DEMONSTRATIO MATHEMATICA
Vol. VII No 1 1974

Henryk Ugowski

ON DIFFERENTIAL INEQUALITIES OF PARABOLIC TYPE WITH MULTIDIMENSIONAL TIME

Mlak [2] has proved a theorem on strong differential inequalities of parabolic type. A theorem on weak differential inequalities was obtained by Szarski [3]. Using Mlak's result Besala [1] proved Szarski's theorem under weaker assumptions than those in [3].

In this paper we extend the above-mentioned results of Mlak and Besala to the case where the time-variable is multidimensional.

1. Preliminaries

In this section we extend the notation and definitions stated in the monograph [4] (§§ 46,47) to the case where the time variable is multidimensional.

Let D be an open domain of the Euclidean space E_{n+m} of the variables $(x,t) = (x_1,\dots,x_n,t_1,\dots,t_m)$ having the following properties:

1^0 D is contained in the strip

$$\left\{(x,t): t_i^0 < t_i < T_i, \quad t_i^0 < T_i \leqslant \infty \qquad (i=1,\dots,m)\right\}$$

and the intersection of the closure $\bar{D}$ with any closed strip

$$\left\{(x,t):\ t_i^0 \leqslant t_i \leqslant t_i^1 < T_i \qquad (i=1,\dots,m)\right\}$$

is bounded.

2^0 For any $\tau=(\tau_1,\dots,\tau_m)$, $\tau_i\in[t_i^0,T_i)$, the projection S_τ on the space $(x_1,\dots,x_n)$ of the intersection of $\bar{D}$ with the hyperplane $t_1=\tau_1,\dots,$ $t_m=\tau_m$ is non-empty.

3^0 For any point $(x,t)\in\bar{D}$ and any sequence $\{t^\nu\}$, $\left(t_i^\nu\in[t_i^0,T_i)\right)$, such that $t^\nu\to t$, there is a sequence $\{x^\nu\}$, $(x^\nu\in S_{t^\nu})$, such that $x^\nu\to x$.

Denote by $R_{t_i^0}$ and R_{T_i}, $(i=1,\dots,m)$, the interiors of those parts of the boundary σD of D which are situated on the planes $t_i=t_i^0$ and $t_i=T_i$, respectively (if $T_i=\infty$, then R_{T_i} is empty).

Let us put

$$\Sigma=\sigma D\setminus\left[\bigcup_{1\le i\le m}(R_{t_i^0}\cup\bar{R}_{T_i})\cup\left\{(x,t^0):x\in S_t\right\}\right]$$

If a function $\alpha(x,t)$ is defined on Σ, then we denote by Σ_α that subset of Σ on which $\alpha(x,t)\neq 0$.

Let functions $f^i(x,t,u,q,r,s)$, $(i=1,\dots,N)$, be defined for $(x,t)\in D$ and for arbitrary $u=(u^1,\dots,u^N)$, $q=(q_1,\dots,q_n)$, $r=(r_{11},r_{12},\dots,r_{nn})$, $s=(s_1,\dots,s_m)$. Assume that the function $u(x,t)=\left(u^1(x,t),\dots,u^N(x,t)\right)$ is defined and possesses derivatives $u_x^i=\left(u_{x_1}^i,\dots,u_{x_n}^i\right)$ $(i=1,\dots,n)$, in the domain D. Under these assumptions, we say that the function $f^i(x,t,u,q,r,s)$ is elliptic with respect to $u(x,t)$ in D if for any r, $\bar{r}$ $\left(r_{jk}=r_{kj},\ \bar{r}_{jk}=\bar{r}_{kj}\right)$ and s, such that the quadratic form in $\lambda_1,\dots,\lambda_n$

$$\sum_{j,k=1}^{n}(r_{jk}-\bar{r}_{jk})\lambda_j\lambda_k$$

is negative, we have

$$f^i(x,t,u(x,t),u_x^i(x,t),r,s)\le f^i(x,t,u(x,t),u_x^i(x,t),\bar{r},s),\ (x,t)\in D.$$

