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ON SOME PROPERTIES OF PERIODIC PROJECTIVE TRANSFORMATIONS ON THE LINE

A projective transformation f is said to be periodic (with period k) if for every point P there holds

$$\underbrace{f(f(\ldots f}_{k}(P)\ldots)) = P.$$

It is known that a projective transformation of a one-dimensional space can be expressed by formulas

$$(1)\qquad \begin{bmatrix} \lambda x_0' = ax_0 + bx_1 \\ \lambda x_1' = cx_0 + dx_1 \end{bmatrix}, \quad \text{where} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

In the sequel the elements of the matrix of the transformation obtained from transformation (1) by n-fold superposition will be denoted by the symbols $a^{(n)}$, $b^{(n)}$, $c^{(n)}$, $d^{(n)}$.

Assume that there are given coefficients $a^{(n-1)}$, $b^{(n-1)}$, $c^{(n-1)}$, $d^{(n-1)}$ for $n \geqslant 2$; then as a result of multiplying the matrices

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a^{(n-1)} & b^{(n-1)} \\ c^{(n-1)} & d^{(n-1)} \end{bmatrix},$$

we obtain the matrix

$$(2)\qquad \begin{bmatrix} aa^{(n-1)} + bc^{(n-1)}, & ab^{(n-1)} + bd^{(n-1)} \\ ca^{(n-1)} + dc^{(n-1)}, & cb^{(n-1)} + dd^{(n-1)} \end{bmatrix}.$$

From this it follows directly that for $n \geqslant 2$

$$a^{(n)} = aa^{(n-1)} + bc^{(n-1)}, \quad b^{(n)} = ab^{(n-1)} + bd^{(n-1)},$$

$$c^{(n)} = ca^{(n-1)} + dc^{(n-1)}, \quad d^{(n)} = cb^{(n-1)} + dd^{(n-1)},$$

where $a^{(1)} = a$, $b^{(1)} = b$, $c^{(1)} = c$, $d^{(1)} = d$.

Now we shall prove

T h e o r e m 1. The coefficients $a^{(n)}$, $b^{(n)}$, $c^{(n)}$, $d^{(n)}$ can be expressed by means of the initial coefficients a,b,c,d as follows

$$a^{(n)} = \sum_{i=0}^{n}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n)} a^{n-i} b^{j} c^{j} d^{i-2j}\right),$$

$$b^{(n)} = \sum_{i=1}^{n}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i} b^{j} c^{j-1} d^{i+1-2j}\right),$$

(3)

$$c^{(n)} = \sum_{i=1}^{n}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i} b^{j-1} c^{j} d^{i+1-2j}\right),$$

$$d^{(n)} = \sum_{i=1}^{n}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n)} d^{n-i} b^{j} c^{j} a^{i-2j}\right),$$

where

$$(4) \qquad A_{ki}^{(n)} = A_{ki}^{(n-1)} + B_{k-1\,i}^{(n-1)}, \quad B_{ki}^{(n)} = A_{k-1\,i-1}^{(n-1)} + B_{k-1\,i}^{(n-1)};$$

as well as $A_{00}^{(1)} = 1$, $B_{11}^{(1)} = 1$; $A_{ki}^{(n)} = 0$ for $k < 0$, or $k > n$, or $i < 0$, or $i > E((k+1)/2)$.

Similarly, $B_{ki}^{(n)} = 0$ for $k < 1$, or $k > n$, or $i < 1$, or $i > E\,(k+1)/2$.

P r o o f. First observe that if in the first formula in (2) we replace a by d, b by c and conversely, we obtain the fourth formula.

Similarly, replacing in the third formula a by d, b by c, and conversely, we obtain the second formula. Hence it suffices to prove, for example, only the first and the third formula in (3).

The proof will proceed by induction. Assume that the following formulas hold:

$$a^{(n)} = \sum_{i=0}^{n}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n)} a^{n-i} b^{j} c^{j} d^{i-2j}\right), \tag{5}$$

$$c^{(n)} = \sum_{i=1}^{n}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i} b^{j-1} c^{j} d^{i+1-2j}\right). \tag{6}$$

We shall prove that

$$a^{(n+1)} = \sum_{i=0}^{n+1}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n+1)} a^{n+1-i} b^{j} c^{j} d^{i-2j}\right), \tag{7}$$

$$c^{(n+1)} = \sum_{i=1}^{n+1}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n+1)} a^{n+1-i} b^{j-1} c^{j} d^{i+1-2j}\right). \tag{8}$$

By (2) we have $a^{(n+1)} = aa^{(n)} + bc^{(n)}$, $c^{(n+1)} = ca^{(n)} + dc^{(n)}$.

