DEMONSTRATIO MATHEMATICA
Vol. VII No 1 1974

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SOME PROBLEMS ON THE EXISTENCE AND BEHAVIOUR OF SOLUTIONS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

1. Let $I=\langle 0,+\infty)$ and let H be a real Hilbert space. Let K be a fixed, not necessarily bounded, linear operator acting in H with dense domain D(K). Assume that K has the following properties:

Z.1. The operator K is symmetric, positive definite, closed, and possesses an infinite set of eigenvalues λ_n, $n=1,2,\ldots$ with corresponding eigenfunctions $x_n \in D(K), n=1,2,\ldots$ such that

(a) $0<\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ \ldots \leqslant \lambda_n \leqslant \ldots \qquad \lim\limits_{n\to\infty} \lambda_n=\infty$

(b) the eigenfunctions x_n of the operator K form a complete system in H. In view of the symmetricity of K we may assume that the system of eigenfunctions $\{x_n\}$ is orthonormal.

Remark 1. In order that a positive definite operator K have eigenvalues and eigenfunctions with properties (a) and (b) it suffices to assume that H is a space in which every subset bounded in the energetic norm of the operator K is compact in H (see [1]).

Z.2. There are defined real scalar functions α, β of the class $C^0(I)$ with the property that there exist constants a,b,A,B such that for every $t\in I$ we have $0<a\leqslant\alpha(t)\leqslant A$, $0<b\leqslant\beta(t)\leqslant B$.

Z.3. Let $x_0\in H$, $\dot{x}_0 \in H$ be elements such that $x_0 \in D(K^3)$, $\dot{x}_0\in D(K^2)$.

2. Under the assumptions above, let us consider the differential equation

$$\ddot{x} + \alpha(t)\dot{x} + \beta(t)x + K^2x = 0, \tag{1}$$

with the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0. \tag{2}$$

In this paper we use the same definition of boundedness and exponential tending to zero for solutions as that given in [2]. On the other hand, we define stability and asymptotical stability of solutions as follows.

Definition 1. A solution x of the problem (1), (2) is said to be stable if it is defined for all $t \in I$, and for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution y of (1) defined for every $t \in I$ satisfying $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$, $(y_0 \in D(K^3), \dot{y}_0 \in D(K^2))$ the inequalities $\|Kx_0 - Ky_0\| < \delta$, $\|\dot{x}_0 - \dot{y}_0\| < \delta$ imply $\|x(t) - y(t)\| < \varepsilon$.

Definition 2. A solution x of the problem (1), (2) is said to be assymptotically stable if it is stable in the sense of definition 1 and moreover it is $\|x(t)-y(t)\| \to 0$ for $t \to \infty$.

3. Theorem. Let Z.1, Z.2, Z.3 hold and assume that the conditions

Z.4. $\lambda_1^2 > \left[\frac{B-b}{a} + \frac{A}{2}\right] - b,$

Z.5. $\lambda_n = O(n),$

hold with the constants a, b, A, B as in Z.2. Then the problem (1), (2) has a solution $x(t)$ that is bounded together with derivatives of the first and second order, and tends exponentially to the trivial solution as $t \to \infty$. The solution $x(t)$ is stable and asymptotically stable.

P r o o f. We shall seek a solution of equation (1) in the form of a series

$$x(t) = \sum_{n=1}^{\infty} T_n(t)x_n$$

where x_n are eigenfunctions of the operator K, $T_n(t) \in R^1$ for every $t \in I$, $n=1,2,\ldots$, and each function $T_n(t)x_n$ is a solution of equation (1).

Under this assumption, for each $n \in N$ we have

$$T_n''(t)x_n + \alpha(t)T_n'(t)x_n + \beta(t)T_n(t)x_n + K^2T_n(t)x_n = 0.$$

The operator K being linear, K^2 is also linear, and hence

$$T_n''(t)x_n + \alpha(t)T_n'(t)x_n + \beta(t)T_n(t)x_n + T_n(t)K^2x_n = 0.$$

Here we used the fact that $Kx_n = \lambda_n x_n$, so that $x_n \in D(K^2)$.

