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SOME APPLICATIONS OF THE APPROXIMATION THEOREM OF ALEXIEWICZ-ORLICZ

1. Introduction

In the paper [1] A.Alexiewicz and W.Orlicz have proved the following approximation theorem.

T h e o r e m (Alexiewicz-Orlicz). Let S be a set dense in $[\alpha,\beta]$ and let $s(t)$ be a measurable function. If a function $f(t,u)$ defined for $a \leqslant t \leqslant b$, $\alpha \leqslant u \leqslant \beta$ is continuous for fixed t and measurable for fixed $u \in S$ and if $|f(t,u)| \leqslant s(t)$, then there exist continuous functions $f_n(t,u)$ such that

$$(1.1) \qquad |f_n(t,u)| \leqslant s(t)$$

and

$$(1.2) \qquad \lim_{n\to\infty} \max_{\alpha\leqslant u\leqslant\beta} |f_n(t,u) - f(t,u)| = 0 \text{ for almost every } t \in [a,b].$$

This theorem has been used by Alexiewicz and Orlicz in the proof of a Carathéodory type theorem for ordinary and partial differential equations ([1], [2]). Subsequently I have used this theorem to prove the existence of solutions of differential-integral equations with a lagging argument and a functional-differential equation of the hyperbolic type ([4],[5]). In this paper the approximation theorem of Alexiewicz-Orlicz has been used in the proof of some smooth approximation theorem for functions satisfying the Carathéodory condition. This theorem may be applied to the proof of Kneser's Theorem for differential equations which satisfy the Carathéodory condition. In section 3 we give a proof of this theorem for some differential-integral equations with a lagging argument. In [3] N.Kikuchi gave a proof of Kneser's theorem for more general functional-differential equations under the assumption that the equation satisfies the Carathéodory condition.

Let R denote the real line, R^n be an n-dimensional linear vector space with the norm $\|x\| = \max(|x_1|,\dots,|x_n|)$, for $x = (x_1,\dots,x_n)$. Let P denote the set in R^{n+1} defined by $P = \{(t,y): t_0 \leqslant t \leqslant T, \ \|y - \eta\| \leqslant a\}$, where $\eta \in R^n$, $a > 0$.

D e f i n i t i o n 1.1. A function $f: P \to R^n$ is said to satisfy the Carathéodory condition on P if $f(t,y)$ is measurable in t for fixed y, continuous in y for each fixed t, and for any fixed $(t,y) \in P$ there is a Lebesgue integrable function $m(t)$ such that $\|f(t,y)\| \leqslant m(t), (t,y) \in P$.

D e f i n i t i o n 1.2. A function $f(t,y)$ defined on a set P is said to be uniformly Lipschitz continuous on P with respect to y if there exists a constant L satisfying $\|f(t,y_2)-f(t,y_1)\| \leqslant L\|y_2 - y_1\|$ for all $(t,y_i) \in P$; $i=1,2$.

2. Smooth approximation theorems

In some situations, it will be convenient to approximate a given function on a closed parallelepiped uniformly by functions which are smooth (C^1 or C^∞) with respect to certain variables. For the continuous function this is accomplished by Lusternik - Stiekłow's method. Using this method and the approximation theorem of Alexiewicz - Orlicz we can extend this result to functions satisfying the Carathéodory condition.

T h e o r e m 2.1. Let $f: P \to R^n$ be a function satisfying the Carathéodory condition on P. For every $\varepsilon > 0$ there exists a function $f^\varepsilon: P \to R^n$ such that

(i) $\max\limits_{\|y-\eta\|\leqslant a} \|f^\varepsilon(t,y) - f(t,y)\| \to 0$ as $\varepsilon \to 0$ for almost every $t \in [t_0, T]$,

(ii) $\|f^\varepsilon(t,y)\| \leqslant m(t)$ for $(t,y) \in P$,

(iii) $f^{\varepsilon}(t,y)$ has continuous partial derivatives of all orders with respect to $y_1,\dots,y_n$.