If a function $u(x,t)$, defined in D and possessing in D derivatives u^i_x, $u^i_{xx}=\left(u^i_{x_1x_1}, u^i_{x_1x_2},\dots,u^i_{x_nx_n}\right)$ and $u^i_t = \left(u^i_{t_1},\dots,u^i_{t_m}\right)$ $(i=1,\dots,N)$, is a solution of the system

$$(1)\quad f^i\left(x,t,u(x,t),u^i_x(x,t),u^i_{xx}(x,t),u^i_t(x,t)\right)=0,\qquad (x,t)\in D,\ i=1,\dots,N$$

and if all the functions f^i are elliptic with respect to $u(x,t)$ in D, then $u(x,t)$ is called a parabolic solution of this system.

We introduce the following condition.

(A) Functions $\alpha^i(x,t)$ $(i=1,\dots,N)$ are defined and non-negative on Σ. For every $(x,t)\in\Sigma_{\alpha^i}$ there is given a direction $l^i(x,t)$ which is ortogonal to all the time axes and some segment, with one extremity at (x,t), of the straight half-line from (x,t) in the direction l^i is contained in $\bar{D}$.

A solution $u(x,t)$ of the system (1) in D will be called Σ_α-regular solution if it is continuous in $\bar{D}$, possesses derivatives $u^i_{x_j}$, $u^i_{x_jx_k}$, $u^i_{t_p}$ $(i=1,\dots,N;\ j,k=1,\dots,n;\ p=1,\dots,m)$ continuous in D and if for every i there exists the derivative $\frac{du^i}{dl^i}$ at each point $(x,t)\in\Sigma_{\alpha^i}$.

Assume that we are given

1. the system (1), where the functions $f^i(x,t,u,q,r,s)$, $(i=1,\dots,N)$, are non-increasing with respect to s;

2. functions $\alpha^i(x,t)$ and directions $l^i(x,t)$, $(i=1,\dots,N)$, satisfying condition (A);

3. functions $\psi^i(x,t)$ defined on Σ and $\beta^i(x,t)$ defined on Σ_{α^i}, $(i=1,\dots,N)$, where $\beta^i(x,t)>0$ on Σ_{α^i};

4. functions $\varphi^i(x)$, $(i=1,\dots,N)$, defined on S_{t^0}.

The first mixed problem for the system (1) consists in finding a Σ_α-regular solution $u(x,t)$ of this system in D, satisfying the initial conditions

$$(2) \qquad u(x,t^o)=\varphi(x), \quad x\in S_{t^o}, \quad (\varphi=(\varphi^1,\dots,\varphi^N))$$

and boundary conditions

$$(3) \begin{cases} \beta^i(x,t)u^i(x,t)-\alpha^i(x,t)\dfrac{du^i}{dl^i}=\Psi^i(x,t), & (x,t)\in\Sigma_{\alpha^i}, \\ u^i(x,t)=\psi^i(x,t), & (x,t)\in\Sigma\setminus\Sigma_{\alpha^i}, (i=1,\dots N). \end{cases}$$

The following lemma will be needed.

L e m m a. Assume that we are given a function $\alpha(x,t)$ and a direction $l(x,t)$ satisfying (for $N=1$) condition (A), and a function $\beta(x,t)$ on Σ_α such that

$$\beta(x,t)>b\geqslant 0 \quad \text{for} \quad (x,t)\in\Sigma_\alpha.$$

Suppose that the function $u(x,t)$, continuous in $\bar{D}$ and possessing the derivative $\frac{du}{dl}$ on Σ_α, satisfies the inequalities

$$\beta(x,t)u(x,t)-\alpha(x,t)\frac{du}{dl}\leqslant b\eta(t) \quad (<b\eta(t)), \ (x,t)\in\Sigma_\alpha,$$

$$u(x,t)\leqslant\eta(t) \quad (<\eta(t)), \quad (x,t)\in\Sigma\setminus\Sigma_\alpha,$$

where $\eta(t)\geqslant 0$. Denote

$$D_o = D\cup\bigcup_{1\leqslant i\leqslant m} R_{t_i^o}.$$

Under these assumptions, if for a point $(x,t)\in D_o\cup\Sigma$ we have

$$\max_{x\in S_{\bar{t}}} u(x,\bar{t}) = u(\bar{x},\bar{t})>\eta(\bar{t}) \qquad (\geqslant\eta(\bar{t})),$$

then $(\bar{x},\bar{t})\in D_o$.

