DEMONSTRATIO MATHEMATICA
Vol. VII No 1 1974

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THE GENERALIZED LÖWNER EQUATION IN THE CLASS OF QUASI-α-STARLIKE MEROMORPHIC k-SYMMETRIC FUNCTIONS

1. Introduction

Let Σ_α^* denote, for any fixed α in the interval $[0,1)$, the class of functions

$$F(z) = \sum_{n=0}^{\infty} A_{n-1} z^{n-1}, \quad A_{-1} = 1, \quad 0<|z|<1$$

satisfying the condition: $\operatorname{re} \frac{-zF'(z)}{F(z)} > \alpha$ when $|z|<1$. The functions of the class Σ_α^* are called α-starlike meromorphic functions. In particular, for $\alpha = 0$ we denote Σ_α^* by Σ^*. This is the class of starlike meromorphic functions. It is easy to see that $\Sigma_\alpha^* \subset \Sigma^*$ for every $\alpha \in [0,1)$.

Next, for any fixed natural k, let $\Sigma_\alpha^{*(k)}$ denote the class of those functions $F \in \Sigma_\alpha^*$ which possess an expansion of the form

$$F(z) = \sum_{n=0}^{\infty} A_{nk-1} z^{nk-1}, \quad A_{-1} = 1, \quad 0 < |z| < 1.$$

The functions of the class $\Sigma_\alpha^{*(k)}$ are called α-starlike meromorphic k-symmetric functions. From the definitions above it follows directly that $\Sigma_\alpha^{*(k)} \subset \Sigma_\alpha^*$ for $k=1,2,\ldots$ and

$$\Sigma_\alpha^{*(1)} \equiv \Sigma_\alpha^* .$$

Consider the class $\Sigma_\alpha^{M(k)}$ of quasi-α-starlike meromorphic k-symmetric functions f defined by the equation

(1) $$F\left(\frac{1}{f}\right) = MF(z), \qquad 0<|z|<1,$$

where F denotes an arbitrary function belonging to the class $\Sigma_{\alpha}^{*(k)}$, and M is a fixed number in the interval $[1,\infty)$. In particular, putting $k=1$ or $\alpha=0$ we obtain classes of functions introduced and investigated previously, namely:

$$\Sigma_0^{M(1)} \equiv \Sigma^{M} \text{ (see [1])}, \Sigma_{\alpha}^{M(1)} \equiv \Sigma_{\alpha}^{M} \text{ (see [2])}, \Sigma_0^{M(k)} \equiv \Sigma^{M(k)} \text{ (see [3])}.$$

Let $f(z,t)$ denote the function defined by the equation:

(2) $$F\left(\frac{1}{f}\right) = e^{t} F(z), \quad 0<|z|<1, \qquad 0 \leqslant t < \infty,$$

where F denotes an arbitrary function of the class $\Sigma_{\alpha}^{*(k)}$. It is easy to see that the function $f(z,T)$ belongs to the class $\Sigma_{\alpha}^{M(k)}$ for $M = e^{T}$. Moreover it can be proved (see [4]) that for the functions f and F in (2), where F is any fixed function of the class $\Sigma_{\alpha}^{*(k)}$, we have

(3) $$\lim_{t\to\infty} e^{-t} f(z,t) = F(z).$$

Hence the functions $F \in \Sigma_{\alpha}^{*(k)}$ can be approximated by functions in the class $\Sigma_{\alpha}^{M(k)}$.

In the sequel, let $\mathcal{P}_{\alpha}^{(k)}$, $\alpha \in [0,1)$, denote the class of the functions

(4) $$p(z) = \sum_{n=0}^{\infty} b_{nk} z^{nk}, \quad b_0 = 1, \quad |z|<1,$$

satisfying the condition: $\operatorname{re} p(z) > \alpha$ if $|z|<1$.

Finally, let $\mathcal{P}^{(k)}(m)$, $\frac{1}{2} < m < \infty$, denote the class of functions of the form (4) satisfying the condition:

$|p(z) - m| < m$, whenever $|z|<1$. Observe that $\mathcal{P}^{(k)}(\infty) \equiv \mathcal{P}_0^{(k)}$.

