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SOME PROBLEMS OF THE EXISTENCE AND BEHAVIOUR OF SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

1. In [1] we considered the problem of existence and behaviour of solutions of homogeneous linear differential equations in real Hilbert space. Under some assumptions we proved there the existence of solutions for considered equations as well as we proved that the solutions tend exponentially to zero. We also demonstrated stability and asymptotical stability of those solutions.

In the present paper we obtain analogical results for non-homogenous equations.

2. Consider the interval $I = \langle 0,+\infty)$ and let H be a real Hilbert space. Let K be a fixed non-bounded linear operator (not depending on t) acting in H with dense domain $D(K)\subset H$.

We aditionally assume that K has the following properties.

Z.1. K is symmetric, positive definite, closed and possesses an infinite set of eigenvalues λ_n $(n=1,2,\dots)$ with corresponding eigenfunctions $X_n \in D\,K$ $(n=1,2,\dots)$ such that

(a) $0<\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots \leqslant \lambda_n \leqslant \dots \qquad \lim\limits_{n\to\infty} \lambda_n = \infty;$

(b) the eigenfunctions x_n of the operator K form a complete system in H. Similarly as in [1], we may assume that the sequence $\{x_n\}$ is orthonormal.

Let α, β be continuous real functions defined on I and satisfying the assumption:

Z.2. There exist constants a,b,A,B such that for every $t \in I$ we have

$$0 < a \leqslant \alpha(t) \leqslant A, \quad 0 < b \leqslant \beta(t) \leqslant B.$$

There is given a function $S: I \rightarrow H$, $S \in C^0$ (I), such that

Z.3. for each $t \in I$, $S(t) \in D(K^3)$ and there is a constant $G > 0$

$$\| K^3 S(t) \| \leqslant G.$$

Moreover we make the following three assumptions:

Z.4. x_0 and $\dot{x}_0$ are any elements of H such that $x_0 \in D(K^3)$, $\dot{x}_0 \in D(K^2)$.

Z.5. $$\lambda_1^2 > \left[\frac{B-b}{a} + \frac{A}{2}\right] - b.$$

Z.6. $$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty .$$

In this paper we use the same definitions of boundedness, exponential tending to zero, stability and asymptotical stability as those given in [1].

3. Consider the following non-homogeneous equation

$$\ddot{x} + \alpha(t)\dot{x} + \beta(t)x + K^2x = S(t), \tag{1}$$

with initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \tag{2}$$

T h e o r e m 1. If assumptions Z.1-Z.6 hold then there exists a solution of problem (1), (2) that is bounded together with its first two derivatives, is stable and asymptotically stable.

P r o o f. We seek a soolution of problem (1), (2) in the form of a sum:

$$x(t) = y(t) + z(t),$$

where $y(t)$ is a solution of the homogeneous equation

$$\ddot{y} + \alpha(t)\dot{y} + \beta(t)y + K^2 y = 0, \tag{3}$$

with initial conditions (2), and $z(t)$ is a solutions of equation (1) with zero initial conditions; that is $z(t)$ is a solution of the problem

$$\ddot{z} + \alpha(t)\dot{z} + \beta(t)z + K^2 z = S(t), \tag{4}$$

$$z(0) = 0, \quad \dot{z}(0) = 0. \tag{5}$$

We know from (1) that $y(t)$, being the solution a homogeneous equation, exists and is bounded together with first two derivatives. Therefore in order to prove that the solution $x(t)$ of equation (1) exists and that $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ are bounded it suffices to show that $z(t)$ exists and is bounded together with $\dot{z}(t)$ and $\ddot{z}(t)$.

We shall seek the solution $z(t)$ of problem (4), (5) in the form of a series

$$z(t) = \sum_{n=1}^{\infty} T_n(t) x_n \tag{6}$$

where x_n are eigenfunctions of the operator K, and $T_n(t) \in R^1$ for every $t \in I$ and $n=1,2,\ldots$

For each $t \in I$ the function $S(t)$, being an element of the Hilbert space H, can be expanded in a Fourier series with respect to the eigenfunctions x_n of the operator K with Fourier coefficients $s_n(t) \in R^1$, $s_n(t) = \left(S(t), x_n\right)$. Hence we have

(7) $$S(t) = \sum_{n=1}^{\infty} s_n(t)x_n.$$

Equation (4) and initial conditions (5) are satisfied provided that the functions $T_n(t)$ fulfill the equations

(8) $$T_n''(t) + \alpha(t)T_n'(t) + \left[\beta(t) + \lambda_n^2\right]T_n(t) = s_n(t),$$

with the initial conditions

(9) $$T_n(0) = 0, \quad T_n'(0) = 0.$$

We shall estimate the functions $s_n(t)$ for all $t \in I$ and $n=1,2,\dots$

$$K^3S(t) = \sum_{n=1}^{\infty} s_n(t)\lambda_n^3 x_n.$$

Using Parserval's identity we obtain

$$\sum_{n=1}^{\infty} s_n^2(t)\lambda_n^6 = \left\|K^3S(t)\right\|^2 \leqslant G^2.$$

From this we infer that for every $n=1,2,\dots$ the following inequality holds

$$s_n^2(t)\lambda_n^6 \leqslant G^2,$$

and because $\lambda_n \geqslant 0$, we get

(10) $$\left|s_n(t)\right| \leqslant \frac{G}{\lambda_n^3}.$$

Hence for equation (8) all the assumptions of lemma 4 in [2] hold. Consequently, this lemma implies that there exist constants C,D,E dependent on a, b, A, B such that for the solution $T_n(t)$ of equation (8) and for the derivatives $T_n'(t)$ and $T''(t)$ the following inequalities hold

(11) $\left|T_n(t)\right| \leqslant CG/\lambda_n^4$, $\left|T'_n(t)\right| \leqslant DG/\lambda_n^3$, $\left|T''_n(t)\right| \leqslant EG/\lambda_n^2$

for every $n=1,2,\ldots$

Making use of inequality (11) we can estimate the norm of the solution $z(t)$ of equation (4) as well as the norms of derivatives $\dot{z}(t)$ and $\ddot{z}(t)$.