P r o o f. The approximation theorem of Alexiewicz-Orlicz implies the existence of a continuous function $f_n(t,y)$ such that conditions (1.1), (1.2) are fulfilled. Hence for every $\varepsilon > 0$ there exists $N_1(\varepsilon)$ such that

$$\max_{\|y-\eta\|\leqslant\alpha} \|f_n(t,y) - f(t,y)\| < \frac{\varepsilon}{2}$$

for $n > N_1(\varepsilon)$ and almost every $t \in [t_0,T]$. Let $f^*(t,y) = f_n(t,y)$ for fixed $n > N_1(\varepsilon)$. The function $f^*(t,y)$ is continuous and such that

$$\|f^*(t,y)\| \leqslant m(t) \quad \text{for} \quad (t,y) \in P.$$

Using the smooth approximation method of Lusternik and Stieklow for the continuous function f^*, we can define for every $\omega>0$ a function f^{ω}: $P \longrightarrow R^n$ satisfying the following conditions on P,

1° $\|f^{\omega}(t,y) - f^*(t,y)\| \longrightarrow 0$ as $\omega \longrightarrow 0$ uniformly on P,

2° $\|f\ (t,y)\| \leqslant m(t)$ for $(t,y) \in P$,

3° $f^{\omega}(t,y)$ has continuous partial derivatives of all orders with respect to $y_1,\dots,y_n$.

Let us take now $\omega = 1/n$ and $f^{\omega}(t,y) = f_n^*(t,y)$. For $\varepsilon > 0$ there exists $N_2(\varepsilon)$ such that

$$\|f^*(t,y) - f_n^*(t,y)\| < \varepsilon/2$$

for $n > N_2(\varepsilon)$ and $(t,y) \in P$. Let us denote $f^{\varepsilon}(t,y) = f^*_{N(\varepsilon)}(t,y)$, where $N(\varepsilon) > \max\big(N_1(\varepsilon),\ N_2(\varepsilon)\big)$. We have

$$\|f\ (t,y) - f^{\varepsilon}(t,y)\| \leqslant$$

$$\leqslant \|f(t,y) - f_{N(\varepsilon)}(t,y)\| + \|f_{N(\varepsilon)}(t,y) - f^{\varepsilon}(t,y)\| \leqslant$$

$$\leqslant \max_{\|y-\eta\|\leqslant a} \|f(t,y) - f_{N(\varepsilon)}(t,y)\| + \varepsilon/2 < \varepsilon$$

for almost every $t\in[t_0,T]$. Hence

$$\max_{\|y-\eta\|\leqslant a} \left\| f(t,y) - f(t,y) \right\| \leqslant \varepsilon$$

for almost every $t\in[t_0,T]$. This completes the proof.

Consider now a system of differential equations of the form

$$(2.1)\qquad \begin{cases} y(t) = \varphi(t) & \text{for } t\leqslant t_0 \\ y'(t) = \int_0^\infty f\ t,y(t-s)\ d_s r(t,s) + g(t) \end{cases}$$

for almost every $t\in[t_0,T]$,

where $f: P\to R^n$ satisfies the Carathéodory condition on P and $y' = (y_1',\ldots,y_n')$, $\varphi = (\varphi_1,\ldots,\varphi_n)$, $g = (g_1,\ldots,g_n)$,

$$\int_0^\infty f\big(t,y(t-s)\big)d_s r(t,s) =$$

$$=\left(\int_0^\infty f_1\big(t,y(t-s)\big)\,d_s r(t,s),\ldots,\int_0^\infty f_n\big(t,y(t-s)\big)\,d_s r(t,s)\right).$$

On the basis of theorem 2.1 we can state the following theorem.

T h e o r e m 2.2. Let $v(t)$ denote a solution of (2.1) defined on $(-\infty, T]$. Suppose that the function $f: P\longrightarrow R^n$ satisfies the Caratheodory condition. For every $\varepsilon>0$ there exists a function $h(t,y)$ satisfying the Carathéodory condition on P and such that

(a) $\|h(t,y)\| \leqslant m(t) + \varepsilon$ for $(t,y)\in P$,

(b) $h(t,y)$ is uniformly Lipschitz continuous with respect to y,

(c) $y = v(t)$ is a solution of the system

$$\begin{cases} y(t) = \varphi(t) \quad \text{for} \quad t \leqslant t_0 \\ y'(t) = \int_0^\infty h\big(t, y(t-s)\big)\, d_s r(t,s) + g(t) \end{cases}$$

for almost every $t \in [t_0, T]$,

(d) $\max_{\|y-\eta\| \leqslant a} \|f(t,y) - h(t,y)\| \leqslant \varepsilon$ for almost every $t_0 \leqslant t \leqslant T$.

