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Helena Jakuszenkow

ON PROPERTIES OF THE GENERALIZED GAMMA DISTRIBUTION

1. Introduction

Let variables $X_0, X_1, \dots, X_n$ be subject to distributions determined by densities $f_j(x)$ for $j = 0, 1, \dots, n$. If variables Y_k, $k = 1, \dots, n$ are known functions of the variables X_j, $j = 0, 1, \dots, n$, then the joint distribution of the n-dimensional random variable $(Y_1, \dots, Y_n)$ is strictly determined by the functions $f_j(x)$. One can ask whether a muldimensional distribution of the variable $(Y_1, \dots, Y_n)$ determines the distributions of the variables X_j, $j = 0, 1, \dots, n$. Of course, in the general case the answer to this question is negative, so there arises a question for what transformations $Y_k = \Psi_k(X_0, X_1, \dots, X_n)$ a distribution of the variable $(Y_1, \dots, Y_n)$ determines the distributions of the variables X_j and whether it determines them uniquely.

The problem formulated above has been dealt with by I. Kotlarski [1], J. Aitchison [2], K. Królikowska [3]. These authors gave theorems concerning the cases where the variables X_j are subject to the gamma distribution or to the generalized gamma distribution, and the variables Y_k are some concrete functions of the random variables X_j.

In the present paper we shall give some theorems that generalize all theorems in the papers mentioned above.

2. A characterization of the generalized gamma distribution

T h e o r e m 1. A necessary and sufficient condition in order that independent random variables $X_0, X_1, \dots, X_n$, $n \geq 2$

have the generalized gamma distribution

$$(1)\quad f_j(x) = \begin{cases} 0 & x \leqslant 0 \\ \dfrac{|\alpha|}{\Gamma\left(\frac{p_j}{\alpha}\right) a^{\frac{p_j}{\alpha}}} x^{p_j-1} \exp\left(-\dfrac{x^{\alpha}}{a}\right) & \text{where } a,\alpha p_j:\ x>0, \\ & j=0,\dots,n \end{cases}$$

is that the n-dimensional random variable $(Y_1,\dots,Y_n)$ defined as follows

$$(2)\qquad Y_k = \Psi_k\left(\frac{X_1}{X_0};\ \dots\ ;\frac{X_n}{X_0}\right) \qquad k=1,\dots,n$$

have a generalized Dirichlet distribution with the density

$$(3)\qquad k(y_1,\dots,y_n) = \frac{|\alpha|\Gamma\left(\frac{1}{\alpha}\sum_0^n p_j\right)}{\prod_0^n \Gamma\left(\frac{p_j}{\alpha}\right)} \prod \eta_k^{p_k-1}(y_1,\dots,y_n)\left[1+\sum_1^n \eta_k^{\alpha}(y_1,\dots,y_n)\right]^{-\frac{1}{\alpha}\sum_0^n p_j} \cdot |J|$$

for $p_j\alpha>0$, $j=0,1,\dots,n$ $(y_1,\dots,y_n)\in\Omega$. Here Ω is the image of $\{x_k|x_k>0,\ k=0,1,\dots,n\}$ under the transformation (2). This transformation is assumed to be one-to-one with respect to the variables $Z_k=\frac{X_k}{X_0}$, $k=1,\dots,n$ for $x_k>0$. The functions $Z_k=\eta_k(Y_1,\dots,Y_n)$ $k=1,\dots,n$ are inverse to the functions (2) and are of class C_1, J is the jacobian of the transformation

$$(4)\qquad Z_k = \frac{X_k}{X_0} = \eta_k(Y_1,\dots,Y_n) \qquad k=1,2,\dots,n$$

and $J\neq 0$ for $(y_1,\dots,y_n)\in\Omega$.

P r o o f. N e c e s s i t y. In the random variables $X_0,X_1,\dots,X_n$ are independent and have distribution (1), then the joint (n+1)- dimensional random variable $(X_0,X_1,\dots,X_n)$ has a distribution with the density

(5) $$f(x_0,x_1,\dots,x_n)=\prod_0^n f_j(x_j)=$$

$$=\frac{|\alpha|^{n+1}}{\prod_0^n\Gamma\left(\frac{p_j}{\alpha}\right)}\,a^{-\frac{1}{\alpha}\sum_0^n p_j}\cdot\prod_0^n x_j^{p_j-1}\cdot\exp\left(-\frac{1}{a}\sum_0^n x_j^{\alpha}\right)$$