Since $K^2x_n = K(Kx_n) = K(\lambda_n x_n) = \lambda_n Kx_n = \lambda_n^2 x_n$ we obtain

$$\left[T_n''(t) + \alpha(t)T_n'(t) + \beta(t)T_n(t) + \lambda_n^2 T_n(t)\right] x_n = 0.$$

As $x_n \neq 0$, this gives

$$T_n''(t) + \alpha(t)T_n'(t) + \left(\beta(t) + \lambda_n^2\right)T_n(t) = 0. \tag{3}$$

The general solution of the linear equation (3) can be written in the form

$$T_n(t) = c_{1n}T_{n1}(t) + c_{2n}T_{n2}(t), \qquad n=1,2,\ldots, \tag{4}$$

where c_{1n}, c_{2n} are arbitrary constants, and $T_{n1}(t)$, $T_{n2}(t)$ are linearly independent solutions. We select the latter such that

(5) $T_{n1}(0) = 1, \quad T'_{n1}(0) = 0, \quad T_{n2}(0) = 0, \quad T'_{n2}(0) = 1.$

Hence we look for a solution of the form

(6) $$x(t) = \sum_{n=1}^{\infty} \left(c_{1n}T_{n1}(t) + c_{2n}T_{n2}(t)\right)x_n.$$

The constants c_{1n}, c_{2n}, $n=1,2,\dots$ are chosen such that the solution (6) satisfies the initial conditions; that is,

(7) $$x_0 = x(0) = \sum_{n=1}^{\infty} c_{1n}x_n, \quad \dot{x}_0 = \dot{x}(0) = \sum_{n=1}^{\infty} c_{2n}x_n .$$

Treating relations (7) as expansions of the elements $x_0, \dot{x}_0$ in Fourier series with respect to the system $\{x_n\}$, we obtain

$$c_{1n} = (x_0, x_n), \quad c_{2n} = (\dot{x}_0, x_n)$$

Hence c_{1n}, c_{2n} are Fourier coefficients in the expansions of $x_0, \dot{x}_0$ in Fourier series with respect to the system $\{x_n\}$.

In view of assumptions Z.2 and Z.4, fom lemma 6 in [3] it follows that there exist constants P,R,S,T,Z depending on a,b,B such that for the solutions of equation (3) under conditions (5) we have

(10) $$\begin{cases} \left|T_{n1}\right| \leqslant Pe^{-\gamma t}, & \left|T_{n2}\right| \leqslant \frac{S}{\lambda_n} e^{-\gamma t}, \\ \left|T'_{n1}\right| \leqslant R\lambda_n e^{-\gamma t}, & \left|T'_{n2}\right| \leqslant Pe^{-\gamma t}, \\ \left|T''_{n1}\right| \leqslant T\lambda_n^2 e^{-\gamma t}, & \left|T''_{n2}\right| \leqslant Z\lambda_n e^{-\gamma t}, \end{cases}$$

where

$$\gamma = \frac{a}{2}\,\frac{2\sqrt{b+\lambda_1^2} - \left[\frac{2(B-b)}{a} + A\right]}{2\sqrt{b+\lambda_1^2} + a} > 0 .$$

Estimating the norm of the solution $x(t)$, in view of (10) we obtain

$$\|x(t)\| \leqslant \sum_{n=1}^{\infty} \left\{ |c_{1n}|\,|T_{n1}(t)| + |c_{2n}|\,|T_{n2}(t)| \right\},$$

and next

$$\|x(t)\| \leqslant e^{-\gamma t} \sum_{n=1}^{\infty} \left(|c_{1n}|P + |c_{2n}|\,\frac{S}{\lambda_n} \right),$$

so that

$$\|x(t)\| \leqslant e^{-\gamma t}\left(P\sum_{n=1}^{\infty} |c_{1n}|\lambda_n \frac{1}{\lambda_n} + S\sum_{n=1}^{\infty} |c_{2n}|\frac{1}{\lambda_n} \right).$$

Making use of Cauchy's inequality we obtain

$$\|x(t)\| \leqslant e^{-\gamma t}\left(P\left[\sum_{n=1}^{\infty} c_{1n}^2\lambda_n^2 \cdot \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right]^{1/2} + S\left[\sum_{n=1}^{\infty} c_{2n}^2 \cdot \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right]^{1/2} \right).$$