P r o o f. In virtue of Theorem 2.1 for every $\varepsilon > 0$ there exists a continuous function $f^{\varepsilon}(t,y)$ such that

1. $\|f^{\varepsilon}(t,y)\| \leqslant m(t)$ for $(t,y) \in P$

2. $\max_{\|y-\eta\| \leqslant a} \|f(t,y) - f^{\varepsilon}(t,y)\| < \varepsilon/2$ for almost every $t \in [t_0, T]$,

3. $f^{\varepsilon}(t,y)$ is uniformly Lipschitz continuous with respect to y.

Let us define a function $h(t,y)$ by the formula

$$h(t,y) = f^{\varepsilon}(t,y) + f\big(t, v(t-s)\big) - f^{\varepsilon}\big(t, v(t-s)\big)$$

for every $(t,y) \in P$ and $s \geqslant 0$. It is easy to see that $h(t,y)$ satisfies the Carathéodory condition and that the condition $\|h(t,y)\| \leqslant m(t) + \varepsilon$ is satisfied. Moreover, $h(t,y)$ is uniformly Lipschitz continuous with respect to y. For almost every $t_0 \leqslant t \leqslant T$ we have

$$\max_{\|y-\eta\| \leqslant a} \|f(t,y) - h(t,y)\| \leqslant$$
$$\leqslant \max_{\|y-\eta\| \leqslant a} \|f(t,y) - f^{\varepsilon}(t,y)\| + \|f\big(t, v(t-s)\big) - f^{\varepsilon}\big(t, v(t-s)\big)\| <$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Since

$$\int_0^\infty h\big(t, v(t-s)\big)\, d_s r(t,s) + g(t) = \int_0^\infty f\big(t, v(t-s)\big)\, d_s r(t,s) + g(t) = v'(t),$$

for almost every $t \in [t_0, T]$, we infer that the theorem holds.

3. H.Kneser's Theorem

As an immediate application of the approximation theorem 2.2 we give a proof of the Theorem of H.Kneser. Consider a function $r: [t_0,T]\times[0,\infty)\ni(t,s)\to r(t,s)\in R$. Suppose that r satisfies the following conditions

(I) $r(t,0) = 0$ for $t\in[t_0,T]$,

(II) there exists a finite number V such that

$$\sum_{s=0}^{\infty} r(t,s)\leqslant V \quad \text{for} \quad t\in[t_0,T],$$

(III) for every $\varepsilon>0$ there is a number $K>0$ such that

$$\sum_{s=K}^{\infty} r(t,s)<\varepsilon \quad \text{for} \quad t\in[t_0,T],$$

(IV) for every $k>0$ and $u\in[t_0,T]$

$$\lim_{t\to u}\int_0^k |r(t,s) - r(u,s)|ds = 0, \quad \text{where} \quad t_0\leqslant t\leqslant T.$$

In the proof of Kneser's theorem we are going to use the following lemmas.

L e m m a 3.1. Let the function r satisfy the assumptions (I)-(IV) and let $\varphi: (-\infty,t_0]\to R^n$ be continuous and such that $\|\varphi(t)-\eta\|\leqslant a$ for $t<t_0$ and $\|\varphi(t_0)-\eta\|<a$. Suppose that $g: [t_0,T]\to R^n$ is Lebesgue integrable on $[t_0,T]$. If $f: P\to R^n$ satisfies the Carathéodory condition on P, then there exists a number $\Delta>0$, where $\int_{t_0}^{t_0+\Delta}\{V\, m(t)+\|g(t)\|\}dt\leqslant a-b$; $b=\|\varphi(t_0)-\eta\|$ such that at least one solution of (2.1) exists on $(-\infty,t_0+\Delta]$.

