DEMONSTRATIO MATHEMATICA
Vol. VII No 1 1974

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AN APPROXIMATE METHOD OF SOLVING A MIXED BOUNDARY VALUE PROBLEM IN THE THEORY OF ELASTICITY FOR PIECEWICE NON-HOMOGENEOUS ANISOTROPIC BODIES

1. Introduction

Consider the boundary value problem given in [3] assuming that the differences τ^{je}_{rs} of the Poisson constants of media are equal to zero, and that the vector functions F and g are independent of the unknown vector function u. In this paper the notation is the same as in [3].

From the results of [3], in particular from Theorem 2 it follows that the solution of this problem is given in the form

$$(1)\quad \overset{(2)}{\beta} u(x) = a\int_l \left[\overset{(1)}{T}G(x,y)\right]^* \varphi(y)dl - \int_l G(x,y)\,\Psi(y)dl - \overset{(1)}{\lambda}(x), \quad x\in B_2,$$

$$(2)\quad \overset{(1)}{\beta} u(x) = -\int_l \left[\overset{(1)}{T}G(x,y)\right]^* \varphi(y)dl + \int_l G(x,y)\,\Psi(y)dl - \overset{(2)}{\lambda}(x), \quad x\in B_1,$$

where $\overset{(1)}{\beta} = 2\pi E$, $\overset{(2)}{\beta} = 2\pi Ea$, the constant a being defined in [3] by (19),

$$(3)\quad \overset{(1)}{\lambda}(x) = -\iint_{B_2} G(x,y)\,F(y)dB + \int_l G(x,y)g(y)dl, \quad x\in B_2$$

$$(4)\quad \overset{(2)}{\lambda}(x) = -\iint_{B_1} G(x,y)F(y)dB - \int_{L'}\left[\overset{(1)}{T}G(x,y)\right] \overset{(1)}{f}(y)dL + \int_{L''} G(x,y)\overset{(2)}{f}(y)dL, \quad x\in B_1$$

and $\varphi(y)$, $\Psi(y)$ is the unique solution of the following system of functional equations

$$(5)\qquad a\int_l \left[\overset{(1)}{T}G(x,y)\right]^{*}\varphi(y)dl - \int_l G(x,y)\Psi(y)dl = \overset{(1)}{\lambda}(x), \qquad x\in B_1$$

$$(6)\qquad -\int_l \left[\overset{(1)}{T}G(x,y)\right]^{*}\varphi(y)dl + \int_l G(x,y)\Psi(y)dl = \overset{(2)}{\lambda}(x), \qquad x\in B_2$$

Theorem 3 in [3] assures that the solution of the above system is unique, since when $x\to x_0\in l$ we can eliminate the vector function $\Psi(y)$ and get the system of integral equation of the form (27), [3].

In this paper we shall present the solution of the problem in the form of uniformly convergent Fourier series which can be constructed effectively. For this purpose we shall apply the method given in [2] and [1].

2. Construction of the system of vector functions

Let l_k' $(k=1,\dots,m)$ be closed curves lying in the domain B_1 in such a way that the domain $B_2^{(k)}$ lies inside each domain $B_2^{(k)'}$ bounded by the curve l_k'. We assume that the curves l_k' $(k=1,\dots,m)$ are disjoint and are disjoint with l_k and L. We introduce the following notation

$$(7)\qquad l' = \sum_{k=1}^{m} l_k' \qquad B_2' = \sum_{k=1}^{m} B_2^{(k)'}.$$

Let l_k'' $(k=1,\dots,m)$ be closed curves lying in the domains $B_2^{(k)}$ $(k=1,\dots,m)$ respectively and having no common points with l_k. Let $B_2^{(k)''}$ denote the domain bounded by the curve l_k''. Consequently we have

$$(8)\qquad l'' = \sum_{k=1}^{m} l_k'', \qquad B_2'' = \sum_{k=1}^{m} B_2^{(k)''}.$$

Let $\left\{\overset{(1)}{x}_r\right\}$, $(r=1,2,\dots)$, $\left\{\overset{(2)}{x}_r\right\}$, $(r=1,2,\dots)$ denote a countable set of points lying everywhere dense on the curves l' and l'' respectively.