By assumption Z.5. the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is convergent. Denote its sum by $\frac{1}{\lambda^2}$, $0<\lambda<\infty$. Then we can write

$$(11)\quad \|x(t)\| \leqslant e^{-\gamma t}\left\{ P\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{1n}^2\lambda_n^2\right]^{1/2} + S\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{2n}^2\right]^{1/2} \right\}.$$

From (7) it follows that $x_0 = x(0) = \sum_{n=1}^{\infty} c_{1n}x_n$. Since $x_0 \in D(K)$ and K is closed, we get

$$Kx_0 = Kx(0) = K\sum_{n=1}^{\infty} c_{1n}x_n = \sum_{n=1}^{\infty} c_{1n}Kx_n = \sum_{n=1}^{\infty} c_{1n}\lambda_n x_n .$$

Here $c_{1n}\lambda_n$ are Fourier coefficients for the expansion of the function $Kx(0)$ with respect to $\{x_n\}$.

Using Parceval's equality we obtain

$$\sum_{n=1}^{\infty} c_{1n}^2 \lambda_n^2 = \|Kx_o\|^2. \tag{12}$$

As well (7) implies

$$\dot{x}_o = \dot{x}(0) = \sum_{n=1}^{\infty} c_{2n} x_n,$$

hence

$$\sum_{n=1}^{\infty} c_{2n}^2 = \|\dot{x}_o\|^2. \tag{13}$$

The number series on the right-hand side of inequality (11) are convergent due to (12) and (13). Hence the series defining $x(t)$ is absolutely and uniformly convergent.

We now estimate the norm of the first derivative $\dot{x}(t)$ with help of inequality (10):

$$\|\dot{x}(t)\| \leqslant e^{-\gamma t} \sum_{n=1}^{\infty}\left(|c_{1n}|R\lambda_n + |c_{2n}|P\right),$$

$$\|\dot{x}(t)\| \leqslant e^{-\gamma t}\left\{R\sum_{n=1}^{\infty}|c_{1n}|\lambda_n^2 \frac{1}{\lambda_n} + P\sum_{n=1}^{\infty}|c_{2n}|\lambda_n \frac{1}{\lambda_n}\right\}.$$

Again using Cauchy's inequality we obtain

$$\|\dot{x}(t)\| \leqslant e^{-\gamma t}\left\{R\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{1n}^2 \lambda_n^4\right]^{1/2} + P\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{2n}^2 \lambda_n^2\right]^{1/2}\right\}. \tag{14}$$

Using (7) and noting that $x_o \in D(K^2)$ (assumption Z.3) and K is closed, we infer that

$$K^2x_o = K(Kx_o) = K\sum_{n=1}^{\infty} c_{1n}\lambda_n x_n = \sum_{n=1}^{\infty} c_{1n}\lambda_n^2 x_n ,$$

so that $c_{1n}\lambda_n^2$ are Fourier coefficients of the expansion of the function K^2x_o with respect to $\{x_n\}$. Therefore

$$\sum_{n=1}^{\infty} c_{1n}^2 \lambda_n^4 = \|K^2x_o\|^2.$$

Similarly from assumption Z.3. for the series $\sum_{n=1}^{\infty} c_{2n}^2 \lambda_n^2$ we get

$$\sum_{n=1}^{\infty} c_{2n}^2 \lambda_n^2 = \|K\dot{x}_o\|^2.$$

The number series on the right-hand side of (14) are convergent, hence the series defining $\dot{x}(t)$ is absolutely and uniformly convergent.