for $p_j\alpha,\ a,x_j>0,\quad j=0,1,\dots,n.$

Next let us introduce new random variables defined as follows

(6) $$Z_k=\frac{X_k}{X_o}.\qquad k=1,2,\dots,n.$$

The density of the n-dimensional random variable $(Z_1,\dots,Z_n)$ has the form

$$h(z_1,\dots,z_n)=\int_0^{\infty}f(x_o,x_oz_1,\dots,x_oz_n)x_o^n\,dx_o,$$

that is

$$h(z_1,\dots,z_n)=\frac{|\alpha|^{n+1}}{\prod_0^n\Gamma\left(\frac{p_j}{\alpha}\right)}\,a^{-\frac{1}{\alpha}\sum_0^n p_j}\cdot\prod_1^n z_k^{p_k-1}\int_0^{\infty}x_o^{\sum_0^n p_j-1}\cdot\exp\left[-\frac{x_o^{\alpha}}{a}\left(1+\sum_1^n z_k^{\alpha}\right)\right]dx_o.$$

Substituting $x_o^{\alpha}=t$ under the integral, we obtain

(7) $$h(z_1,\dots,z_n)=\frac{|\alpha|^{n}\Gamma\left(\frac{1}{\alpha}\sum_0^n p_j\right)}{\prod_0^n\Gamma\left(\frac{p_j}{\alpha}\right)}\prod_1^n z_k^{p_k-1}\cdot\left(1+\sum_1^n z_k^{\alpha}\right)^{-\frac{1}{\alpha}\sum_0^n p_j}$$

where $p_j\alpha,z_k>0$ for $j=0,1,\dots,n.\quad k=1,\dots,n.$

Applying (6) to (2), we get

(8) $$Y_k=\Psi_k(Z_1,\dots,Z_n)\qquad k=1,2,\dots,n.$$

From the hypotheses of the theorem it follows that there exists a transformation inverse to (8), namely (4). Moreover, by assumption the jacobian J of this transformation is different from 0. Hence the density of the n-dimensional random variable $(Y_1,\dots,Y_n)$ can be expressed by the formula

$$k(y_1,\dots,y_n) = h\left[\eta_1(y_1,\dots,y_n),\dots, \eta_n(y_1,\dots,y_n)\right]\cdot|J|$$

where J is the Jacobian of transformation (4). This shows that the condition is necessary.

P r o o f o f s u f f i c i e n c y. By assumption, the n-dimensional random variable $(Y_1,\dots,Y_n)$ is subject to the generalized Dirichlet distribution (3), where the variables Y_k, $k=1,1,\dots,n$ are functions (2) of independent and positive random variables X_j, $j=0,1,\dots,n$. Then the random variable $(Z_1,\dots,Z_n)$ has the distribution with density (7), since the transformations (4) and (8) are one-to-one in view of the hypotheses of the theorem.

Next we make use of a theorem on characteristic functions of random variables $\ln X_0$, $\ln X_1,\dots,\ln X_n$ and the random variable $(\ln Z_1,\dots,\ln Z_n)$, where $Z_k=\frac{X_k}{X_0}$, $k=1,\dots,n$. This theorem says (see [1] and [3]) that if random variables X_0, $X_1,\dots,X_n$ are independent and positive, and the characteristic functions of the variables $\ln X_0,\ln X_1,\dots,\ln X_n$ do not vanish at any point, then the characteristic function of the n-dimensional random variable $(\ln Z_1,\dots,\ln Z_n)$ determines the characteristic functions of the variables $\ln X_0,\ln X_1,\dots,\ln X_n$ up to a factor $\exp(itb)$ which is the same for all the characteristic functions in question.

Accordingly, let $\varphi(t_1,\dots,t_n)$ be the characteristic function of the variable $(\ln Z_1,\dots,\ln Z_n)$, and let $\varphi j(t)$, $j=0,1,\dots,n$ denote the characteristic functions of the variables $\ln X_j$, respectively. As is easy to show, these functions satisfy the condition

(9) $$\varphi(t_1,\dots,t_n) = \varphi_1(t_1)\dots\varphi_n(t_n)\,\varphi_0\left(-\sum_1^n t_k\right).$$

The random variable $(Z_1,\dots,Z_n)$ is subject to the distribution with density (7), so that the characteristic function of the variable $(\ln Z_1,\dots,\ln Z_n)$ has the form

$$\varphi(t_1,\dots,t_n) =$$

$$= \frac{|\alpha|^n\Gamma\left(\frac{1}{\alpha}\cdot\sum_0^n p_j\right)}{\prod_0^n\Gamma\left(\frac{p_j}{\alpha}\right)}\int_0^\infty\dots\int_0^\infty\prod_1^n z_k^{p_k+it_k-1}\cdot\left(1+\sum_1^n z_k^\alpha\right)^{-\frac{1}{\alpha}\sum_0^n p_j}\cdot\prod_1^n dz_k.$$

