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INTEGRABILITY CONDITIONS OF (1,1) TENSOR FIELD f SATISFYING $f^3 - f = 0$

Structures on C^∞-manifolds, by introducing vectorvalued linear functions satisfying some algebraic relations, have been extensively studied by many mathematicians under various topics, such as complex and almost complex spaces, almost product spaces, f-structure spaces, contact and almost contact spaces. Yano [1] generalised the idea of almost complex structure by defining an f-structure which is a (1,1) tensor field f satisfying $f^3 + f = 0$. He also established [2] integrability conditions of the f-structure. Recently [3] the present authors generalised the idea of an almost product structure by defining an f-(3, -1)-structure which is given by (1,1) tensor field f which satisfies $f^3 - f = 0$; $f \neq 0, 1, \ldots$

In the present paper, we study integrability conditions of an f-(3,-1)-structure. First section is introductory. In second and third section we get some conditions of integrability of the distributions of f-(3,-1)-structure manifold.

1. Preliminaries. Let M^n be an n-dimensional C^∞-manifold equipped with a (1,1) tensor field $f \neq 0$; $f \neq I$ satisfying the condition

$$f^3 - f = 0; \tag{1.1}$$

I being the unit tensor field. We call such a manifold an f-(3,-1)-manifold [3].

Let l and m be two operators given by [3]

$$l = f^2; \qquad m = 1 - f^2;$$

then

$$(1.2) \qquad l^2 = l; \quad m^2 = m; \quad lm = ml = 0; \quad l + m = 1.$$

Thus these operators are complementary projection operators on M^n.

T h e o r e m 1.1. It is easy to verify the following

$$(1.3) \qquad fl = lf = f; \quad f^2 l = l; \quad fm = mf = 0; \quad f^2 m = 0.$$

Let rank $f = r$, then there exist two complementary distributions D_L and D_M corresponding to the projection operators l and m respectively. The dimensions of D_L and D_M are r and $(n-r)$ respectively. Also the tensor field f acts as an almost product structure on D_L and as a null operator on D_M.

Throughout this paper, we assume that our space is an f-(3,-1)-manifold and all the vector fields X, Y, Z etc. are differential and belong to M^n.

Let $N(X,Y)$ be the Nijenhuis tensor corresponding to the (1,1) tensor field f, i.e.

$$(1.4) \qquad N(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + l[X,Y].$$

Nijenhuis tensor satisfies the following relations:

$$(1.5) \qquad N(mX, mY) = l[mX, mY] = lN(mX, mY),$$

$$(1.6) \qquad mN(lX, lY) = m[fX, fY],$$

$$(1.7) \qquad mN(fX, fY) = m[lX, lY],$$

$$(1.8) \qquad mN(X,Y) = m[fX, fY].$$

T h e o r e m 1.2. In an f-(3,-1)-manifold, the following conditions are equivalent:

$$mN(X,Y)=0$$

$$mN(fX,fY)=0,$$

$$mN(lX,lY)=0.$$

P r o o f. The proof follows from (1.6), (1.7) and (1.8).

T h e o r e m 1.3. In an f-(3,-1)-manifold, $N(fX,fY)=0$ iff $N(lX,lY)=0$.

P r o o f. The proof is a consequence of (1.2) and (1.3).

Let $(\mathcal{L}_Y f)$ be the Lie-derivative of the (1,1) tensor field f with respect to the vector field Y, then by definition [6]

$$(\mathcal{L}_Y f)(X) = f[X,Y] - [fX,Y], \tag{1.9}$$

and using (1.4), we have

$$N(lX,mY) = f(\mathcal{L}_{mY} f)(lX) = f\{l(\mathcal{L}_{mY} f)lX\}, \tag{1.10}$$

and

$$fN(lX,mY) = l(\mathcal{L}_{mY} f)(lX). \tag{1.11}$$

2. In this section, we discuss integrability conditions of the distributions D_L and D_M.

The distribution D_L is integrable [5] iff

$$m[lX,lY] = 0; \tag{2.1}$$

while D_M is integrable [5] iff

$$l[mX,mY] = 0. \tag{2.2}$$

These definitions in view of relation (1.5) and Theorem (1.2) provide the proof of the following

T h e o r e m 2.1. The distribution D_M is integrable iff

$$N(mX,mY) = 0, \quad \text{or} \quad \iota N(mX,mY) = 0.$$

T h e o r e m 2.2. The distribution D_L is integrable iff one of the conditions stated in the Theorem (1.2) is satisfied.

T h e o r e m 2.3. A necessary and sufficient condition for both the distributions D_L and D_M to be integrable is

$$N(X,Y) = \iota N(\iota X,\iota Y) + N(\iota X,mY) + N(mX,\iota Y).$$

P r o o f. Theorems (2.1) and (2.2) imply that both the distributions D_L and D_M are integrable iff $mN(\iota X,\iota Y) = 0$ and $N(mX,mY) = 0$ respectively. Also, Nijenhuis tensor can be written as

$$N(X,Y) = \iota N(\iota X,\iota Y) + mN(\iota X,\iota Y) + N(\iota X,mY) + N(mX,\iota Y) + N(mX,mY).$$

Thus, if both the distributions are integrable, then

$$N(X,Y) = \iota N(\iota X,\iota Y) + N(\iota X,mY) + N(mX,\iota Y).$$

Conversely

$$N(X,Y) = \iota N(\iota X,\iota Y) + N(\iota X,mY) + N(mX,\iota Y) \Longrightarrow N(mX,mY) = 0,$$

and also $mN(\iota X,\iota Y) = 0$.