The system of functional equations (5) - (6) may be written in the form

$$\int_l K(\overset{(1)}{x}, \overset{(2)}{x}, y)\Psi(y)dl = \lambda(\overset{(1)}{x}, \overset{(2)}{x}), \tag{9}$$

where $x = \overset{(i)}{x}$ at $x \in B_i$ and

$$K\left(\overset{(1)}{x}, \overset{(2)}{x}, y\right) = \left\| \begin{matrix} a\left[T G(\overset{(1)}{x}, \overset{(1)}{y})\right]^* & -G(\overset{(1)}{x}, y) \\ \left[T G(\overset{(1)}{x}, \overset{(2)}{y})\right]^* & G(\overset{(2)}{x}, y) \end{matrix} \right\| = \left\| \begin{matrix} K_1 \\ K_2 \end{matrix} \right\|, \tag{10}$$

where $K_i = \left\| \begin{matrix} \overset{(1)}{K_i} \\ \overset{(2)}{K_i} \end{matrix} \right\|$, $i=1,2$ and

$$\Psi(y) = \left[\varphi(y), \psi(y)\right] \tag{11}$$

$$\lambda(\overset{(1)}{x}, \overset{(2)}{x}) = \left[\overset{(1)}{\lambda}(x), \overset{(2)}{\lambda}(x)\right]. \tag{12}$$

Consider the following system of vector functions

$$\nu_{je}(y) = \begin{cases} K_1^{(e)}\left(\overset{(1)}{x}_r, y\right), & \text{if } r = \frac{j+1}{2} \text{ is integer} \\ K_2^{(e)}\left(\overset{(2)}{x}_r, y\right), & \text{if } r = \frac{j}{2} \text{ is integer} \end{cases} \tag{13}$$

$e = 1,2, \quad j = 1,2,3,\dots$

T h e o r e m 1. The system of vector functions $\{\nu_{je}(y)\}$ is linearly independent and complete in the space $C_{\alpha}^{L_2}(l)$ *).

P r o o f. Let us assume that the system (13) is linearly dependent i.e. there exist constants C_{je}, at least one not zero, such that

$$\sum_{e=1}^{2}\sum_{j=1}^{N} C_{je}\,\nu_{je}(y) \equiv 0. \tag{14}$$

This identity can be written in the form

$$\sum_{e=1}^{2}\sum_{\alpha=1}^{N'} C'_{\alpha e}\, \overset{(e)}{K_1}\left(\overset{(1)}{x}_{p_\alpha}, y\right) + \sum_{e=1}^{2}\sum_{\alpha=1}^{N''} C''_{\alpha e}\, \overset{(e)}{K_2}\left(\overset{(2)}{x}_{q_\alpha}, y\right) \equiv 0, \tag{15}$$

where $N' + N'' = N$ and p_α, q_α are integer. From (15) it follows directly that

$$\sum_{e=1}^{2}\sum_{\alpha=1}^{N'} C'_{\alpha e}\left[\overset{(1)(e)}{TG}\left(\overset{(1)}{x}_{p_\alpha}, y\right)\right]^{*} - \sum_{e=1}^{2}\sum_{\alpha=1}^{N''} C''_{\alpha e}\left[\overset{(1)(e)}{TG}\left(\overset{(2)}{x}_{q_\alpha}, y\right)\right]^{*} \equiv 0 \tag{16}$$

and

$$\sum_{e=1}^{2}\sum_{\alpha=1}^{N'} C'_{\alpha e}\, \overset{(e)}{G}\left(\overset{(1)}{x}_{p_\alpha}, y\right) - \sum_{e=1}^{2}\sum_{\alpha=1}^{N''} C''_{\alpha e}\, \overset{(e)}{G}\left(\overset{(2)}{x}_{q_\alpha}, y\right) \equiv 0. \tag{17}$$

Now we introduce the vector function $v(x) = \overset{(i)}{v}(x)$ at $x \in B_i$, where

$$\overset{(1)}{v}(x) = \sum_{e=1}^{2}\sum_{\alpha=1}^{N''} C''_{\alpha e}\, \overset{(e)}{G}\left(\overset{(2)}{x}_{q_\alpha}, x\right), \qquad x \in B_1, \tag{18}$$

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*) $C_{\alpha}^{L_2}(l)$ denotes the space of vector functions defined on l and satisfying the Hölder condition.

(19) $$\overset{(2)}{v}(x) = \sum_{e=1}^{2} \sum_{\alpha=1}^{N'} C'_{\alpha e} \overset{(e)}{G} \overset{(1)}{\left(x_{p_\alpha}, x\right)}, \qquad x \in B_2$$

From the definition of G we get

(20) $$A\left(\frac{\partial}{\partial x}\right) v(x) = 0, \qquad x \in B$$

(21) $$\overset{(1)}{v}(x)\Big|_{L'} = 0, \quad \overset{(1)}{T}\overset{(1)}{v}(x)\Big|_{L''} = 0.$$