In the present work we shall obtain the generalized Löwner's equation to the functions of the class $\Sigma_{\alpha}^{M(k)}$. Moreover, using this equation we shall obtain an estimation for basic functionals in the class $\Sigma_{0}^{M(k)} \equiv \Sigma^{M(k)}$ of quasi-starlike meromorphic functions.

2. The generalized Löwner equation

T h e o r e m 1. A function $f(z)$ belongs to the class $\Sigma_{\alpha}^{M(k)}$ if and only if $f(z) = f(z,T)$, where $f(z,t)$ is a solution of the equation

$$(5)\quad \frac{\partial f(z,t)}{\partial t} = f(z,t)P\left(\frac{1}{f(z,t)}\right), \qquad 0 < |z| < 1, \quad 0 \leqslant t \leqslant T$$

with the initial condition $f(z,0) = 1/z$, where P in (5) denotes a function of the class $\wp^{(k)}(m)$, with $m = 1/2\alpha$ and $T = \log M$.

P r o o f. Take an arbitrary function $F \in \Sigma_{\alpha}^{*(k)}$. It is known that the function $p(z) = \frac{-zF'(z)}{F(z)}$ belongs to the class $\wp_{\alpha}^{(k)}$. With the above notation we have

$$\frac{F'(z)}{F(z)} + \frac{1}{z} = \frac{1-p(z)}{z}.$$

After integrating this equation from 0 to z, we get

$$(6)\qquad F(z) = \frac{1}{z}\exp\left(-\int_0^z \frac{p(\zeta)-1}{\zeta}\, d\zeta\right).$$

From (6) and (2) it follows that

$$f(z,t)\exp\left(-\int_0^{\frac{1}{f(z,t)}} \frac{p(\zeta)-1}{\zeta}\, d\zeta\right) = \frac{1}{z}\exp\left(t - \int_0^z \frac{p(\zeta)-1}{\zeta}\, d\zeta\right).$$

From this, taking the logarithm of both sides and then differentiating with respect to t, we obtain

$$(7) \qquad \frac{\partial \log f(z,t)}{\partial t} + \frac{p(1/f(z,t))-1}{f(z,t)} \cdot \frac{\partial f(z,t)}{\partial t} = 1$$

and consequently

$$(8) \qquad \frac{\partial f(z,t)}{\partial t} = f(z,t)\frac{1}{p(1/f(z,t))} .$$

Upon denoting $P(z) = 1/p(z)$, equation (8) can be rewritten in the form (5). Making use of the fact that $p \in \mathcal{P}_{\alpha}^{(k)}$ we easily show that the function $P(z)$ satisfies the conditions $P(0)=1$, $\left|P(z) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$ whenever $|z|<1$. Hence the function P belongs to the class $\mathcal{P}^{(k)}(m)$ where $m = 1/2\alpha$. If $\alpha = 0$, then $P \in \mathcal{P}^{(k)}(\infty) = \mathcal{P}_{0}^{(k)}$.

From the consideration above it follows that each function $f \in \sum_{\alpha}^{M(k)}$ can be represented in the form $f(z) = f(z,T)$, $T = \log M$, where $f(z,t)$ is a solution of equation (5) and $P \in \mathcal{P}^{(k)}(m)$, $m = 1/2\alpha$ (when $\alpha=0$ we take $m=\infty$).

It is not difficult to prove the converse theorem (see [5]).