P r o o f. In virtue of the approximation theorem of Alexiewicz-Orlicz there exist continuous functions $f_n: P\to R^n$ such that conditions (1.1), (1.2) are fulfilled. Consider, for fixed $n=1,2,\dots$, the equation

$$(3.1)\quad \begin{cases} y(t) = \varphi(t) & \text{for} \quad t \leqslant t_0 \\ y'(t) = \int_0^{\infty} f_n\big(t, y(t-s)\big) d_s r(t,s) + g_n(t) & \text{for} \quad t \in [t_0, T], \end{cases}$$

where the sequence $\{g_n(t)\}$ is such that: 1° $g_n(t)$ is continuous on $[t_0,T]$, 2° $g_n(t) \to g(t)$ for almost every $t \in [t_0,T]$. It is easy to verify that (3.1) is equivalent to following condition

$$(3.2)\quad \begin{cases} \varphi(t) & \text{for} \quad t \leqslant t_0 \\ \varphi(t_0) + \int_{t_0}^{t} \left\{ \int_0^{\infty} f_n\big(u, y(u-s)\big) d_s r(u,s) + g_n(u) \right\} du & \text{for} \quad t_0 \leqslant t \leqslant T. \end{cases}$$

Let $\Delta > 0$ be chosen in such a way that $t_0 + \Delta \leqslant T$ and let B denote the Banach space of continuous and bounded functions $y: (-\infty, t_0+\Delta] \to R^n$ with the norm $\|y\|_B = \sup_{t \leqslant t_0+\Delta} \|y(t)\|$. Denote by K the set of all functions $y \in B$ which satisfy the following conditions

1) $y(t) = \varphi(t)$ for $t \leqslant t_0$

2) $\|y(t) - \eta\| \leqslant a$ for $t_0 \leqslant t \leqslant T$.

It is easy to verify that K is a non-empty, closed and convex subset of B. We define on K an operator A by the formula

$$(Ay)(t) = \begin{cases} \varphi(t) & \text{for} \quad t \leqslant t_0 \\ \varphi(t_0) + \int_{t_0}^{t_0+\Delta} \left\{ \int_0^{\infty} f_n\big(u, y(u-s)\big) d_s r(u,s) + g_n(u) \right\} du & \text{for} \quad t_0 \leqslant t \leqslant T. \end{cases}$$

In a similar way as in [4] it is easy to verify that the operation A satisfies the conditions of Schauder's fixed point theorem, provided that the number Δ satisfies the condition $\int_{t_0}^{t_0+\Delta} \left\{ V\, m(t) + \|g(t)\| \right\} dt \leqslant a-b$, where $b = \|\varphi(t_0) - \eta\|$. Then for fixed $n = 1,2,\dots$ there exists at least one solution of (3.1), say $y_n(t)$, defined on $(-\infty, t_0+\Delta]$. It is easy to see that there exists a subsequence $\{y_k(t)\}$ of $y_n(t)$ which is uniformly convergent on $(-\infty, t_0+\Delta]$ to the solution of (2.1).

L e m m a 3.2. Let the assumptions of Lemma 3.1 hold and let $f(t,y,\lambda)$ be a function that satisfies for every $\lambda \in (\alpha,\beta)$ the Carathéodory condition on P and is uniformly Lipschitz continuous with respect to y. Suppose that there exists a Lebesque integrable function K(t) such that

$$\left\| f(t,y,\lambda_2) - f(t,y,\lambda_1) \right\| \leqslant K(t) \left| \lambda_2 - \lambda_1 \right| .$$

Then the solution $y(t,\lambda)$ of

$$\begin{cases} y(t) = \varphi(t) & \text{for } t \leqslant t_0 \\ y'(t) = \int_{t_0}^{t} f\big(t, y(t-s), \lambda\big) d_s r(t,s) + g(t) & \text{for almost every } t \in [t_0, T] \end{cases}$$

is continuous with respect to $\lambda \in (\alpha,\beta)$.

