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APPLICATION OF SOME SYSTEM OF FUNCTIONAL EQUATIONS TO THE INVESTIGATION OF BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

A considerable part of boundary-value problems for second-order partial differential equations of elliptic or parabolic type can be solved by applying the methods of potential theory [4]. These methods allow us to reduce the matter of solving various boundary-value problems to that of solving an integral equation or a system of such. The systems of integral equations obtained by this procedure can be, after some modification, treated as special cases of the system of p functional equations of the form

$$(1)\quad \varphi_i(x)=G_i(x,\varphi(x),f_{i1}[x,\varphi(x)],f_{i2}[x,\varphi(x)],\dots,f_{iq}[x,\varphi(x)]),$$

$i=1,2,\dots,p$, where $x(x_1,x_2,\dots,x_n)$ is a point of the n-dimensional Euclidean space R^n $(n\geqslant 1)$, $\varphi(x)=(\varphi_1(x),\varphi_2(x),\dots,\varphi_p(x))$ denotes a system of p unknown functions $\varphi_1,(x),\varphi_2(x),\dots,\varphi_p(x)$, and G_i as well as $f_{i1},f_{i2},\dots,f_{iq}$ are given functions.

In a concrete boundary-value problem the given functions f_{ij}, $i=1,2,\dots,p$, $j=1,2,\dots,q$, represent various- type definite integrals depending on parameters, and the form of the functions G_i are determined by boundary conditions stated in the problem as well as by the limit properties of potentials and their derivatives used in the course of solving the problem.

In the present paper we shall formulate sufficient conditions in order that there exist a unique (discontinuous in the sense of W. Pogorzelski's class $\mathfrak{H}_{\alpha}^{h}$) solution of system (1). Next we shall give examples to illustrate the possibility of applying this system to the investigation of some boundary-value problems for second-order partial differential equations.

2. Existence of a discontinuous solution of system (1)

At the beginning we recall the definition (see [3] p.445) of W. Pogorzelski's class $\mathfrak{H}_{\alpha}^{h}$ of functions defined on a bounded set $\Omega \in R^n$.

By the class $\mathfrak{H}_{\alpha}^{h}(\Omega)$ we understand the set of all real (or complex) functions $f(x)$ defined and continuous on the set Ω and satisfying the inequalities

$$|f(x)| \leqslant \frac{M}{|x-x_s|^{\alpha}}, \tag{2}$$

$$|f(x)-f(\tilde{x})| \leqslant \frac{k}{|x-x_s|^{\alpha+h}} |x-\tilde{x}|^{h}, \tag{3}$$

where $|x-\tilde{x}|$ denotes the Euclidean distance between points x and $\tilde{x}$, $|x-x_s|$ - the distance from the point x to the boundary S of the set Ω (i.e. $|x-x_s| = \inf_{y \in S}|x-y|$). The exponents α and h satisfy the conditions: $\alpha \geqslant 0$, $0 < h \leqslant 1$; and the coefficients M and k are positive contants. Without loss of generality we may assume that $|x-x_s| \leqslant |\tilde{x}-\tilde{x}_s|$.

The proof of existence of a discontinuous solution in the sense of class $\mathfrak{H}_{\alpha}^{h}$ for the system of functional equations (1) will be carried out with the help of well-known Schauder's theorem (see e.g. [4], p.18) on fixed point for a continuous operation in a Banach space.

To this aim assume that given functions f_{ij} and G_i are real (or complex) and satisfy the following requirements:

1° The functions $f_{ij}(x, u_1, u_2, \ldots, u_p)$, $i = 1,2,\ldots,p$; $j = 1,2,\ldots,q$, are defined and uniformly continuous relatively to the totality of their arguments respectively in the domains

(4) $\left[x \in \Omega_i;\ |u_k| < +\infty,\ k=1,2,\ldots,p\right],\ (i=1,\ldots,p),$

where $\Omega_1,\ldots,\Omega_p$ are disjoint bounded open sets in the space R^n.

2^0. The functions $G_i(x,u_1,u_2,\ldots,u_p,y_{i1},y_{i2},\ldots,y_{iq})$ are defined and continuous respectively in the domains

(5) $$\left[x \in \Omega_i;\ |u_k| < +\infty,\ k=1,2,\ldots,p;\right.$$
$$\left.|y_{ij}| < +\infty,\ j=1,2,\ldots,q\right],\ (i=1,\ldots,p).$$

3^0. If $u_k (k=1,2,\ldots,p)$ belong to the classes $\mathfrak{H}_\alpha^h\ (\Omega_k)$, respectively, i.e. if they satisfy the inequalities