3. Estimation of some functionals in the class $\sum^{M(k)}$

Theorem 2. For any arbitrary function $f \in \sum^{M(k)}$ the following inequalities hold

$$(9) \qquad m(r) \leqslant |f(z)| \leqslant M(r) \qquad \text{for} \quad 0<|z|=r<1,$$

where

$$(9') \qquad m(r) = \frac{M}{r}\left(\frac{1}{2}(1-r^k)^2 + \frac{r^k}{M^k} + \frac{1}{2}(1-r^k)\sqrt{(1-r^k)^2 + \frac{4r^k}{M^k}}\right)^{1/k},$$

$$(9'') \qquad M(r) = \frac{M}{r}\left(\frac{1}{2}(1+r^k)^2 - \frac{r^k}{M^k} + \frac{1}{2}(1+r^k)\sqrt{(1+r^k)^2 - \frac{4r^k}{M^k}}\right)^{1/k},$$

In particular, for the function f defined by the equation

$$(10') \qquad \frac{1}{f}\,(f^k - 1)^{2/k} = \frac{M}{z}\,(z^k - 1)^{2/k}$$

we have $|f(z)| = m(r)$, $|z| = r$, and for the function f satisfying the equation

$$(10'') \qquad \frac{1}{f}\,(f^k + 1)^{2/k} = \frac{M}{z}\,(z^k + 1)^{2/k}$$

we have the equality $|f(z)| = M(r)$, $|z| = r$.

P r o o f. From Theorem 1 we know that a function f belongs to $\sum^{M(k)}$ if and only if $f(z) = f(z,T)$, $T = \log M$, where $f(z,T)$ is a solution of equation (5) in which P denotes a function of the class $\mathcal{P}^{(k)}(\infty) = \mathcal{P}_0^{(k)}$. From equation (5) it follows that

$$(11) \qquad d_t \log f(z,t) = P\,(1/f(z,t))dt\,,$$

and next

$$(11') \qquad d_t \log |f(z,t)| = \mathrm{re}\, P(1/f(z,t))dt\,.$$

It is well-known that every function $P \in \mathcal{P}_0^{(k)}$ can be represented in the form

$$(12) \qquad P(z) = \int_0^{2\pi} \frac{e^{i\theta} + z^k}{e^{i\theta} - z^k}\, d\mu(\theta),$$

where $\mu(\theta)$ is a non-decreasing function in the interval $[0,2\pi]$ such that $\int_0^{2\pi} d\mu(\theta) = 1$. From this we obtain the inequality

$$(13) \qquad \frac{1-r^k}{1+r^k} \leqslant \mathrm{re}\, P(z) \leqslant \frac{1+r^k}{1-r^k} \quad \text{for } |z| = r < 1.$$

The following inequalities follow from (11′) and (13′):

$$(14) \qquad \frac{|f(z,t)|^k-1}{|f(z,t)|^k+1}\,dt \leqslant d_t \log|f(z,t)| \leqslant \frac{|f(z,t)|^k+1}{|f(z,t)|^k-1}\,dt.$$

By integrating (14) in the interval $[0,T]$, $T=\log M$, and taking into account (11′) we obtain

$$\frac{\left(|f(z,T)|^k+1\right)^2}{|f(z,T)|^k} \leqslant e^{kT}\,\frac{(r^k+1)^2}{r^k}$$

and

$$\frac{\left(|f(z,T)|^k-1\right)^2}{|f(z,T)|^k} \geqslant e^{kT}\,\frac{(r^k-1)^2}{r^k}.$$

The assertion of Theorem 2 follows directly from the inequalities above.

Theorem 2 implies the following theorem on covering for quasi-starlike meromorphic k-symmetric functions.

T h e o r e m 3. Each function $w=f(z)$ belonging to the class $\sum^{M(k)}$ maps the unit disc $K_1 \equiv \{z: |z|<1\}$ onto a region D_f containing the $D_0=\{w: |w|>R_0\}$, where $R_0=\left(2M^k-1+2\sqrt{M^k(M^k-1)}\right)^{1/k}$.

The function f defined by the equation

$$\frac{1}{f}\,(f^k+1)^{2/k} = \frac{M}{z}(z^k+1)^{2/k}$$

maps the disc K_1 onto the domain $D=E_2\,(K_2 \cup \bigcup_{n=0}^{K=0} I_n)$, where E_2 is the complex plane of Gauss, $K_2=\{w: |w|<1\}$, $I_n = \left\{w:\ w=te^{i\frac{2\pi n}{k}},\ 1\leqslant t\leqslant R_0\right\}$.