Taking into account (5) and (6) as well as formulas (4), we obtain

$$a^{(n+1)} = \sum_{i=0}^{n}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n)} a^{n+1-i} b^{j} c^{j} d^{i-2j}\right) +$$

$$+ \sum_{i=1}^{n}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i} b^{j} c^{j} d^{i+1-2j}\right) = \sum_{i=0}^{n} A_{i0}^{(n)} a^{n+1-i} d^{i} +$$

$$+\sum_{i=2}^{n}\sum_{j=1}^{E(i/2)}\left(A_{ij}^{(n)} + B_{i-1\,j}^{(n)}\right) a^{n+1-i}b^{j}c^{j}d^{i-2j} +$$

$$+\sum_{j=1}^{E((n+1)/2)} B_{nj}^{(n)} b^{j}c^{j}d^{n+1-2j} = \sum_{i=0}^{n+1} A_{i0}^{(n+1)} a^{n+1-i}d^{i} +$$

$$+\sum_{i=2}^{n}\left(\sum_{j=1}^{E((n+1)/2)} A_{ij}^{(n+1)} a^{n+1-i}b^{j}c^{j}d^{i-2j}\right) +$$

$$+\sum_{j=1}^{E((n+1)/2)} A_{n+1\,j}^{(n+1)} b^{j}c^{j}d^{n+1-2j} = \sum_{i=0}^{n+1}\left(\sum_{j=0}^{E(i/2)} A_{ij}^{(n+1)} a^{n+1-i}b^{j}c^{j}d^{i-2j}\right),$$

since in view of (4)

$$A_{ki}^{(n)} = 0 \quad \text{for} \quad k > n, \quad \text{and} \quad A_{k0}^{(n)} = 0 \quad \text{for} \quad k \geqslant 0.$$

Similarly,

$$c^{(n+1)} = \sum_{i=0}^{n}\left(\sum_{j=1}^{E((i+1)/2)} A_{ij}^{(n)} a^{n-i}b^{j}c^{j+1}d^{i-2j}\right) +$$

$$+\sum_{i=1}^{n}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i}b^{j-1}c^{j}d^{i+2-2j}\right) = \sum_{i=0}^{n} A_{i0}^{(n)} a^{n-i}cd^{i} +$$

$$+\sum_{i=1}^{n} B_{i1}^{(n)} a^{n-i}cd^{i} + \sum_{i=2}^{n}\left(\sum_{j=1}^{E(i/2)}\left(A_{ij}^{(n)} + B_{ij+1}^{(n)}\right) a^{n-i}b^{j}c^{j+1}d^{i-2j}\right) =$$

$$= B_{11}^{(n+1)} a^{n}c + \sum_{i=1}^{n} B_{i+11}^{(n+1)} a^{n-i}cd^{i} +$$

$$+\sum_{i=2}^{n}\left(\sum_{j=1}^{E(i/2)} B_{i+1\ j+1}^{(n+1)}\ a^{n-1}b^{j}c^{j+1}d^{i-2j}\right) =$$

$$=\sum_{i=1}^{n+1}\left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n+1)} a^{n+1-i}b^{j-1}c^{j}d^{i+1-2j}\right),$$ since according to (4) $A_{i0}^{(n)} = 0$ for $i \geqslant 1$ and $A_{00}^{(n)} = 1 = B_{11}^{(n)}$ as well as $B_{i1}^{(n)} = 1$ for $i \leqslant n$. This ends the proof.

L e m m a. The following relation holds

$$\left(a^{(n)} - d^{(n)}\right)b \equiv (a-d)b^{(n)}. \tag{9}$$

P r o o f. If $n=1$, then obviously $(a-d)b \equiv (a-d)b$. Similarly, for $n=2$ we have $a^{(2)} = a^2+bc$ $b^{(2)} = ab+bd$, $c^{(2)} = ac+cd$, $d^{(2)} = bc+d^2$. Hence the left-hand side of formula (9) has the form $(a^2-d^2)b$, and the right-hand side has the form $(a-d)b(a+d)$, which shows that the formula holds.