(6) $$|u_k| \leqslant \frac{\varrho_1}{|x-x_{s_k}|^\alpha},$$

(7) $$|u_k-\tilde{u}_k| \leqslant \frac{\varrho_2}{|x-x_{s_k}|^{\alpha+h}}\,|x-\tilde{x}|^h,$$

where $x,\tilde{x} \in \Omega_k$ and $|x-x_{s_k}|$ denotes the distance from the point x to the boundary S_k of the domain Ω_k, then the functions $G_i(x,u_1,u_2,\ldots,u_p,y_{i1},y_{i2},\ldots,y_{iq})$ belong (relatively to the point x) to the classes $\mathfrak{H}_{\alpha_G}^{h_G}(\Omega_i)$, respectively, and satisfy the inequaiities

(8) $$|G_i(x,u_1,\ldots,u_p,y_{i1},\ldots,y_{iq})| \leqslant \frac{B_1(\varrho_1,\varrho_2)}{|x-x_{s_i}|^{\alpha_G}},$$

(9) $$|G_i(x,u_1,\ldots,u_p,y_{i1},\ldots,y_{iq})-G_i(\tilde{x},\tilde{u}_1,\ldots,\tilde{u}_p,\tilde{y}_{i1},\ldots,\tilde{y}_{iq})| \leqslant$$
$$\leqslant \frac{B_2(\varrho_1,\varrho_2)}{|x-x_{s_i}|^{\alpha_G+h_G}}\,|x-\tilde{x}|^{h_G},\quad (i=1,\ldots,p),$$

where the exponents α, α_G, h and h_G satisfy the inequalities

$$\alpha \geqslant \alpha_G \geqslant 0, \quad 0 < h \leqslant h_G \leqslant 1, \tag{10}$$

and ϱ_1, ϱ_2 denote positive constants.

4°. The function $B_1(\varrho_1, \varrho_2)$ and $B_2(\varrho_1, \varrho_2)$ appearing in inequalities (8) and (9) are defined and continuous in a bounded set $D \in R^2$ and satisfy, for sufficiently large positive values of the variables ϱ_1 and ϱ_2, the system of inequalities

$$\begin{cases} 0 < B_1(\varrho_1, \varrho_2) \leqslant \varrho_1, \\ 0 < B_2(\varrho_1, \varrho_2) \leqslant \varrho_2. \end{cases} \tag{11}$$

We shall show that under the assumptions accepted above the system (1) possesses at least one solution.

Accordingly, let us consider the Banach space Λ (see e.g. [5] p.56) whose points φ are all system $\varphi = (\varphi_1(x), \varphi_2(x), \ldots, \ldots, \varphi_p(x))$ of functions defined and continuous in the sets $\Omega_1, \ldots, \Omega_p$ respectively, satisfying the condition

$$\max_{1 \leqslant i \leqslant p} \sup_{x \in \Omega_i} \left\{ \left| x - x_{s_i} \right|^{\alpha + h} |\varphi_i(x)| \right\} < +\infty \tag{12}$$

where α and h denote the exponents appearing in assumption 3°.

The norm of a point φ in the space Λ is defined by the equality

$$\|\varphi\| \overset{df}{=} \max_{1 \leqslant i \leqslant p} \sup_{x \in \Omega_i} \left\{ \left| x - x_{s_i} \right|^{\alpha + h} |\varphi_i(x)| \right\}, \tag{13}$$

and the distance $\delta(\varphi, \tilde{\varphi})$ of two points $\varphi, \tilde{\varphi} \in \Lambda$ is the norm of their difference

$$\delta(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\| .$$

In the space Λ let Z be the set of all points φ satisfying the inequalities

$$(14)\qquad \begin{cases} |x-x_{s_i}|^{\alpha}\,|\varrho_i(x)| \leq \varrho_1 \\ |x-x_{s_i}|^{\alpha+h}\,|\varphi_i(x)-\varphi_i(\tilde{x})| \leq \varrho_2 |x-\tilde{x}|^h , \end{cases}$$

$i=1,2,\dots,p$, where ϱ_1,ϱ_2 denote any positive numbers satisfying conditions (11).

Of course, the set Z is closed, and it is easy to check that it is also convex.

Moreover, Z.Rojek and W.Żakowski have proved in [5] that the set Z is compact.

Taking into account the form of considered system (1), we transform the set Z by means of the operation $\psi = \hat{A}\varphi$ defined (for $i=1,2,\dots,p$) by the equalities

$$(15)\qquad \psi_i(x) = G_i(x,\varphi(x),\ f_{i1}[x,\varphi(x)],\ f_{i2}[x,\varphi(x)],\dots, f_{iq}[x,\varphi(x)]).$$

The operation $\hat{A}$ associates, with each point

$\varphi = (\varphi_1(x),\varphi_2(x),\dots,\varphi_p(x))$ of the set Z, a point

$\psi = (\psi_1(x),\psi_2(x),\dots,\psi_p(x))$ of some set Z'.

From assumptions 3^o and 4^o, inequalities (14) and Theorem 1 in paper [3], we infer that the inclusion $Z' \subset Z$ holds.