By an analogous reasoning for the second derivative we get

$$\|\ddot{x}(t)\| \leqslant e^{-\gamma t}\left\{\sum_{n=1}^{\infty}|c_{1n}|T\lambda_n^2 + |c_{2n}|Z\lambda_n\right\},$$

and

$$(15)\quad \|\ddot{x}(t)\| \leqslant e^{-\gamma t}\, T\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{1n}^2 \lambda_n^6\right]^{1/2} + Z\frac{1}{\lambda}\left[\sum_{n=1}^{\infty} c_{2n}^2 \lambda_n^4\right]^{1/2}$$

Since

$$\sum_{n=1}^{\infty} c_{1n}^2 \lambda_n^6 = \|K^3x_o\|^2, \qquad \sum_{n=1}^{\infty} c_{2n}^2 \lambda_n^4 = \|K^2x_o\|^2,$$

the series defining $\ddot{x}(t)$ is absolutely and uniformly convergent, in analogy to the case of $\dot{x}(t)$ and $x(t)$. Thus $x(t)$ in formula (6) is the classical solution of problem (1), (2). The uniqueness of the solution of problem (1),(2) follows from consideration analogous to that given e.g. in [4], p.167.

The convergence of the number series in (11), (14), (15) implies the boundedness of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$. In view of the factor $e^{-\gamma t}$ that appear in these inequalities, it follows that the functions $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ tend exponentially to zero when $t \to \infty$.

We now show that the solutions are stable. Let ε be an arbitrary number greater than 0, and let

$$\delta = \left[\frac{1}{\lambda}(P+S)\right]^{-1}\varepsilon$$

Let $y(t)$ be another solution of equation (1) such that

$$y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0, \quad \left(y_0 \in D(K^3),\ \dot{y}_0 \in D(K^2)\right),$$

and

$$(16) \qquad \left\| Kx(0) - Ky(0) \right\| < \delta, \qquad \left\| \dot{x}(0) - \dot{y}(0) \right\| < \delta.$$

Let us form the difference $z(t) = x(t) - y(t)$ which is also a solution of equation (1) (by its linearity) with the initial condition

$$(17) \quad z(0) = x(0) - y(0) = z_0 \in D(K^3), \quad \dot{z}(0) = \dot{x}(0) - \dot{y}(0) = \dot{z}_0 \in D(K^2).$$

We have by (16)

$$(18) \qquad \left\| Kz(0) \right\| < \delta \quad \text{and} \quad \left\| \dot{z}(0) \right\| < \delta.$$

The function $z(t)$, being a solution of (1), (17), can be written in the form of a series:

$$z(t) = \sum_{n=1}^{\infty} \left(\tilde{c}_{1n} T_{n1}(T) + \tilde{c}_{2n} T_{n2}(t) \right) x_n ,$$

where $\tilde{c}_{1n} = (z_0, x_n)$, $\tilde{c}_{2n} = (\dot{z}_0, x_n)$.

From (10) we obtain

$$\| z(t) \| \leqslant e^{-\gamma t} \left\{ P \sum_{n=1}^{\infty} |\tilde{c}_{1n}| + S \sum_{n=1}^{\infty} \frac{1}{\lambda_n} |\tilde{c}_{2n}| \right\}$$

and

$$(19) \qquad \|z(t)\| \leqslant e^{-\gamma t}\left\{P\,\frac{1}{\lambda}\left[\sum_{n=1}^{\infty}\tilde{c}_{1n}^2\lambda_n^2\right]^{1/2} + S\,\frac{1}{\lambda}\left[\sum_{n=1}^{\infty}\tilde{c}_{2n}^2\right]^{1/2}\right\}$$

On the other hand we have

$$\sum_{n=1}^{\infty}\tilde{c}_{1n}^2\,\lambda_n^2 = \|Kz_0\|^2, \qquad \sum_{n=1}^{\infty}\tilde{c}_{2n}^2 = \|\dot{z}_0\|^2.$$

In view of (19) we have

$$(20) \qquad \|z(t)\| \leqslant e^{-\gamma t}\left\{\frac{1}{\lambda}\,\delta\,(P+S)\right\} = \varepsilon\, e^{-\gamma t},$$

so that

$$(21) \qquad \|z(t)\| < \varepsilon.$$

Hence the solution $x(t)$ is stable. The asymptotical stability follows from (21).

The method of investigation of the existence of solutions for differential equations discussed above is similar to known methods [5], [6], but here we considered the solution of equation (1) for the infinite interval $t \in \langle 0, +\infty)$ and our primary aim has been the investigation of boundedness and stability of this solution.

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Received November 2nd, 1972.