Proceeding by induction, assume that the following formulas hold

$$\left(a^{(k)} - d^{(k)}\right)b \equiv (a-d)b^{(k)}, \tag{10}$$

$$\left(a^{(k-1)} - d^{(k-1)}\right)b \equiv (a-d)b^{(k-1)}. \tag{11}$$

Making use of (2) we can rewrite (10) in the form

$$aba^{(k-1)} + b^2c^{(k-1)} - bcb^{(k-1)} - bdd^{(k-1)} \equiv$$

$$\equiv a^2b^{(k-1)} + abd^{(k-1)} - dab^{(k-1)} - bdd^{(k-1)}.$$

Multiplying both sides of the above identity by a we get

$$(12)\qquad \begin{aligned} &a^2ba^{(k-1)} + ab^2c^{(k-1)} , abcb^{(k-1)} - abdd^{(k-1)} \equiv \\ &\equiv a^3b^{(k-1)} + a^2bd^{(k-1)} - a^2db^{(k-1)} - abdd^{(k-1)} . \end{aligned}$$

Similarly, multiplying both sides of the same identity by d we get

$$(13)\qquad \begin{aligned} &abda^{(k-1)} + b^2dc^{(k-1)} - bcdb^{(k-1)} - bd^2d^{(k-1)} \equiv \\ &\equiv a^2db^{(k-1)} + abdd^{(k-1)} - ad^2b^{(k-1)} - bd^2d^{(k-1)}. \end{aligned}$$

Adding (12) to (13) side by side we obtain

$$(14)\qquad \begin{aligned} &a^2ba^{(k-1)} + ab^2c^{(k-1)} - abcb^{(k-1)} - abdd^{(k-1)} + abda^{(k-1)} + \\ &+ b^2dc^{(k-1)} - bd^2d^{(k-1)} - bcdb^{(k-1)} \equiv a^3b^{(k-1)} + a^2bd^{(k-1)} - \\ &- a^2db^{(k-1)} - abdd^{(k-1)} + a^2db^{(k-1)} + abdd^{(k-1)} - ad^2b^{(k-1)} - \\ &- bd^2d^{(k-1)}. \end{aligned}$$

Let us multiply identity (11) by $a\cdot d$, then $abda^{(k-1)} - abdd^{(k-1)} = a^2db^{(k-1)} - ad^2b^{(k-1)}$. Next we sibtract this identity from (14) side by side. This gives

$$(15)\qquad \begin{aligned} &a^2ba^{(k-1)} + ab^2c^{(k-1)} - abcb^{(k-1)} + b^2dc^{(k-1)} - bd^2d^{(k-1)} - \\ &-bcdb^{(k-1)} \equiv a^3b^{(k-1)} + a^2bd^{(k-1)} - a^2db^{(k-1)} - abdd^{(k-1)} + \\ &+ abdd^{(k-1)} - bd^2d^{(k-1)}. \end{aligned}$$

Finally, multiplying both sides of (11) by $b\cdot c$ and adding it to (15) side by side, we obtain

$$a^2ba^{(k-1)} + ab^2c^{(k-1)} - abcb^{(k-1)} + b^2dc^{(k-1)} - bd^2d^{(k-1)} -$$

$$(16)\quad - bcdb^{(k-1)} - b^2cd^{(k-1)} \equiv a^3b^{(k-1)} + a^2bd^{(k-1)} - a^2db^{(k-1)} -$$

$$- abdd^{(k-1)} + abdd^{(k-1)} - bd^2d^{(k-1)} + abcb^{(k-1)} - bcdb^{(k-1)}.$$

Taking into account formulas (2) we obtain from (16) $aba^{(k)} + b^2c^{(k)} - bcb^{(k)} - bdd^{(k)} \equiv a^2b^{(k)} + abd^{(k)} - adb^{(k)} - bdd^{(k)}$, and next $b\,a^{(k+1)} - d^{(k+1)} \equiv (a-d)b^{(k+1)}$, which concludes the proof.

T h e o r e m 2. In order that a non-identity transformation (1) be periodic with period n it is necessary and sufficient that the following relation hold

$$(17)\qquad \sum_{i=1}^{n} \left(\sum_{j=1}^{E((i+1)/2)} B_{ij}^{(n)} a^{n-i} b^{j-1} c^{j-1} d^{i+1-2j} \right) = 0.$$

P r o o f. It is clear that a necessary and sufficient condition for a transformation f to be periodic with period n is that its n-fold superposition with itself be the identity; that is, $a^{(n)} - d^{(n)} = 0$, $b^{(n)} = 0$, $c^{(n)} = 0$. Let us denote the left-hand side of (17) by φ. Observe that in agreement with (3) and the proved lemma, the following identities hold: $b^{(n)} = b\varphi$, $c^{(n)} = c\varphi$, $a^{(n)} - d^{(n)}\ b = b(a-d)\varphi$. This shows that $a^{(n)} - d^{(n)} = 0$, whenever $a=d$ or $\varphi=0$. However, if we had $\varphi \neq 0$, than in order that $b^{(n)} = 0 = c^{(n)}$ we would have to have $b=c=0$, in contradiction to the assumption that (1) is not the identity transformation. Hence the theorem is fully proved.

In the special case $n=2$ we obtain a necessary and sufficient condition for a projective transformation (1) to be an involution. Namely, this condition is $a+d=0$.

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