From (17) we have

(22) $$\overset{(1)}{v}(x_0) = \overset{(2)}{v}(x_0), \qquad x_0 \in l$$

and from (16) we obtain

(23) $$\overset{(1)}{T}\overset{(1)}{v}(x_0) = a \overset{(1)}{T}\overset{(2)}{v}(x_0), \qquad x_0 \in l.$$

It is easily shown that from (20) - (23) it follows directly that

(24) $$v(x) \equiv 0 \quad \text{at} \quad x \in B.$$

Thus $\overset{(1)}{v}(x) = 0$ at $x \in B - \bar{B}''_2$. Let us suppose that $C''_{\alpha e} \neq 0$. If a point $x \in B_2$ is placed sufficiently close to the point $\overset{(2)}{x}_{q_\alpha}$, then the term $\sum_{e=1}^{2} C''_{\alpha e} \overset{(e)}{G} (\overset{(2)}{x}_{q_\alpha}, y)$ can be made large while the others are bounded. Similarly $\overset{(2)}{v}(x) = 0$ at $x \in B'_2$ and if we suppose that $C'_{\alpha e} \neq 0$, then the term $\sum_{e=1}^{2} C'_{\alpha e} \overset{(e)}{G} (\overset{(1)}{x}_{p_\alpha}, y)$ can be made large while the others are bounded. These facts contradict the identity (24). Hence, we obtain that $C''_{\alpha e} = 0$, $C'_{\alpha e} = 0$. Thus the system $\left\{\nu_{je}(y)\right\}$ is linearly independent.

Now, we shall prove that the system $\left\{\vartheta_{je}(y)\right\}$ is complete in the space $C_\alpha^{L_2}(l)$. Let $\delta(y)$ be an arbitrary vector of the space $C_\alpha^{L_2}(l)$ orthogonal to all the vectors of system (13)

$$\int_l \left(\vartheta_{je}(y)\,\delta(y)\right)dl = 0 \qquad e=1,2, \qquad j=1,2,\ldots \tag{25}$$

Let us consider the vector

$$W\left(\overset{(1)}{x},\overset{(2)}{x}\right) = \int_l K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\delta(y)dl. \tag{26}$$

At the points $\left\{\overset{(1)}{x}_r\right\}, \left\{\overset{(2)}{x}_r\right\}$ the vector

$$W\left(\overset{(1)}{x}_r,\overset{(2)}{x}_r\right) = \int_l K\left(\overset{(1)}{x}_r,\overset{(2)}{x}_r,y\right)\delta(y)dl = 0 \tag{27}$$

and its derivatives are continuous on l' and l''. Hence, taking into account the fact that the points $\left\{\overset{(1)}{x}_r\right\}, \left\{\overset{(2)}{x}_r\right\}$ are located everywhere dense on the l' and l'' we obtain that

$$W\left(\overset{(1)}{x},\overset{(2)}{x}\right) = 0 \quad \text{for} \quad \overset{(1)}{x} \in l', \quad \overset{(2)}{x} \in l''. \tag{28}$$

From the well-known property of Green's tensor $G(x,y)$ and the theorems of uniqueness we have that

$$W\left(\overset{(1)}{x},\overset{(2)}{x}\right) = 0 \quad \text{for} \quad \overset{(1)}{x} \in B_1 - \bar{B}_2', \quad \overset{(2)}{x} \in B_2''. \tag{29}$$

Thus, $W\left(\overset{(1)}{x},\overset{(2)}{x}\right) = 0$ in the whole domain B, and

$$\int_l K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\delta(y)dl = 0 \quad \text{at} \quad x \in B. \tag{30}$$

The above system of functional equations is homogeneous and $\delta(y) = 0$ at $y \in l$, which proves that the system (16) is complete in the space $C_\alpha^{L_2}(l)$.

From that it follows directly that the system (13) is linearly independent and complete in the space L_2,[2].

3. The approximate solution of the problem

Let us introduce the notation

$$(31)\qquad \vartheta_{j1}(y) = \overset{(2j-1)}{\mu}(y), \quad \vartheta_{j2}(y) = \overset{(2j)}{\mu}(y).$$

Applying the Schmidt orthogonalization procedure we can construct a new orthogonal sequence of vectors

$$(32)\qquad \overset{(s)}{\Phi}(y) = \sum_{r=1}^{s} B_{sr}\overset{(r)}{\mu}(y) \qquad s=1,2,3,\ldots,$$

where B_{sr} are known constants.

