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ON SOME PROPERTIES OF FLAT GRAPHS

Let π be a fixed plane. We shall consider graphs $G=(X,U)$, where X is a finite set of points of this plane called vertices of the graph, U is a finite set of arcs lying in the plane π, and the end-points of each arc in the set U are distinct vertices of the graph. The arcs in the set U are called the edges of the graph. The terminology used in the sequel is consistent with paper [1].

Let $G=(X,U)$ be a fixed graph. For any set $C\subseteq U$ we form a graph $G_C=(X',C)$, where

$$X' = \left\{ x \in X: \text{ there exists an edge } u \in C \text{ incident with } x \right\}.$$

Definition 1. a) A set $C\subseteq U$ is called a quasicycle if every vertex of the graph G_C is incident with an even number of edges. b) A set $C\subseteq U$ is called a cycle if G_C is a connected graph and its every vertex is incident with its two edges. An edge u of the graph G is said to be a cyclic edge if it belongs to some quasicycle of the graph G. In the set $Q(G)$ of all quasicycles of the graph G we define addition of quasicycles

$$C_1 + C_2 = (C_1 \cup C_2)-(C_1 \cap C_2), \quad C_1,C_2 \in Q(G).$$

Moreover we define multiplication of quasicycles by the elements of the field of remainders mod 2

$$0\cdot C = \emptyset, \quad 1\cdot C = C, \quad C \in Q(G).$$

It is easy to see that the set $Q(G)$ with the so-defined operations forms a linear space over the field of remainders mod 2. This linear space is called the space of cycles of the graph G. It can be shown (see [1] p.166) that the dimension of the space $Q(G)$ equals the cyclomatic number λ of the graph.

We shall use the following lemma.

Lemma 1. If quasicycles $C_1,\dots,C_\lambda$ form a basis of the space $Q(G)$, then the set of quasicycles $\{U_1,\dots,U_\lambda\}$ defined by any of the following three conditions

(W_1) $U_i = C_i + C_{i+1}$ for $i=1,\dots,\lambda-1$ and $U_\lambda = C_\lambda$;

(W_2) $U_i = C_1 + \dots + C_i$ for $i=1,\dots,\lambda$;

(W_3) $U_i = C_i$ for $i=1,\dots,k-1,k+1,\dots,\lambda$ and $U_k = C_1+\dots+C_\lambda$;

form a basis for $Q(G)$.

Definition 2. A graph G is said to be flat if there exists a graph $L=(X,U)$ isomorphic to G such that

$$\bigwedge_{x \in \pi} \bigwedge_{u,v \in U} x \in u \cap v \Longrightarrow x \in X.$$

Flat graphs are characterized by the following classical theorem.

Theorem 1. (Kuratowski). A graph G is flat if and only if its skeleton does not contain parts homeomorphic to F_5 or parts homeomorphic to full König's graph $K_{3,3}$.*)

*) The terms: part of a graph, homeomorphic part, the graph F_5, the graph $K_{3,3}$ are also taken from the monograph [1] p.428.

The present work is based on the following characterization of flat graphs given by Mac Lane.

T h e o r e m 2. ([1] p.428). A graph G is flat if and only if the space of cycles Q(G) has a basis such that each edge of the graph G belongs to at most two quasicycles of this basis.

The basis defined by Theorem 2 will be called a Mac Lane basis.

D e f i n i t i o n 3. We say that a graph $G=(X,U)$ satisfies condition (K) if the space of cycles $Q(G)$ has a basis $\mathcal{R} = \{C_1, \ldots, C_\lambda\}$ such that

$$\text{(K)} \qquad \bigwedge_{u \in U} \ \bigwedge_{1 \leq \alpha \leq \beta \leq \gamma \leq \lambda} u \in C_\alpha \wedge u \in C_\gamma \Longrightarrow u \in C_\beta .$$

It turns out that property (K) characterizes flat graphs. In fact, we have the following theorem.