P r o o f. The inequality of the form

$$\left\| y(t,\lambda_2) - y(t,\lambda_1) \right\| \leqslant V \left| \lambda_2 - \lambda_1 \right| \int_0^t K(u) du \leqslant$$

$$\leqslant V \left| \lambda_2 - \lambda_1 \right| \int_{t_0}^{t_0+\Delta} K(t) dt ; \qquad t_0 \leqslant t \leqslant t_0 + \Delta$$

implies that $\max_{t_0 \leqslant t \leqslant t_0+\Delta} \left\| y(t,\lambda_2) - y(t,\lambda_1) \right\| \longrightarrow 0$ as $\lambda_2 \to \lambda_1$.

L e m m a 3.3. Let the functions r, φ and g satisfy the assumptions of Lemma 3.1. Suppose that $f_0(t,y), f_1(t,y), \dots$ satisfy the Carathéodory condition on P and let

$$\lim_{n\to\infty} \max_{\|y-\eta\| \leqslant a} \left\| f_n(t,y) - f_0(t,y) \right\| = 0$$

for almost every $t \in [t_0, t_0+\Delta]$. Let $y_n(t)$ be a solution of the system

$$(3.3_n)\quad \begin{cases} y(t) = \varphi(t) & \text{for } t \leqslant t_o \\ y'(t) = \int_0^\infty f_n\big(t, y(t-s)\big)\, d_s r(t,s) + g(t) & \text{for almost every } t \in [t_o, t_o+\Delta] \end{cases}$$

where $n=1,2,\ldots$. Suppose that there exists a Lebesgue integrable function $M(t)$ such that $\|f_n(t,y)\| \leqslant M(t)$ for $n = 1,2,\ldots$ and $(t,y) \in P$. Then there is a subsequence $\{y_k(t)\}$ of the sequence $\{y_n(t)\}$ which is uniformly convergent on the interval $(-\infty, t_o+\Delta]$. For every such subsequence the limit $y_o(t) = \lim_{k\to\infty} y_k(t)$ is a solution of (4_o).

P r o o f. Since for every $n=1,2,\ldots$ and $t, t_1, t_2 \in [t_o, t_o+\Delta]$ we have

$$\|y_n(t) - \eta\| \leqslant \|\varphi(t_o) - \eta\| + \int_{t_o}^{t_o+\Delta} \big\{V M(t) + \|g(t)\|\big\} dt \leqslant a$$

and

$$\|y_n(t_1) - y_n(t_1)\| \leqslant \left| \int_{t_1}^{t_2} \big\{V M(t) + \|g(t)\|\big\}\, dt \right|,$$

we infer that the functions $y_n(t)$ are equicontinuous and uniformly bounded. Hence, by Arzela's theorem, there exists a subsequence $\{y_k\}$ of the sequence $\{y_n\}$ converging uniformly on the interval $(-\infty, t_o+\Delta]$ to $y_o(t)$. It remains to show that $y_o(t)$ is a solutions of (4_o). For $t_o \leqslant t \leqslant t_o+\Delta$ we have

$$y_o(t) - \varphi(t_o) - \int_{t_o}^{t} \Big\{ \int_0^\infty f_o\big(u, y_o(u-s)\big) d_s r(u,s)\ d_s r(u,s) + g(u) \Big\} du = \sum_{i=1}^{3} \Lambda_i(t),$$

where

$$\Lambda_1(t) = y_o(t) - y_k(t),$$

$$\Lambda_2(t) = \int_{t_0}^{t}\left\{\int_0^{\infty}\left[f_k\left(u,y_k(u-s)\right) - f_o\left(u,y_k(u-s)\right)\right]d_s r(u,s)\right\}du,$$

$$\Lambda_3(t) = \int_{t_0}^{t}\left\{\int_0^{\infty}\left[f_o\left(u,y_k(u-s)\right) - f_o\left(u,y_o(u-s)\right)\right]d_s r(u,s)\right\}du.$$

It is easy to verify that $\Lambda_1(t)\to 0$, $\Lambda_2(t)\to 0$, $\Lambda_3(t)\to 0$ for $k\to\infty$ ([4]). This completes the proof.

L e m m a 3.4. Let the function $f(t,y)$ be uniformly Lipschitz continuous with respect to y on P and let the assumptions of Lemma 3.1 hold. Then (2.1) has a unique solution on the interval $[t_o,t_o+\Delta]$.