The proof of this lemma is the same as that of Lemma 47.1 of [4].

2. Strong differential inequalities

We introduce the following assumptions:

(I) The functions $f^i(x,t,u,q,r,s)$ $(i=1,\dots,N)$: satisfy condition W_+ with respect to u (see § 4 of [4]) and they are non-increasing with respect to s.

(II) The functions $\alpha^i(x,t)$ and the direction $l^i(x,t)$ $(i=1,\dots,N)$ satisfy condition (A), while $\beta^i(x,t)$ are defined and positive on Σ_{α^i}.

(III) Functions $u(x,t)=\big(u^1(x,t),\dots,u^N(x,t)\big)$ and $v(x,t)=\big(v^1(x,t),\dots,v^N(x,t)\big)$ are Σ_α-regular in D and every function f^i is elliptic with respect to $u(x,t)$. The following differential inequalities are fulfilled in the domain D

(4) $$f^i\big(x,t,u(x,t),u^i_x(x,t),u^i_{xx}(x,t),u^i_t(x,t)\big) > f^i\big(x,t,v(x,t),v^i_x(x,t),v^i_{xx}(x,t),v^i_t(x,t)\big), \quad (i=1,\dots,N).$$

Moreover, the initial inequalities

(5) $$u(x,t^0) < v(x,t^0), \quad x\in S_{t^0}$$

and the boundary inequalities

(6) $$\beta^i(x,t)\big[u^i(x,t)-v^i(x,t)\big]-\alpha^i(x,t)\frac{d[u^i-v^i]}{dl^i}<0, \quad (x,t)\in\Sigma_{\alpha^i},$$
$$u^i(x,t)-v^i(x,t)<0, \quad (x,t)\in\Sigma\setminus\Sigma_{\alpha^i}, \ (i=1,\dots,N),$$

hold true.

T h e o r e m 1. If assumptions (I)-(III) are satisfied, then $u(x,t)<v(x,t)$ in D [1)].

[1)] Theorem 1 is true if, instead of the ellipticity with respect to $u(x,t)$, we assume the ellipticity with respect to $v(x,t)$.

P r o o f. We proceed similarly as in the proof of Theorem 63.1 of [4]. Since the set $\{(x,t^0): x\in S_{t^0}\}$ is compact, there is, by (5) and by continuity of $u(x,t)$ and $v(x,t)$, a point $t^1=(t_1^1,\dots,t_m^1)$, $t_i^0<t_i^1<T_i$ $(i=1,\dots,m)$ such that

(7) $$u(x,t)<v(x,t),\qquad t^0\leqslant t<t^1,\qquad x\in S_t.$$

Let $\bar\varepsilon\in\left(0,\ \min\limits_i(t_i^1-t_i^0)\right)$ be an arbitrary fixed number and denote

$$t_i^\varepsilon=\begin{cases}\varepsilon(T_i-t_i^0-\bar\varepsilon)+t_i^0+\bar\varepsilon & \text{if}\quad T_i<\infty,\\[2mm] \dfrac{\varepsilon}{1-\varepsilon}+t_i^0+\bar\varepsilon & \text{if}\quad T_i=\infty,\end{cases}\qquad (i=1,\dots,m)$$

where $\varepsilon\in(0,1)$. Hence, by (7), there is a number $\varepsilon\in(0,1)$ such that $u(x,t)<v(x,t)$ in the domain

$$D^\varepsilon=\bar D\cap\left\{(x,t):\ t_i^0+\bar\varepsilon\leqslant t_i<t_i^\varepsilon\quad (i=1,\dots,m)\right\}.$$