T h e o r e m 3. A graph $G=(X,U)$ is flat if and only if it satisfies condition (K).

P r o o f. Assume that a set of cycles $\{U_1, \ldots, U_\lambda\}$ constitutes a Mac Lane basis of $Q(G)$. We shall show that the sequence of sets $\{C_\lambda\}$, where $C_\lambda = U_1 + \ldots + U_\lambda$, satisfies the condition (K). To this aim we shall prove that for any cyclic edge u, the indices of the quasicycle C_i containing this edge form a sequence of consecutive numbers. Observe that the following two cases are possible:

1) u belongs to U_k and only to U_k. Then u belongs to all quasicycles in the sequence $C_k, C_{k+1}, \ldots, C_\lambda$ and only to those quasicycles;

2) $u \in U_k$ and $u \in U_m$ (let $k < m$). Then u belongs to all quasicycles $C_k, C_{k+1}, \ldots, C_{m-1}$ and only to them.

Conversely, let a family $\mathcal{R} = \{C_1, \ldots, C_\lambda\}$ form a basis from Definition 3. We form a basis $\mathcal{U} = \{U_1, \ldots, U_\lambda\}$ such that

$$U_i = C_i + C_{i+1} \quad \text{for} \quad i=1,2,\ldots,\lambda-1 \quad \text{and} \quad U_\lambda = C_\lambda .$$

In view of Theorem 2 it suffices to show that an arbitrary edge u belongs to at most two quasicycles of the basis $\mathcal{U}$. Then one of the following cases holds:

1) u belongs to no cycle of the basis $\mathcal{R}$, which implies that u belongs to no quasicycle of the basis $\mathcal{U}$;

2) u belongs to all quasicycles $C_1, \ldots, C_\alpha$, $(\alpha \geqslant 1)$, and only to them. Then u belongs to the quasicycle U_α only;

3) u belongs only to all quasicycles $C_\alpha, C_{\alpha+1}, \ldots, C_\beta$, $(1 \leqslant \alpha \leqslant \beta \leqslant \lambda)$. Then the edge u belongs to the quasicycles $U_{\alpha-1}$ and U_β and only to them.

L e m m a 2. For every cyclic edge u of a flat graph (X,U) there exists a Mac Lane basis such that the edge u belongs only to one quasicycle of this basis.

P r o o f. Let $\mathcal{U} = \{U_1, \ldots, U_\lambda\}$ be an arbitrary Mac Lane basis for the graph G. Since the numeration of the cycles in this basis is not essential, it suffices to consider the case $u \in U_1 \cap U_2$. Then the basis $\mathcal{U}' = \{U'_1, \ldots, U'_\lambda\}$, where $U'_i = U_i$ for $i \neq 2$, $U'_2 = U_1 + \ldots + U_\lambda$, satisfies the required conditions.

C o r o l l a r y 1. For every cyclic edge u of a flat graph $G=(X,U)$ there exists a basis $\mathcal{R} = \{C_1, \ldots, C_\lambda\}$ satisfying condition (K) and such that

$$\bigwedge_{1 \leqslant i \leqslant \lambda} u \in C_i .$$

P r o o f. Let $u \in U$ be a cyclic edge of the graph G. Then there exists a Mac Lane basis $\{U_1 \ldots, U_\lambda\}$ such that the edge u belongs to U_1 only. Repeating the reasoning given in the proof of Theorem 3 we can show that the basis $C_i = U_1 + \ldots + U_i$, $(i=1,\ldots,\lambda)$, satisfies condition (K).

D e f i n i t i o n 4. A basis C_f for the space of cycles of a graph G is said to be foundamental if every quasicycle of this basis has an edge not belonging to any other quasicycle of the basis C_f.

It can be shown ([1] p.166) that every graph with at least one quasicycle has a foundamental basis and the quasicycles of any foundamental basis are cycles.