In order to compute the above n-fold integral we make use of formula 4.638 in [4]

$$\varphi(t_1,\dots,t_n) = \frac{\Gamma\left(\frac{1}{\alpha}\sum_0^n p_j\right)}{\prod_0^n\Gamma\left(\frac{p_j}{\alpha}\right)}\cdot\frac{\prod_1^n\Gamma\left(\frac{it_k}{\alpha}+\frac{p_k}{\alpha}\right)\cdot\Gamma\left(\frac{1}{\alpha}\sum_0^n p_j-\sum_1^n\left(\frac{it_k+p_k}{\alpha}\right)\right)}{\Gamma\left(\frac{1}{\alpha}\sum_0^n p_j\right)}$$

hence

$$\varphi(t_1,\dots,t_n) = \frac{\Gamma\left(\frac{p_0}{\alpha}-\frac{i}{\alpha}\sum_1^n t_k\right)}{\Gamma\left(\frac{p_0}{\alpha}\right)}\cdot\prod_1^n\frac{\Gamma\left(\frac{p_k+it_k}{\alpha}\right)}{\Gamma\left(\frac{p_k}{\alpha}\right)}.$$

Taking into account formula (9) and the theorem cited above we see that the characteristic function of the variable $\ln X_j$, $j=0,1,\dots,n$ has the form

$$\varphi_j(t) = \frac{\Gamma\left(\frac{it_jp_j}{\alpha}\right)}{\Gamma\left(\frac{p_j}{\alpha}\right)}\cdot\exp\left(\frac{it_j\ln a}{\alpha}\right),\quad p_j\alpha,\quad a>0,\quad j=0,1,\dots,n$$

where $\exp\left(\frac{it_j\ln a}{\alpha}\right)$ is the arbitrary factor equal for each characteristic function of the variable $\ln X_j$, and the parameters p_j and α are determined by the n-dimensional distribution (3).

It is easy to show that the density of the random variable X_j is of the form (1), q.e.d.

The following transformations are special cases of this theorem

1°. for $n=2$ and $\alpha=1$

$$Y = \frac{X_k - X_o}{\sqrt{X_k X_o}} , \sqrt{\frac{p_k}{2}} , \qquad k=1,2,$$

which was dealt with by I.Kotlarski [1].

$$2^{\circ} \qquad Y_k = \frac{X_k - Y_o}{\sqrt{X_k X_o}} , \sqrt{\frac{p_k}{2}} \qquad k=1,2,\ldots,n \qquad n \geqslant 2, \alpha > 0.$$

This case was dealt with by the Author in [5]. In this case the density of the variable $(Y_1,\ldots,Y_n)$ is of the form

$$\text{(3a)} \qquad k(y_1,\ldots,y_n) =$$

$$= \frac{(2\alpha)^n \cdot \Gamma\left(\frac{1}{\alpha} \sum_0^n p_j\right)}{\prod_0^n \Gamma\left(\frac{p_j}{\alpha}\right)} \cdot \prod_1^n \frac{\left(\frac{y_k}{\sqrt{2p_k}} + \sqrt{\frac{(y_k)^2}{2p_k} + 1}\right)^{2p_k}}{\sqrt{2p_k y_k^2 + 2p_k}} \cdot$$

$$\cdot \left[1 + \sum_1^n \left(\frac{y_k}{\sqrt{2p_k}} + \sqrt{\frac{y_k^2}{2p_k} + 1}\right)^{2\alpha}\right]^{-\frac{1}{\alpha}\sum_0^n p_j}$$

The density (3a) determines densities of the variables X_j, $j=0,1,\ldots,n$ up to a constant parameter a which is arbitrary and equal for all variables X_j.

3° $n \geqslant 2$, $\alpha = 1$. For example, the transformations

$$Y_k = \frac{X_k}{\sum_0^n X_j} \qquad \text{or} \qquad Y_k = \frac{\sum_0^n X_i}{\sum_0^{k+1} X_i}$$

and others were considered by J. Aitchison in [2].

4°
$$Y_k = \frac{X_{k-1}}{X_n} \qquad k=1,\ldots,n.$$

The density of the variable $(Y_1,\ldots,Y_n)$ was found by K. Królikowska [3].

T h e o r e m 2. A necessary and sufficient condition for independent random variables $X_0,X_1,\ldots,X_n$, $n \geqslant 2$ to have the generalized gamma distribution

$$(10) \quad f_j(x) = \frac{|\alpha_j|}{\Gamma\left(\frac{p_j}{\alpha_j}\right) a^{\frac{p_j}{\alpha_j}}} \, x^{p_j-1} \exp\left(-\frac{x}{a}^{\alpha_j}\right), \qquad \alpha_j p_j, a, x > 0, \quad (j=0,1,\ldots,n)$$