Denote by ε_0 the least upper bound of all $\varepsilon\in(0,1)$ with this property. We shall prove that $\varepsilon_0=1$. Suppose to the contrary, that $\varepsilon_0<1$. This implies (as in [4]) the inequality

(8) $$u(x,t)\leqslant v(x,t),\quad (x,t)\in\overline{D^{\varepsilon_0}}.$$

Moreover, for some index j and some point $(\bar x,\bar t)\in\overline{D^{\varepsilon_0}}$ such that $\bar t_i=t_i^{\varepsilon_0}$ for some i $(1\leqslant i\leqslant m)$, we have

(9) $$u^j(\bar x,\bar t)=v^j(\bar x,\bar t).$$

If this equality was not true, we would have (by (8))

$$u(x,t)<v(x,t),\quad (x,t)\in\overline{D^{\varepsilon_0}}.$$

Hence it follows that inequality $u<v$ holds true in D^{ε_1} for some $\varepsilon_1 \in (\varepsilon_0, 1)$, which contradicts the definition of ε_0.

Relations (8), (9) imply that

$$\max_{x\in S_{\bar{t}}} \left[u^j(x,\bar{t}) - v^j(x,\bar{t})\right] = u^j(\bar{x},\bar{t}) - v^j(\bar{x},\bar{t}) = 0$$

whence, by (6) and by Lemma, we conclude that $(\bar{x},\bar{t}) \in D$. Thus the function $u^j(x,\bar{t}) - v^j(x,\bar{t})$ attains its maximum at the interior point $\bar{x} \in S_{\bar{t}}$. This implies that

$$(10) \qquad u^j_x(\bar{x},\bar{t}) = v^j_x(\bar{x},\bar{t})$$

and the quadratic form

$$(11) \qquad \sum_{k,l=1}^{n} \left[u^j_{x_k x_l}(\bar{x},\bar{t}) - v^j_{x_k x_l}(\bar{x},\bar{t})\right]\lambda_k \lambda_l \text{ is negative.}$$

In view of (4) we have, by (8)-(11), condition W_+ and by ellipticity of f^j, the inequality

$$(12) \qquad f^j\left(\bar{x},\bar{t},u(\bar{x},\bar{t}),u^j_x(\bar{x},\bar{t}),v^j_{xx}(\bar{x},\bar{t}),u^j_t(\bar{x},\bar{t})\right) >$$

$$> f^j\left(\bar{x},\bar{t},u(\bar{x},\bar{t}),u^j_x(\bar{x},\bar{t}),v^j_{xx}(\bar{x},\bar{t}),v^j_t(\bar{x},\bar{t})\right).$$

On the other hand, (8) and (9) imply the relation

$$\max_{t\leqslant \bar{t}} \left[u^j(\bar{x},t) - v^j(\bar{x},t)\right] = u^j(\bar{x},\bar{t}) - v^j(\bar{x},\bar{t}) = 0.$$

Therefore we have

$$u^j_t(\bar{x},\bar{t}) - v^j_t(\bar{x},\bar{t}) \geqslant 0$$

which implies, by the monotonicity of $f^j(x,t,u,q,r,s)$ with respect to s, the inequality

$$f^j\left(\bar{x},\bar{t},u(\bar{x},\bar{t}),u^j_x(\bar{x},\bar{t}),v^j_{xx}(\bar{x},\bar{t}),u^j_t(\bar{x},\bar{t})\right) \leqslant$$

$$\leqslant f^j(\bar{x},\bar{t},u(\bar{x},\bar{t}),u_x^j(\bar{x},\bar{t}),v_{xx}^j(\bar{x},\bar{t}),v_t^j(\bar{x},\bar{t})),$$

but this contradicts (12).