D e f i n i t i o n 5. A graph G=(X,U) is said to be elementary if G has a foundamental basis satisfying condition (K) such that there exists an edge of G belonging to every cycle of this basis.

It is easy to see that every elementary graph is a flat graph without disconnecting points and bridges. An example of a flat graph that is not elementary is the graph represented in fig.1.

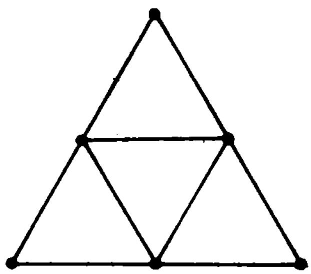

Fig.1

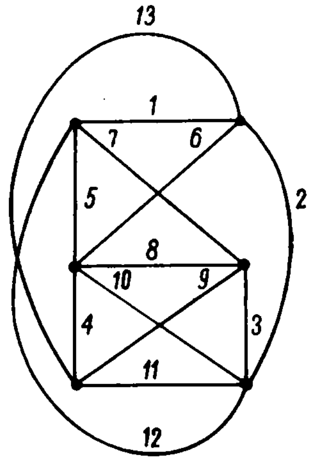

Fig.2

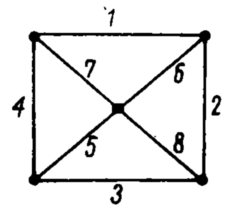

Fig.3

An essential property of flat graphs that are not elementary is given in the following theorem.

T h e o r e m 4. If a graph G=(X,U) possesses a foundamental basis C_f for which there exists an edge of the graph belonging to every cycle C_f, and if the cycle of this basis cannot be numbered to satisfy condition (K), then the graph is not flat.

P r o o f. Let $C_f = \{C_1,\dots,C_\lambda\}$ be a foundamental basis and u_0 an edge belonging to each of the cycles C_f. Suppose that G is flat. Let $\mathcal{U} = \{U_1,\dots,U_\lambda\}$ be its Mac Lane basis such that u belongs only to the quasicycle U_1. The existence of a basis with this property is guaranteed by Lemma 2. The quasicycles of the basis $\mathfrak{U}$ can be expressed in the form of sums cycles of the basis C_f. Moreover, from the properties of the basis $\mathfrak{U}$ and C_f it directly follows that

1) every quasicycle of the basis $\mathfrak{U}$, with exception of the quasicycle U_1, is the sum of an even number of cycles of the basis C_f;
2) every cycle of the basis C_f appears as a component in at most two cycles of the basis $\mathfrak{U}$.

First we shall show that from conditions 1) and 2) it follows that

R_1) every quasicycle of the basis $\mathfrak{U}$, except U_1, is a sum of two quasicycles of the basis C_f. The cycle U_1 equals some cycle of the basis C_f;

R_2) every cycle of the basis C_f, except one, appears as a component in two quasicycles of the basis $\mathfrak{U}$.

Let m_i denote the number of those elements of the basis C_f whose sum is equal to U_i. From condition 1) it follows that $m_1 \geq 1$, $m_i \geq 2$, $(i=2,3,\dots,\lambda)$. This implies $\sum_1^{\lambda} m_i \geq 2(\lambda-1)+1 = 2\lambda - 1$. Next, let k_i denote the number of quasicycles of the basis $\mathfrak{U}$ which contain the cycle C_i as a component. From condition 2) it follows that $1 \leq k_i \leq 2$, $(i=1,2,\dots,\lambda)$, where at least one of the numbers k_i must be least than 2, because otherwise the sum of all quasicycles of the basis $\mathfrak{U}$ would be the empty set. Hence we have $\sum_1^{\lambda} k_i \leq 2\lambda - 1$. But clearly the following equality holds: $\sum_1^{\lambda} m_i = \sum_1^{\lambda} k_i$. Consequently, the numbers m_i and k_i satisfy the equations