Let Φ_s be the Fourier coefficients of an expansion of an unknown vector function $\Psi(y)$ in a series

$$(33)\qquad \Phi_s = \int_l \Psi(y)\,\overset{(s)}{\Phi}(y)dl.$$

In order to find the s-th Fourier coefficient, we substitute step by step the points $\overset{(1)}{x}_1, \overset{(2)}{x}_1; \overset{(1)}{x}_2, \overset{(2)}{x}_2; \ldots; \overset{(1)}{x}_p, \overset{(2)}{x}_p$ $(p=1,2,\ldots,k)$ in (9), where $k = \left[\frac{s+3}{4}\right]$ ($\left[\frac{s+3}{4}\right]$ denotes the integer part of the quotient) and multiply by the vectors

$$\vec{B}_{sp} = B_{s,4p-3}\,\vec{i}_1 + B_{s,4p-2}\,\vec{i}_2 + B_{s,4p-1}\,\vec{i}_3 + B_{s,4p}\,\vec{i}_4,$$

where $\vec{i}_1, \vec{i}_2, \vec{i}_3, \vec{i}_4$ are the unity axial vectors.

After summation, in virtue of (31) and (13), we obtain

$$(34)\quad \int_l \sum_{r=1}^{s} B_{sr}\,\overset{(r)}{\mu}(y)\,\Psi(y)dl = \sum_{p=1}^{k} \vec{B}_{sp}\lambda\left(\overset{(1)}{x}_p, \overset{(2)}{x}_p\right)$$

and from (32) and (33) we have

$$\Phi_s = \sum_{p=1}^{k} \vec{B}_{sp}\lambda\left(\overset{(1)}{x}_p, \overset{(2)}{x}_p\right). \tag{35}$$

Thus, the Fourier series of the vector $\Psi(y)$ has the following form

$$\sum_{s=1}^{\infty}\sum_{p=1}^{k}\vec{B}_{sp}\,\lambda\left(\overset{(1)}{x}_p, \overset{(2)}{x}_p\right)\overset{(s)}{\Phi}(y). \tag{36}$$

In view of Theorem 1 this series is convergent in average to the vector $\Psi(y)$.

T h e o r e m 2. The sequence of partial sums

$$\overset{(N)}{u}(x) = \frac{1}{\beta}\sum_{s=1}^{N}\Phi_s\int_l K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\overset{(s)}{\Phi}(y)\,dl - \frac{1}{\beta}\lambda\left(\overset{(1)}{x},\overset{(2)}{x}\right) \tag{37}$$

is uniformly convergent to the solution $u(x)$ of the given problem.

P r o o f. The exact solution of the problem can be written in the form

$$u(x) = \frac{1}{\beta}\int_l K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\Psi(y)\,dl - \frac{1}{\beta}\lambda\left(\overset{(1)}{x},\overset{(2)}{x}\right). \tag{38}$$

The difference of (38) and (37) has the following form

$$u(x) - \overset{(N)}{u}(x) = \frac{1}{\beta}\int_l K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\left[\Psi(y) - \sum_{s=1}^{N}\Phi_s\overset{(s)}{\Phi}(y)\right]dl\,. \tag{39}$$

Making use of the Buniakowski-Schwarz inequality we obtain

$$\left|u(x) - \overset{(N)}{u}(x)\right| \leqslant \frac{1}{\beta}\left\{\int_l\left[K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\right]^2 dl\int_l\left[\Psi(y) - \sum_{s=1}^{N}\Phi_s\overset{(s)}{\Phi}(y)\right]^2 dl\right\}^{\frac{1}{2}}. \tag{40}$$

The first integral is bounded in any closed domain lying entirely inside B and

$$\text{(41)} \qquad \left\{ \int_{l} \left[K\left(\overset{(1)}{x},\overset{(2)}{x},y\right)\right]^2 dl \right\}^{\frac{1}{2}} < C, \quad \text{(C being const.).}$$

From the fact that the series (36) is convergent in average to the vector function $\Psi(y)$ it results that there exists one N such that

$$\text{(42)} \qquad \left\{ \int_{l} \left[\Psi(y) - \sum_{s=1}^{N} \Phi_s \overset{(s)}{\Phi}(y) \right]^2 dl \right\}^{\frac{1}{2}} < \frac{\varepsilon_N}{C}.$$

Hence

$$\text{(43)} \qquad \left| u(x) - \overset{(N)}{u}(x) \right| < \varepsilon_N$$

and

$$\text{(44)} \qquad u(x) = \lim_{N \to \infty} \frac{1}{\beta} \int_{l} K\left(\overset{(1)}{x},\overset{(2)}{x},y\right) \left[\sum_{S=1}^{N} \Phi_s \overset{(s)}{\Phi}(y) \right] dl - \frac{1}{\beta} \lambda\left(\overset{(1)}{x},\overset{(2)}{x}\right)$$

which is what we had to prove.

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Received October 30th, 1972.