Thus we have proved that $\varepsilon_0=1$; i.e.

$$u(x,t)<v(x,t) \quad \text{in} \quad \bar{D}\cap\{(x,t):\ t_i^0+\bar{\varepsilon}\leqslant t_i<T_i \quad (i=1,\dots,m)\}.$$

Since $\bar{\varepsilon}$ can be taken arbitrarily small, this completes the proof.

3. Weak differential inequalities

We make the following assumptions:

(IV) The functions $f^i(x,t,u,q,r,s)$ $(i=1,\dots,N)$ satisfy assumption (I) and each f^i is decreasing with respect to s_i.

(V) Assumption (III) with ">" and "<" replaced by "$\geqslant$" and "$\leqslant$", respectively.

(VI) For $u\leqslant\bar{u}$ the following inequalities are filfilled

$$f^i\Big(x,t,u,q,r,s+\sigma^i\Big(\sum_{j=1}^m(t_j-t_j^0),\bar{u}-u\Big)\Big)\leqslant f^i(x,t,u,q,r,s) \quad \Big(s+\sigma^i=\{s_k+\sigma^i\}_{k=1}^m\Big),$$

where $\sigma^i(\tau,y_1,\dots,y_N)$ $(i=1,\dots,N)$ are non-negative and continuous for

$$0\leqslant\tau<\tau_0=\sum_{j=1}^m(T_j-t_j^0), \quad y_k\geqslant 0 \quad (k=1,\dots,N).$$

Moreover, we assume that $\sigma^i(\tau,0)\equiv 0$ $(i=1,\dots,N)$ and that $y(\tau)\equiv(0,\dots,0)$ is the unique solution of the problem

$$\frac{dy_i}{d\tau}=\sigma^i(\tau,y), \quad y_i(0)=0 \quad (i=1,\dots,N).$$

T h e o r e m 2. If assumptions (II), (IV)-(VI) hold, then $u(x,t)\leqslant v(x,t)$ in D.

P r o o f. We apply the same method as in the proof of Theorem 1 of [1].

Let $y^{\delta}(\tau) = \left(y_1^{\delta}(\tau), \ldots, y_N^{\delta}(\tau)\right)$, $\delta > 0$ be a solution of the problem

$$\frac{dy_i}{d\tau} = \sigma^i(\tau, y) + \delta, \quad y_i(0) = \delta \quad (i=1,\ldots,N).$$

For any $\varepsilon \in (0, \tau_0)$ there is $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ the functions $y_i^{\delta}(\tau)$ are defined in the interval $[0, \tau_0 - \varepsilon)$ and

$$\lim_{\delta \to 0} y_i^{\delta}(\tau) = 0 \quad (i=1,\ldots,N),$$

where the convergence is uniform in this interval.

Denote by D^{ε}, Σ^{ε} and $\Sigma_{\alpha^i}^{\varepsilon}$ the parts of D, Σ and Σ_{α^i} respectively, contained in the strip

$$\left\{(x,t):\ t_j^o \leqslant t_j < T_j - \frac{\varepsilon}{m} \quad (j=1,\ldots,m)\right\},$$

where

$$0 < \frac{\varepsilon}{m} < \max_j\ (T_j - t_j^o).$$

Let us put

$$v^{\delta}(x,t) = v(x,t) + \omega^{\delta}(t),$$

where

$$\omega^{\delta}(t) = y^{\delta}\left(\sum_{j=1}^{m} (t_j - t_j^o)\right).$$

Then we have

$$f^i\left(x,t,u,u_x^i,u_{xx}^i,u_t^i\right) - f^i\left(x,t,v^{\delta},v_x^{i\delta},v_{xx}^{i\delta},v_t^{i\delta}\right) =$$

$$= \left[f^i\left(x,t,u,u_x^i,u_{xx}^i,u_t^i\right) - f^i\left(x,t,v,v_x^i,v_{xx}^i,v_t^i\right)\right] +$$

$$+ \left[f^i(x,t,v,v_x^i,v_{xx}^i,v_t^i) - f^i\left(x,t,v^{\delta},v_x^i,v_{xx}^i,v_t^i + \sigma^i\left(\sum_{j=1}^{m} (t_j - t_j^o),\ \omega^{\delta}(t)\right) + \delta\right]\right].$$