P r o o f. Suppose that $y_1(t)$ and $y_2(t)$ are solutions of (2.1). It is easy to see that

$$\|y_2(w) - y_1(w)\| \leqslant VL\int_{t_0}^{w} \max \|y_2(v) - y_1(v)\|\, du \leqslant$$

$$\leqslant VL\int_{t_0}^{t} \max_{t_0\leqslant v\leqslant u}\|y_2(v) - y_1(v)\|du; \qquad t_o \leqslant w \leqslant t.$$

Therefore

$$\max_{t_0\leqslant v\leqslant t}\|y_2(v) - y_1(v)\| \leqslant LV\int_{t_0}^{t}\max_{t_0\leqslant v\leqslant u}\|y_2(v) - y_1(v)\|\,du$$

for $t_o\leqslant t\leqslant t_o+\Delta$. In virtue of Gronwall's inequality we have $\max\limits_{t_0\leqslant v\leqslant t}\|y_2(v) - y_1(v)\| = 0$ for every $t_o\leqslant t\leqslant t_o+\Delta$.

T h e o r e m 3.5. Let the function r satisfy assumptions (I) - (IV) and let $\varphi: (-\infty,t_o] \to R^n$ be continuous and such that $\|\varphi(t) - \eta\|\leqslant a$ for $t<t_o$ and $\|\varphi(t_o) - \eta\| < a$. Suppose that $g(t)$ is a Lebesgue integrable function on $[t_o, T]$. Let $f: P\to R^n$, where $P = \{t_o\leqslant t\leqslant T;\ \|y-\eta\|\leqslant a\}$, satisfy the Carathéodory condition on P. Finally, let S_c be a subset of R^n defined by $S_c = \{y(c),\ y(t)\in\Sigma\}$, where Σ denote the set of all solutions of (2.1); $c\in(t_o,t_o+\Delta)$, where $\int_{t_0}^{t_0+\Delta}\left\{Vm(t) + \|g(t)\|\right\}dt \leqslant a - b$, $b = \|\varphi(t_0) - \eta\|$. Then S_c is a continuum, i.e., a convex connected set.

P r o o f. The set S_c is bounded in R^n. To see that S_c is a closed let $y_{nc} \to y_c$, $n \to \infty$ and $y_{nc} \in S_c$. Then $y_{nc} = y_n(c)$ for some $y_n(t) \in \Sigma$. By Lemma 3.3 the sequence $\{y_n(t)\}$ has a subsequence which is uniformly convergent to some $y(t) \in \Sigma$ on the interval $(-\infty, t_0+\Delta]$. Clearly $y_c = y(c)$.

Suppose to the contrary that S_c is not connected. Then $S_c = S^1 \cup S^2$, where S^1, S^2 are nonempty, disjoint, closed sets. Since S_c is bounded, $\delta = \operatorname{dist}(S^1,S^2) > 0$, where $\operatorname{dist}(S^1,S^2) = \inf\left\{\|y^1-y^2\|;\ y^1 \in S^1,\ y^2 \in S^2\right\}$. For any y, put $e(y) = \operatorname{dist}(y,S^1) - \operatorname{dist}(y,S^2)$, so that $e(y) \geqslant \delta > 0$ if $y \in S^2$ and $e(y) \leqslant -\delta < 0$ if $y \in S^1$. The function $e(y)$ is continuous and $e(y) \neq 0$ for $y \in S_c$. Let $\varepsilon > 0$ and let $y_1(t)$, $y_2(t) \in \Sigma$ where $y_1(c)$, $y_2(c)$ are in S^1 and S^2, respectively. In virtue of Theorem 2.2 there exist functions $h_1(t,y)$ and $h_2(t,y)$ satisfying the Carathéodory condition on $Q = \left\{(t,y) : t_0 \leqslant t \leqslant t_0+\Delta;\ \|y\| \infty\right\}$ with the properties (a) - (d), where $v(t) = y_1(t)$, $y_2(t)$, respectively. Consider the one-parameter family of differential - integral equations