By taking limit (3) in the above theorems we obtain previously known theorems for starlike meromorphic k-symmetric functions.

T h e o r e m 2′. For any function $F \in \Sigma^{*(k)}$ we have

$$\frac{1}{r}(1-r^k)^{2/k} \leqslant |F(t)| \leqslant \frac{1}{r}(1+r^k)^{2/k}, \qquad 0<|z| = r<1,$$

where the equality holds for the function $F(z)=\frac{1}{z}(z^k \pm 1)^{2/k}$.

T h e o r e m 3′. Each function $F \in \Sigma^{*(k)}$ maps the disc $K_1=\{z: |z|<1\}$ onto a region D_F containing $D_0=\{w: |w|>\sqrt[k]{4}\}$. In particular, the function $F(z)=\frac{1}{2}(z^k+1)^{2/k}$ maps the disc K_1 onto the region $D = E_2 \setminus \bigcup_{n=0}^{k-1} I_n$, where $I_n=\{w: w=te^{i2\pi n/k}, 0 \leqslant t \leqslant \sqrt[k]{4}\}$.

In the sequel we shall estimate the argument of a function in the class $\Sigma^{M(k)}$, where $\arg zf(z)$ will denote that branch of a multiple valued function which satisfies the condition $\lim_{z\to 0} \arg zf(z) = 0$.

T h e o r e m 4. For any function $f \in \Sigma^{M(k)}$ the following inequalities hold

$$(15) \qquad |\arg zf(z)| \leqslant \frac{1}{k} \log \frac{(|f(z)|^k-1)(1+r^k)}{(|f(z)|^k+1)(1-r^k)}, \qquad 0<|z|=r<1,$$

$$(16) \qquad |\arg zf(z)| \leqslant \frac{1}{2k} \log\left(1 + \left(1-\frac{1}{M^k}\right)\frac{4r^k}{(1-r^k)^2}\right), \qquad 0<|z|=r<1.$$

The equality in (15) holds for functions f defined by the equation

$$(17) \qquad \frac{1}{f}(f^k+\sigma)^{2/k} = \frac{M}{z}(1+\sigma z^k)^{2/k}, \qquad |\sigma|=1,$$

where σ is selected in such a way that $\mathrm{im}(\sigma\overline{f^k(z)})=|f(z)|^k$ or $\mathrm{im}\left(\sigma\overline{f^k(z)}\right) = -|f(z)|^k$.

P r o o f. From (11) it follows that

$$d_t \arg f(z,t) = \mathrm{im}\left(1/f(z,t)\right) dt.$$

From this, taking into account (11'), we get

$$d_t \arg f(z,t) = \frac{\operatorname{im} P(1/f(z,t))}{\operatorname{re} P(1/f(z,t))} d_t \log |f(z,t)|. \tag{18}$$

On the other hand, using (12) we can show that

$$\left|\frac{\operatorname{im} P(z)}{\operatorname{re} P(z)}\right| \leqslant \frac{2\,r^k}{1-r^{2k}}, \qquad |z| = r < 1, \tag{19}$$

where the equality holds for the function $P(z) = \frac{1 \pm iz^k}{1 \pm iz^k}$ with $z=r$. Consequently,

$$\left|\frac{\operatorname{im} P(1/f(z,t))}{\operatorname{re} P(1/f(z,t))}\right| \leqslant \frac{2\,|f(z,t)|^k}{|f(z,t)|^{2k}-1} \quad \text{for} \quad |z|<1, \tag{20}$$

where the equality holds for the function $P\left(\frac{1}{f}\right) = \frac{f^k + \sigma}{f^k - \sigma}$ with σ being a number such that $|\sigma| = 1$ and $\operatorname{im}\left(\sigma \overline{f^k(z,t)}\right) = |f(z,t)|^k$ or $\operatorname{im}\left(\sigma \overline{f^k(z,t)}\right) = -|f(z,t)|^k$. From (18) and (20) we obtain the inequality

$$|d_t \arg f(z,t)| \leqslant \frac{2\,|f(z,t)|^k}{|f(z,t)|^{2k}-1} d_t \log |f(z,t)|.$$

By integrating it in the interval $[0,T]$, $T = \log M$, and taking into account $f(z,0) = \frac{1}{z}$, $f(z,T) = f(z)$, we get inequality (15). From this and from (9) - (9'') we obtain (16).