is that the n-dimensional random variable

$$(11) \quad Y_k = \Psi_k\left(\frac{X_1^{\alpha_1}}{X_0^{\alpha_0}}, \ldots, \frac{X_n^{\alpha_n}}{X_0^{\alpha_0}}\right) \qquad k=1,\ldots,n$$

have a generalized Dirichlet distribution with the density

$$(12) \quad k(y_1,\ldots,y_n) =$$

$$= \frac{\Gamma\left(\sum_0^n \frac{p_j}{\alpha_j}\right)}{\prod_0^n \Gamma\left(\frac{p_j}{\alpha_j}\right)} \prod_1^n \eta_k^{\frac{p_k}{\alpha_k}-1}(y_1,\ldots,y_n) \cdot \left[1 + \sum_1^n \eta_k(y_1,\ldots,y_n)\right]^{-\sum_0^n \frac{p_j}{\alpha_j}} |J|$$

for $p_j\alpha_j > 0$, $j=0,1,\dots,n$, $(y_1,\dots,y_n)\in\Omega$, where the functions $\Psi_k(y_1,\dots,y_n)$ are of class C_1 and there exists transformation inverse to transformation (11) with respect to $Z_k = \frac{X_k^{\alpha_k}}{X_o^{\alpha_o}}$, $k=1,\dots,n$ and $x_k > 0$. As asual, J is the jacobian of the transformation

$$Z_k = \frac{X_k^{\alpha_k}}{X_o^{\alpha_o}} = \eta_k(Y_1,\dots,Y_n), \qquad k=1,\dots,n \text{ and } J\neq 0$$

for $(y_1,\dots,y_n)\in\Omega$, and $\eta_k(y_1,\dots,y_n)$ are the functions of the transformation inverse to (11).

P r o o f. If the variables X_j, $j=0,1,\dots,n$ are independent and subject to the generalized gamma distribution (10), then the variables $V_j = X_j^{\alpha_j}$ are also independent and have a gamma distribution with the density

$$g_j(v) = \frac{1}{\Gamma\left(\frac{p_j}{\alpha_j}\right) a^{\frac{p_j}{\alpha_j}}}\, v^{\frac{p_j}{\alpha_j}-1} \cdot \exp\left(-\frac{v}{a}\right) \qquad p_j\ \alpha_j, a, v > 0, \quad j=0,1,\dots,n.$$

Therefore the variables V_j satisfy the hypotheses of Theorem 1, where we put $\alpha=1$ and replace p_j by the parameter $\frac{p_j}{\alpha_j}$. Theorem 2 generalizes several theorems of paper [3].

3. A characterization of the beta distribution of second kind

T h e o r e m 3. A necessary and sufficient condition in order that variables $Z_1,\dots,Z_n$ have the generalized beta distribution of second kind with density

$$(13) \qquad g_k(z) = \frac{z^{p_k-1}}{(1+z)^{p_k+p_o}} \cdot \frac{\Gamma(p_k+p_o)}{\Gamma(p_k)\Gamma(p_o)} \qquad k=1,\dots,n, j=0,\dots n \quad p_j > 0, \quad z > 0$$

is that the n-dimensional random variable $(Y_1,\dots,Y_n)$, where

(14) $$Y_k = \Psi_k(Z_1,\dots,Z_k), \qquad k=1,\dots,n$$

have a generalized Dirichlet distribution with the density

(15) $$k(y_1,\dots,y_n) =$$

$$= \frac{\Gamma\left(\sum_0^n p_j\right)}{\prod_0^n \Gamma(p_j)} \prod_1^n \eta_k^{p_k-1} \cdot k(y_1,\dots,y_n) \cdot \left[1+\sum_1^n \eta_k(y_1,\dots,y_n)\right]^{-\sum_0^n p_j} |J|$$

$p_k > 0, \quad k=0,1,\dots,n, \quad (y_1,\dots,y_n) \in \Omega.$

Moreover, the transformation (14) is one-to-one with respect to Z_k, $k=1,\dots,n$, the functions $Z_k = \eta_k(Y_1,\dots,Y_n)$ of class C_1 are inverse to the functions of (14), and J is the jacobian of the transformation

$Z_k = \eta_k(Y_1,\dots,Y_n)$, $k=1,\dots,n$ where $J \neq 0$ for $(y_1,\dots,y_n) \in \Omega$.

P r o o f. It suffices to observe that if in Theorem 1 we put $\alpha=1$, then the variables X_j have the gamma distribution

$$f_j(x) = \frac{1}{\Gamma(p_j)a^{p_j}} x^{p_j-1} \exp\left(-\frac{x}{a}\right), \quad a, p_j, x > 0, \quad j=0,\dots,n.$$

Hence the variables $Z_k = \frac{X_k}{X_0}$, $k=1,\dots,n$ have the generalized beta distribution of second kind with density (13). This ends the proof of Theorem 3.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF ŁÓDŹ

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