Thus both the distributions are integrable, Q.E.D.

Let the distribution D_L be integrable and let X' be an arbitrary non-zero vector field which is tangent to an integral submanifold of D_L. We define an operator f' by

$$f'X' = fX'$$

then f' leaves invariant tangent spaces of every integral submanifold D_L. Also, Theorem 1.1 implies that f' acts as an almost product-structure on each integral sub-manifold of D_L.

Let X', Y' be any two vector-fields tangent to the integral submanifold of the distribution D_L, then

$$(2.3)\qquad N(X',Y') = [f'X', f'Y'] - f'[f'X', Y'] - f'[X', f'Y'] + [X', Y'].$$

Thus, $N(X',Y')$ can be regarded as a vector-valued two-form in each integral submanifold of the distribution D_L and its values are tangent to the integral submanifold. Let $N'(X',Y')$ denote the Nijenhuis tensor of the structure f'-induced on each integral submanifold of D_L. Then by (2.3),

$$(2.4)\qquad N(lX, lY) = N'(lX', lY').$$

D e f i n i t i o n . We call an f-(3,-1)-structure to be partially integrable if the distribution D_L is integrable and both the distributions of integral submanifold, of D_L, considered as an almost product space with the induced almost product structure f', are integrable.

T h e o r e m 2.4. An f-(3,-1)-structure is partially integrable iff

$$N(lX, lY) = 0$$

or equivalently,

$$N(fX, fY) = 0.$$

P r o o f. Let f-(3,-1)-structure be partially integrable then (2.4) implies $N(lX, lY) = 0; \Longleftrightarrow N(fX, fY) = 0$. Conversely: $N(lX, lY) = 0; \Longrightarrow N'(lX', lY') = 0 \Longrightarrow f'$-structure is integrable. Also $N(lX, lY) = 0; \Longrightarrow mN(lX, lY) = 0; \Longrightarrow D_L$ is integrable. Thus f-(3,-1)-structure is partially integrable, Q.E.D.

T h e o r e m 2.5. The distribution D_M is integrable and the f-(3,-1)-structure is partially integrable iff

$$N(X,Y) = N(lX, mY) + N(mX, lY).$$

P r o o f. Theorems (2.1), (2.2) and (2.4) provide the proof.

3. In this section, we prove certain theorems on integrability of the distributions.

T h e o r e m 3.1. A necessary and sufficient condition for the tensor field $\left\{ l(\underset{mY}{\mathcal{L}} f)l \right\}$ to be zero is

$$N(lX, mY) = 0.$$

P r o o f. The proof follows directly from (1.10) and (1.11).

Let the distributions D_L and D_M be integrable, then N(X,Y) will be of the form given in Theorem 2.3.

In this case we can choose a local coordinate system in such a way that the integral submanifolds of D_L are represented by taking the last (n-r) coordinates constant while the integral submanifolds of D_M are given by taking the first r coordinates constant. We call such a system, an adapted coordinate system.

In this coordinate system, the projection operators l and m have components

$$l = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}; \quad m = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

where I_r and I_{n-r} are unit matrices of order r and (n-r) respectively.

Since, $lf = fl = f$, and $mf = 0$, therefore

$$f = \begin{pmatrix} f_r & 0 \\ 0 & 0 \end{pmatrix}$$

in an adapted coordinate system; where f_r is an $r \times r$ square matrix. Also, the Lie-derivative $(\underset{mY}{\mathcal{L}} f)$ has components given by

$$f = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$

where $*$ stands for the components of f in the form of f^a_b; $(a,b,c = 1,2,3, \ldots, r)$.

Thus, if the tensor field $\{l(\underset{mY}{\mathcal{L}} f)l\}$ vanishes identically, then $(\underset{mY}{\mathcal{L}} f) = 0$, which shows that the components of f are independent of $(n-r)$-coordinates which are constant in the integral submanifold of D_L in an adapted frame. Converse also holds.

Thus, by Theorem 3.1, we have,

T h e o r e m 3.2. Let the distributions D_L and D_M be integrable and the adapted coordinate system be chosen, then a necessary and sufficient condition for the components of f to be independent of the coordinates which are constant in the integral submanifold of D_L is that one of the following conditions is satisfied

$$N(lX,mY) = 0 \quad \text{and} \quad N(X,Y) = lN(lX,lY).$$

D e f i n i t i o n. We call f-(3,-1)-structure to be integrable if

(i) the f-(3,-1)-structure is partially integrable,

(ii) the distribution D_M is integrable,

and

(iii) the components of f are independent of the coordinates which are constant in the integral submanifold of D_L.

T h e o r e m 3.3. The structure f-(3,1) is integrable iff

$$N(X,Y) = 0.$$

P r o o f. The proof is a consequence of the Theorems 2.1, 2.3, 3.2 and the definition of f-(3,-1)-structure to be integrable.

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