The estimation (15) is sharp. We get extremal functions in (15) by taking $P\left(\frac{1}{f}\right) = \frac{f^k + \sigma}{f^k - \sigma}$ and selecting σ in such a way that $\operatorname{im}\left(\sigma \overline{f^k(z,t)}\right) = |f(z,t)|^k$, or else $\operatorname{im}\left(\sigma \overline{f^k(z,t)}\right) = -|f(z,t)|^k$.

This shows that the extremal functions are defined by equation (17) with a suitably selected δ, which ends the proof.

In subsequent consideration our aim will be to find an estimation for the modulus of the derivative and the modulus of the logarithmic derivative of a quasi-starlike meromorphic k-symmetric function.

T h e o r e m 5. For any function $f\in\Sigma^{M(k)}$ the following inequalities hold

$$(21)\quad \frac{|f(z)|\left(|f(z)|^k+1\right)}{|f(z)|^k-1}\cdot\frac{1-r^k}{r(1+r^k)}\leqslant|f'(z)|\leqslant\frac{|f(z)|\left(|f(t)|^k-1\right)}{|f(z^k)|+1}\cdot\frac{1+r^k}{r(1-r^k)},\qquad 0<|z|=r<1,$$

$$(22)\quad \frac{|f(z)|^k+1}{|f(z)|^k-1}\cdot\frac{1-r^k}{1+r^k}\leqslant\left|\frac{zf'(z)}{f(z)}\right|\leqslant\frac{|f(z)|^k-1}{|f(z)|^k+1}\cdot\frac{1+r^k}{1-r^k},\qquad 0<|z|=r<1,$$

$$(23)\quad \left|\frac{f'(z)}{f^2(z)}\right|\leqslant\begin{cases}\dfrac{\left(\frac{1}{2}(1-r^k)^2+\frac{r^k}{M^k}+\frac{1}{2}(1-r^k)^2\sqrt{1+\frac{4r^k}{M^k(1-r^k)^2}}\right)^{-1/k}}{M\sqrt{1-\left(1-\frac{1}{M^k}\right)\frac{4r^k}{(1+r^k)^2}}}, & 0<|z|=r\leqslant r.\\[2ex] \dfrac{k\left(\sqrt{k^2+1}-k\right)^{1/k}}{\left(\sqrt{k^2+1}+1\right)}\cdot\dfrac{1+r^k}{r(1-r^k)}, & r_0<|z|=r<1,\end{cases}$$

$$(24)\quad \left|\frac{f'(z)}{f^2(z)}\right| \geqslant \frac{\left(\frac{1}{2}(1+r^k)^2-\frac{r^k}{M^k}+\frac{1}{2}(1+r^k)^2\sqrt{1-\frac{4r^k}{M^k(1+r^k)^2}}\right)^{-1/k}}{M\sqrt{1+\left(1-\frac{1}{M^k}\right)\frac{4r^k}{(1-r^k)^2}}},$$

$$0<|z|=r<1,$$

$$(25)\quad \frac{1}{\sqrt{1+\left(1-\frac{1}{M^k}\right)\frac{4r^k}{(1-r^k)^2}}} \leqslant \left|\frac{z\,f'(z)}{f(z)}\right| \leqslant \sqrt{1+\left(1-\frac{1}{M^k}\right)\frac{4r^k}{(1-r^k)^2}},$$

$$0<|z|=r<1,$$

where $\quad r_o=\left(\lambda-\sqrt{\lambda^2-1}\right)^{1/k}, \quad \lambda=1+\frac{\sqrt{k^2+1}-1}{M^k}.$

We obtain the equality sign on the right-hand sides of (21) and (22) for the function f defined by equation (10′), and on the left-hand sides for the function f in (10″). The estimation (23) is sharp for $0<|z|=r\leqslant r_o$. The function f in (10′) is an extremal function. We obtain the equality sign in (24) and on the left-hand side of (25) for the function f in (10″).

