{FE_i}('G)(FE_i,X) + $$

$$+ \nabla_{GE_i}('G)(GE_i,X) + \nabla_{HE_i}('G)(HE_i,X)\Big\} -$$

$$-\sum_{i=1}^{m}\Big\{GE_i(\sigma)<E_i,X> - E_i(\sigma)\,'G(E_i,X) + 'G(E_i,E_i)X(\sigma) -$$

$$-<E_i,E_i>G(X)(\sigma) + GFE_i(\sigma)<FE_i,X> - FE_i(\sigma)\,'G(FE_i,X) +$$

$$+ 'G(FE_i,FE_i)X(\sigma) - <FE_i,FE_i>GX(\sigma) +$$

$$+ G^2E_i(\sigma)<GE_i,X> - GE_i(\sigma)\,'G(GE_i,X) +$$

$$+ 'G(GE_i,GE_i)X(\sigma) - <GE_i,GE_i>GX(\sigma) +$$

$$+ GHE_i(\sigma)<HE_i,X> - HE_i(\sigma)\,'G(HE_i,X) +$$

$$+ 'G(HE_i,HE_i)X(\sigma) - <HE_i,HE_i>GX(\sigma)\Big\} =$$

$$= \delta\,'G(X) + 4mGX(\sigma) - 2\sum_{i=1}^{m}\Big\{a_iGE_i(\sigma) - c_iE_i(\sigma) -$$

$$- b_iHE_i(\sigma) + d_iFE_i(\sigma)\Big\},$$

where

$$X = \sum_{j=1}^{m}\Big\{a_jE_j + b_jFE_j + c_jGE_j + d_jHE_j\Big\}.$$

Hence

$$\delta^{o}\,'G^{o}(X^{o})\cdot\Phi = \delta\,'G(X) + (n-2)GX(\sigma).$$

The equality (4.10) is a direct consequence of (4.6)

By means of Theorem 4.1 and the lemma preceeding it we can prove the following

Theorem 4.2. Let $\Phi: M \to M^{o}$ be a conformal diffeomorphism between almost Hermitian quaternion manifolds M and M^{o}.(1) - The manifold $M \in H.Q$ iff $M^{o} \in H.Q$. (2) If dim $M \geq 8$, Φ is not nomothetic (i.e. σ is non-constant), and M is in one of the classes: K.Q., ^{F}(N.K.Q.), ^{G}(N.K.Q), ^{F}(Q.K.Q), ^{G}(Q.K.Q), ^{F}(S.K.Q), ^{G}(S.K.Q), then M^{o} is never in any of these classes.

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