$$(r_1) \qquad m_1 + m_2 + \dots + m_\lambda = 2\lambda - 1,$$

$$(r_2) \qquad k_1 + k_2 + \dots + k_\lambda = 2\lambda - 1.$$

It is easy to see that after taking into account the limitations put on the numbers m_i and k_i, we obtain, as the only solution of (r_1): $m_1=1$, $m_2=m_3=\dots=m_\lambda=2$. Similarly, the only solution of (r_2) is as follows: $k_1=k_2=\ldots=k_{\alpha-1}=k_{\alpha+1}=\ldots=k_\lambda=2$, $k_\alpha=1$. This shows the validity of (R_1) and (R_2).

We now number the quasicycles of the basis $\mathcal{U}$ and the basis C_f as follows.

U'_1 denotes the quasicycle U_1. The cycle of the basis C_f equal to U'_1 is denoted by C'_1. From conditions (R_1) and (R_2) it follows that there exists exactly one quasicycle of the basis $\mathcal{U}$, to be denoted by U'_2, which is the sum of the cycle C'_1 and another cycle C'_2 of the basis C_f. Suppose that for some $n<\lambda$ we have numbered the quasicycles of the basis $\mathcal{U}$ and the cycles of the basis C_f such that $U'_1 = C'_1$, $U'_2 = C'_1 + C'_2, \ldots, U'_n = C'_{n-1} + C'_n$, where for $i \neq j$, $(i,j = 1,2,\ldots,n)$, we have $U'_i \neq U'_j$ and $C'_i \neq C'_j$.

Let U'_{n+1} denote that of the remaining quasicycles of the basis $\mathcal{U}$ which contains C'_n as a component. If such a cycle did not exist, then by (R_2) the sum of all quasicycles $\mathcal{U}$ different from $U'_1,\ldots,U'_n$ would be empty, which is obviously impossible. The uniqueness of this choice is guaranteed by (R_1).

Let C'_{n+1} denote the other component of the quasicycle U'_{n+1}. From (R_2) it follows that $C'_{n+1} \neq C'_i$ for $i=1,\ldots,n$.

Continuing this numbering we obtain after λ steps:

$$U'_1 = C'_1,\quad U'_2 = C'_1 + C'_2,\ldots,U'_i = C'_{i-1} + C'_i,\ldots,U'_\lambda = C'_{\lambda-1} + C'_\lambda,$$

where

$$\mathcal{U} = \left\{ U'_1,\ldots,U'_\lambda \right\},\quad C_f = \left\{ C'_1,\ldots,C'_\lambda \right\}.$$

Repeating the reasoning given in the proof of Theorem 3 one can show that the sequence of sets

$$C'_1 = U'_1,\quad C'_2 = U'_1 + U'_2,\ldots,C'_\lambda = U'_1 + U'_2 + \ldots + U'_\lambda$$

satisfies condition (K). This contradiction ends the proof.

Concluding this paper let us observe that

a) the class of graphs defined in Theorem 4 is not empty (for example, the graph in fig.2 has a basis $C_1 = \{1, 5, 6\}$, $C_2 = \{1,2,3,7\}$, $C_3 = \{1,2,3,5,8\}$, $C_4 = \{1,2,3,4,5,9\}$, $C_5 = \{1,2,5,10\}$, $C_6 = \{1,2,4,5,11\}$, $C_7 = \{1,2,12\}$, $C_8 = \{1,4,5,13\}$ satisfying the hypothesis of Theorem 4);

b) Theorem 4 cannot be strengthened by rejecting the hypothesis that the basis is foundamental (e.g. the graph in fig.3 has a basis $C_1 = \{1,2,3,4\}$, $C_2 = \{1,2,7,8\}$, $C_3 = \{1,4,5,6\}$, $C_4 = \{1,6,7\}$. This graph is clearly flat; moreover, it is an elementary graph).

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