P r o o f. From equation (5) it follows that

$$(26)\quad d_t\log|f'_z(z,t)| = \left(\text{re } P\,(1/f(z,t)) - \text{re}\left(\frac{P\,(1/f(z,t))}{f(z,t)}\right)\right)dt.$$

From this, in view of (11′), we get

$$(27)\quad d_t\log|f'_z(z,t)| = \left(1-\frac{\text{re }\left(1/f(z,t)P(1/f(z,t))\right)}{\text{re } P(1/f(z,t))}\right) d_t\log|f(z,t)|.$$

Using formula (12) it can be shown that

$$\frac{-2kr^k}{1-r^{2k}}\,\mathrm{re}\,P(z) \leqslant \mathrm{re}\big(z\,P'(z)\big) \leqslant \frac{2kr^k}{1-r^{2k}}\,\mathrm{re}\,P(z), \qquad |z| = r<1.$$

This implies

$$(28) \qquad \frac{-2kr^k}{1-r^{2k}} \leqslant \frac{\mathrm{re}\big(z\,P'(z)\big)}{\mathrm{re}\,P(z)} \leqslant \frac{2kr^k}{1-r^{2k}}, \qquad |z| = r<1.$$

From (27) and (28) we obtain the inequalities

$$\left(1-\frac{2k\,|f(z,t)|^k}{|f(z,t)|^{2k}-1}\right) d_t \log|f(z,t)| \leqslant d_t \log|f'_z(z,t)| \leqslant$$

$$\leqslant \left(1+\frac{2k\,|f(z,t)|^k}{|f(z,t)|^{2k}-1}\right) d_t \log|f(z,t)|.$$

Upon integrating it with respect to t in the interval $[0,T]$, $T=\log M$ we obtain

$$(29) \qquad \frac{|f(z,T)|\big(|f(z,T)|^k+1\big)}{|f(z,T)|^k-1}\cdot\frac{1-r^k}{r(1+r^k)} \leqslant |f'(z,T)| \leqslant$$

$$\leqslant \frac{|f(z,T)|\big(|f(z,T)|^k-1\big)}{|f(z,T)|^k-1}\cdot\frac{1+r^k}{r(1-r^k)}.$$

Putting $f(z,T) = f(z)$ in (29) we immediately obtain inequalities (21) and (22). The equality sign in (21) and (22) is obtained for the functions defined by equations (10″) and (10′).

Inequality (21) implies directly the following inequality

$$(30) \quad \frac{|f(z)|^k+1}{|f(z)|\big(|f(z)|^k-1\big)}\cdot\frac{1-r^k}{r(1+r^k)} \leqslant \left|\frac{f'(z)}{f^2(z)}\right| \leqslant \frac{|f(z)|^k-1}{|f(z)|\big(|f(z)|^k+1\big)}\cdot\frac{1+r^k}{r(1-r^k)}$$

$$0<|z| = r<1.$$

Denoting $x = |f(z)|$, $A_1 = \frac{1+r^k}{r(1-r^k)}$, we can rewrite the right-hand side of inequality (30) in the form