$$(3.4)\quad \begin{cases} y(t) = \varphi(t) & \text{for } t \leqslant t_0 \\ y'(t) = \int_0^\infty h\big(t, y(t-s), \lambda\big)\, d_s r(t,s) + g(t) & \text{for almost every } t_0 \leqslant t \leqslant t_0 + \Delta, \end{cases}$$

where $0 \leqslant \lambda \leqslant 1$ and $h(t,y,\lambda) = \lambda h_1(t,y) + (1-\lambda)h_2(t,y)$. Since $h(t,y,\lambda)$ satisfies the assumptions of Lemma 3.4 and Lemma 3.2 it follows that (3.4) has a unique solution $y = y(t,\lambda)$ which continuously depends on λ in the interval $[0,1]$. As a consequence of Lemma 3.1, where a is an arbitrary, this solution exists on the interval $(-\infty, t_0+\Delta_\lambda]$ for some $\Delta_\lambda \leqslant \Delta$. Since $a > 0$ is arbitrary, we can assume that $y(t,\lambda)$ is extended to $t_0+\Delta$.

Note that $\|h(t,y,\lambda)\| \leqslant m(t)+\varepsilon$ implies that $\|y(t,\lambda)-\eta\| \leqslant a+\varepsilon V\Delta$ for $t_0 \leqslant t \leqslant t_0+\Delta$. Lemma 3.2 implies that $y(t,\lambda) \to y(t,\lambda_0)$ as $\lambda \to \lambda_0$ uniformly on $(-\infty, t_0+\Delta]$. In particular, $y(c,\lambda)$ and consequently $e(y(c,\lambda))$, is continuous with respect to λ. Since $y(c,0) = y_2(c) \in S^2$, $y(c,1) = y_1(c) \in S^1$, so that $e(y(c,0)) \leqslant 0$, $e(y(c,1)) > 0$, there exists a value of $\lambda = \Theta$, $0 < \Theta < 1$ such that $e(y(c,\Theta))=0$. If $y_1(t)$, $y_2(t)$ are fixed, the choice of Θ depends only on ε, say $\Theta = \Theta(\varepsilon)$. Let $\varepsilon = 1/n$, $n > 1$ and let $h_n(t,y) = h(t,y,\lambda)$ where $\lambda = \Theta(1/n)$. We have $\max_{\|y-\eta\| \leqslant a} \|f(t,y) - h_n(t,y)\| \leqslant 1/n$ for almost every $t_0 \leqslant t \leqslant t_0+\Delta$ and, by the choice of $\lambda = \Theta(1/n)$ the system

$$\begin{cases} y(t) = \varphi(t) & \text{for } t \leqslant t_0 \\ y'(t) = \int_0^\infty h_n(t, y(t-s))\, d_s r(t,s) + g(t) & \text{for almost every } t_0 \leqslant t \leqslant t_0+\Delta \end{cases}$$

has a unique solution $y = y\, t,\Theta(1/n) = y^{(n)}(t)$ on $(-\infty, t_0+\Delta]$ such that $e(y^{(n)}(c)) = 0$ for $n > 1$. Since $\|h_n(t,y)\| \leqslant m(t) + 1/n < m(t) + 1$ for $n > 1$, the sequence $\{y^{(n)}(t)\}$ has a subsequence $\{y_k(t)\}$ which is uniformly convergent to $y_0(t)$ for $t \leqslant t_0+\Delta$. Furthermore, $\|y_k(t) - \eta\| \leqslant a + 1/nV\Delta$ and $e(y_k(c)) = 0$ for $k=1,2,\ldots$, The function $y_0(t)$ satisfies the equation

$$\begin{cases} y(t) = \varphi(t) & \text{for } t \leqslant t_0 \\ y'(t) = \int_0^\infty f(t, y(t-s))\, d_s r(t,s) + g(t) & \text{for almost every } t_0 \leqslant t \leqslant t_0+\Delta \end{cases}$$

and $\|y_0(t) - \eta\| \leqslant a$; $t_0 \leqslant t \leqslant t_0+\Delta$. Hence $y_0(t) \in \Sigma$. Also $e\, y_0(c) = \lim_{k\to\infty} e\, y_k(c) = 0$. But then $y_0(c) \in S_c$ and $e(y_0(c)) = 0$. This contradiction proves the theorem.

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