Accordingly, we have

$$\left\|z(t)\right\| \leqslant \sum_{n=1}^{\infty}\left|T_n(t)\right| \left\|x_n\right\| = \sum_{n=1}^{\infty} \left|T_n(t)\right|$$

(12)

$$\left\|z(t)\right\| \leqslant \sum_{n=1}^{\infty} CG/\lambda_n^4 = C\cdot G \sum_{n=1}^{\infty} 1/\lambda_n^4 .$$

Analogously, for the first derivative

(13) $$\left\|\dot{z}(t)\right\| \leqslant \sum_{n=1}^{\infty}\left|T'_n(t)\right| \leqslant D\cdot G \sum_{n=1}^{\infty} 1/\lambda_n^3 ,$$

and for the second derivative

(14) $$\left\|\ddot{z}(t)\right\| \leqslant \sum_{n=1}^{\infty}\left|T''_n(t)\right| \leqslant E\cdot G \sum_{n=1}^{\infty} 1/\lambda_n^2 .$$

By assumption Z.6 the series on the right-hand sides of inequalities (12) - (14) are convergent, and consequently the functional series defining $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$ are uniformly and absolutely convergent. This implies that $z(t)$ is a classical solution of equation (4) with initial conditions (5).

Inequalities (12) - (14) show that $z(t)$, $\dot{z}(t)$ and $\ddot{z}(t)$ are bounded. Hence the solution $x(t)$ of problem (1),(2) is also bounded as well as its derivatives $\dot{x}(t)$ and $\ddot{x}(t)$.

We now show the stability and asymptotical stability of solutions of equation (1).

Let us take another solution $u(t)$ of equation (1) with initial conditions $u(0) = u_o$, $\dot{u}(0)=\dot{u}_o$ $\left(u_o \in D(K^3),\ \dot{u}_o \in D(K^2)\right)$.

The difference $v(t) \overset{df}{=} x(t) - u(t)$ is a solution of homogeneous equation (3) with the initial conditions

$$v(0) \overset{df}{=} x(0) - u(0) = v_0 \in D(K^3),$$
(15)
$$\dot{v}(0) \overset{df}{=} \dot{x}(0) - \dot{u}(0) = \dot{v}_0 \in D(K^2).$$

From [1] we know that for any solution $v(t)$ of equation (1) with initial conditions (15) there exist constants P, S, γ dependent on a, b, B such that

(16) $$\|v(t)\| \leqslant e^{-\gamma t} \frac{1}{\lambda} (\|Kv_0\|P + \|\dot{v}_0\|S),$$

where

$$\frac{1}{\lambda^2} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \qquad 0 < \lambda < \infty$$

and $\gamma > 0$ is defined in [1] and [2].

Hence from (16) we obtain

(17) $$\|x(t) - u(t)\| \leqslant e^{-\gamma t} \frac{1}{\lambda}(P\|Kx_0 - Ku_0\| + S\|\dot{x}_0 - \dot{u}_0\|),$$

which shows the stability and asymptotical stability of equation (1).

4. Theorem 2. If the assumptions of Theorem 1 hold and the functions α, β, S are periodic with respect to t with a common period T, then there exists a solution of (1) that is periodic with period T.

Proof. Since $S(t)$ is T - periodic by assumption, $s_n(t) = (S(t), x_n)$ is also T- periodic. Hence in equation (8) for every $n=1,2,\ldots$ the coefficients and the free term are T-periodic. In view of the accepted assumptions it follows by [3] that equation (8) has a particular T-periodic solution $\omega_n(t)$ $n=1,2,\ldots$ estimated by relation (11). Let us denote

(18) $$\omega_n(0) = a_n, \quad \dot{\omega}_n(0) = b_n.$$

Let

(19) $$x(t) = \sum_{n=1}^{\infty} \omega_n(t)x_n.$$

From the estimation of $\omega_n(t)$ by inequalities (11) it follows (as in the proof of Theorem 1) that the series defining $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are uniformly convergent. Since $\omega_n(t)$ satisfies equation (8), $x(t)$ satisfies equation (1).

Next, from formula (19) we obtain

(20) $$x(0) = \sum_{n=1}^{\infty} a_n x_n, \qquad \dot{x}(0) = \sum_{n=1}^{\infty} b_n x_n,$$

(where a_n, b_n can be treated as Fourier coefficients of the expansion of the elements $x(0) = x_0$, and $\dot{x}(0) = \dot{x}_0$ with respect to the system $\{x_n\}$). Hence $x(t)$ defined in (19) is a solution of equation (1) with initial conditions (20). In view of the periodicity of $\omega_n(t)$, the solution $x(t)$ is also T-periodic.

R e m a r k. It is evident that if the hypotheses of theorem 2 in [1] are satisfied, the periodic solution $x(t)$ is unique and all solutions tend to it as $t \to \infty$.

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