$$h_1(x) = A_1 \frac{x^k-1}{x(x^k+1)}$$

where, according to (9), $x \in [m(r), M(r)] \subset (1, \infty)$. It is easy to check that the function h_1 is increasing for $x \in (1, x_0)$ and decreasing for $x \in (x_0, \infty)$, where $x_0 = (k+\sqrt{k^2+1})^{1/k}$. If $m(r) \geq x_0$ i.e. $0 < r \leq r_0 = \left(\lambda - \sqrt{\lambda^2-1}\right)^{1/k}$, $\lambda = 1 + \frac{\sqrt{k^2+1}-1}{M^k}$, the function $h_1(x)$ attains its maximum in the interval $[m(r), M(r)]$ for $x = m(r)$. Consequently, putting $x = |f(z)| = m(r)$ into the right-hand side of inequality (30), we obtain, in view of (9′), the first inequality in (23). Equality is realized by the function f defined in (10′). If $m(r) < x_0$, i.e. $r_0 < r < 1$, we substitute $x = |f(z)| = x_0 = \left(k+\sqrt{k^2-1}\right)^{1/k}$, into the right-hand side of (30) and we obtain the second inequality in (23). Let us mention that r_0 depends upon two parameters: M and k. It is easy to see that $\lim_{M\to\infty} r_0 = 1$ and $\lim_{k\to\infty} r_0 = 1$. Hence inequality (23) gives a sharp estimation for the considered functional in the class $\Sigma^{M(k)}$ in an arbitrary disc of the form $|z| \leq r < 1$ for sufficiently large k.

Denoting $x = |f(z)|$ and $A_2 = \frac{1-r^k}{r(1+r^k)}$ we can rewrite the left-hand side of inequality (30) as follows

$$h_2(x) = A_2 \frac{x^k+1}{x(x^k-1)}$$

where, as previously, $x \in [m(r), M(r)] \subset (1, \infty)$. The function h_2 is decreasing in the interval $(1, \infty)$. This shows that the function $h_2(x)$ attains a minimum in the interval $[m(r), M(r)]$

for $x=M(r)$. Substituting $x=|f(z)|=M(r)$ into the left-hand side of (30) and taking into account (9″) we obtain inequality (24). This estimation is sharp. We obtain equality for the function f defined by equation (10″).

Finally let us prove inequality (25). The right-hand side of (22) can be written in the form

$$h_3(x) = A_3 \frac{x^k-1}{x^k+1},$$

where $x=|f(z)|$, $A_3 = \frac{1+r^k}{1-r^k}$, $x\in[m(r),\ M(r)]$. It is easy to check that the function h_3 is increasing in the interval $[m(r),\ M(r)]$. This implies that $h_3(x)$ attains a maximum in the considered interval for $x=M(r)$. Putting $x=|f(z)|=M(r)$ into the right-hand side of inequality (22) we obtain, in view of (9″) the equality on the right-hand side of (25). Analogously, the left-hand side of inequality (22) can be written in the form

$$h_4(x) = A_4 \frac{x^k+1}{x^k-1},$$

where $x=|f(z)|$, $A_4=(1-r^k)/(1+r^k)$, $x\in[m(r),\ M(r)]$. Since h_4 is decreasing in the interval $[m(r),\ M(r)]$, we obtain a minimum for $h_4(x)$ in the considered interval putting $x=M(r)$. Substituting $x=|f(z)|=M(r)$ into the left-hand side of inequality (22) and taking into account (9″), we obtain an estimation from below for the modulus of the logarithmic derivative. This estimation is sharp. We obtain equality for the function f in (10″). This ends the proof.

From the above theorem one obtains known inequalities for starlike meromorphic k-symmetric functions.

T h e o r e m 5′. For any function $F\in\Sigma^{*(k)}$ the following sharp estimation hold

$$\frac{1-r^k}{1+r^k} \leqslant \left|\frac{zF'(z)}{F(z)}\right| \leqslant \frac{1+r^k}{1-r^k}, \qquad 0<|z|=r<1,$$

$$\frac{1-r^k}{(1+r^k)^{1+2/k}} \leqslant \left|\frac{F'(z)}{F^2(z)}\right| \leqslant \frac{1+r^k}{(1-r^k)^{1+2/k}}, \qquad 0<|z|=r<1$$

and we obtain a minimum for the considered functionals for the function $F(z) = \frac{1}{z}(z^k+1)^{2/k}$, and a maximum for $F(z) = \frac{1}{z}(z^k-1)^{2/k}$.

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Received May 19th, 1973.