The first difference in brackets is, by assumption (V), non-negative. According to assumptions (IV), (VI) the second difference is positive. Thus we have proved that

$$(13)\qquad f^i\left(x,t,u,u_x^i,u_{xx}^i,u_t^i\right) > f^i\left(x,t,v^\delta,v_x^{i\delta},v_{xx}^{i\delta},v_t^{i\delta}\right)$$

in D^ε (for $0<\delta<\delta_0$), $i=1,\dots,N$.

Since $\omega^{i\delta}(t)\geqslant\delta>0$, we obtain (by assumption (V)) the following inequalities

$$(14)\quad \begin{cases} u(x,t^0) < v^\delta(x,t^0), & x\in S_{t^0}, \\ \beta^i(x,t)\left[u^i(x,t)-v^{i\delta}(x,t)\right]-\alpha^i(x,t)\dfrac{d\left[u^i-v^{i\delta}\right]}{dl^i}<0, & (x,t)\in\Sigma_{\alpha^i}^\varepsilon, \\ u^i(x,t)-v^{i\delta}(x,t)<0, & (x,t)\in\Sigma^\varepsilon\setminus\Sigma_{\alpha^i}^\varepsilon \ (i=1,\dots,N). \end{cases}$$

Inequalities (13),(14) imply, by Theorem 1, that

$$u(x,t)<v^\delta(x,t), \qquad (x,t)\in D^\varepsilon,$$

whence, letting $\delta\to 0$, we have

$$u(x,t)\leqslant v(x,t), \qquad (x,t)\in D^\varepsilon.$$

Since ε is arbitrary, the theorem is proved.

As an immediate consequence of Theorem 2 we obtain the following corollaries.

C o r o l l a r y 1 (Uniqueness criterion). If assumptions (IV) and (VI) are satisfied, then the first mixed problem (1)-(3) admits at most one parabolic, Σ_α -regular solution.

C o r o l l a r y 2 (Maximum principle). Let assumptions (IV) and (VI) be satisfied. Assume that for $u\geqslant 0,(x,t)\in D$ we have

$$f^i(x,t,u,0,0,0)\leqslant 0 \quad (i=1,\dots,N).$$

Suppose that the function $u(x,t)$ is a Σ_α-regular and parabolic solution of the system of differential inequalities

$$f^i\left(x,t,u,u^i_x,u^i_{xx},u^i_t\right) \geqslant 0, \qquad (x,t)\in D \quad (i=1,\dots,N)$$

and, moreover, the initial inequalities

$$u(x,t^0) \leqslant c = (c_1,\dots,c_N), \qquad x\in S_{t^0}$$

and the boundary inequalities

$$\beta^i(x,t)u^i(x,t)-\alpha^i(x,t)\,\frac{du^i}{dl^i} \leqslant c_i\beta^i(x,t), \quad (x,t)\in \Sigma_{\alpha^i},$$

$$u^i(x,t) \leqslant c_i, \qquad (x,t)\in \Sigma \setminus \Sigma_{\alpha^i} \quad (i=1,\dots,N)$$

hold true, where c_i are non-negative constants. Under these assumptions we have $u(x,t) \leqslant c$ in D.

BIBLIOGRAPHY

[1] P. B e s a l a: On weak differential inequalities, Ann.Polon.Math. 16(1965) 185-194.

[2] W. M l a k: Differential inequalities of parabolic type, Ann.Polon. Math. 3(1957) 349-354.

[3] J. S z a r s k i: Sur un système non linéaire d'inégalités différentielles paraboliques, Ann.Polon. Math. 15(1964) 15-22.

[4] J. S z a r s k i: Differential inequalities. Warszawa 1965.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF GDAŃSK

Received May 29th, 1973.