Since by assumption $1^o - 3^o$ the operation (15) is continuous in the space Λ, we see that all the assumptions of Schauder's theorem are satisfied. By means of that theorem it follows that in the set Z there exists at least one point $\varphi^* = (\varphi_1^*(x),\varphi_2^*(x),\dots,\varphi_p^*(x))$ invariant under the operation (15), i.e. there exists at least one solution $\varphi^*(x)$ of system (1) satisfying inequalities (14).

Hence we can formulate the following theorem.

T h e o r e m 1. If conditions $1^o - 4^o$ hold, then there exists at least one solution $\varphi(x) = (\varphi_1(x),\dots,\varphi_p(x))$ of the functional system (1) such that each function $\varphi_i(x)$ belongs to the class $\mathcal{H}_\alpha^h(\Omega_i)$, $(i=1,\dots,p)$.

3. Uniqueness

The proof of uniqueness of a solution for system (1) will be carried out with the help of Banach-Cacciopoli's theorem on contractive transformations.

For this purpose, instead of assumptions 1^o–4^o formulated in section 2, we accept the following stronger assumptions 5^o–7^o:

5^o. The functions $f_{ij}(x,u_1,u_2,\dots,u_p)$ are defined and continuous in the domains (4), respectively, and satisfy the conditions

$$(16)\qquad \left|f_{ij}(x,u_1u_2,\dots,u_p)\right| \leq \frac{M_f}{\left|x-x_{s_i}\right|^{\alpha_f}} + \tilde{M}_f \sum_{k=1}^{p} |u_k|,$$

$$(17)\qquad \left|f_{ij}(x,u_1u_2,\dots,u_p)-f_{ij}(x,\tilde{u}_1,\tilde{u}_2,\dots,\tilde{u}_p)\right| \leq k_f \sum_{k=1}^{p} |u_k-\tilde{u}_k|,$$

$(i=1,2,\dots,p;\ \ j=1,2,\dots,q)$, where $\alpha_f \geq 0$ and M_f, $\tilde{M}_f, k_f$ denote some positive constants.

6^o. The functions $G_i(x,u_1,u_2,\dots,u_p,y_{i1},y_{i2},\dots,y_{iq})$ are defined and continuous in the domains (5), respectively, and satisfy the inequalities

$$(18)\qquad \left|G_i(x,u_1,u_2,\dots,u_p,y_{i1},y_{i2},\dots,y_{iq})\right| \leq \frac{M_G}{\left|x-x_{s_i}\right|^{\alpha_G}} + \tilde{M}_G\left(\sum_{k=1}^{p} |u_k| + \sum_{j=1}^{q} |y_{ij}|\right),$$

$$(19)\qquad \left|G_i(x,u_1,\dots,u_p,y_{i1},\dots,y_{iq})-G_i(x,\tilde{u}_1,\dots,\tilde{u}_p,\tilde{y}_{i1},\dots,\tilde{y}_{iq})\right| \leq k_G\left(\sum_{k=1}^{p} |u_k-\tilde{u}_k| + \sum_{j=1}^{q} |y_{ij}-\tilde{y}_{ij}|\right),$$

$(i=1,2,\dots,p)$, where $\alpha_G \geq 0$ and M_G, $\tilde{M}_G$ and k_G are positive constants.

7° The constants $M_G, M_f, \tilde{M}_f, k_f$ may be chosen arbitrarily in the problem, but the coefficients $\tilde{M}_G$ and k_G must satisfy the inequalities

(20) $$\tilde{M}_G < \frac{1}{p(1+q\tilde{M}_f)},$$

(21) $$k_G < \frac{1}{p(1+qk_f)}.$$

To show that the solution of system (1) is unique, consider a class of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)$, satisfying the inequality

(22) $$\sup_{x\in\Omega_i}\left[\left|x-x_{s_i}\right|^{\alpha}\left|\varphi_i(x)\right|\right] \leqslant \varrho$$

$(i=1,2,\ldots,p)$, where ϱ is any fixed finite number fulfilling the condition

(23) $$\varrho \geqslant \frac{b_1M_G+b_2q\tilde{M}_GM_f}{1-p\tilde{M}_G(1+q\tilde{M}_f)},$$

in which b_1, b_2 denote some positive constants.

Then the following theorem holds.

T h e o r e m 2. Under assumptions 5° - 7°, the system (1) possesses exactly one solution in the class of functions satisfying inequality (22).

The proof of this theorem does not differ essentially from that of paper [1].

4. Applications

We shall give two examples indicating the possibility of applying the system of equation (1) to the investigation of boundary-value problems for partial differential equations.

As the first example we consider a boundary-value problem examined by A.Hać in paper [2]. This problem consists in determining a real function $u(x)$ that satisfies, at every interior point $X(x_1,\dots,x_n)$ of some bounded n-dimensional region Ω in the space R^n, a quasi-linear partial differential equation of elliptic type (see [2], formula (1)) with coefficients depending on the unknown function u and unlimited at the boundary of Ω. Besides, the function $u(X)$ should satisfy, at every point $P \in S$, a non-linear boundary condition (see [2], formula (2)) with tangential derivatives.

In paper [2] the above problem has been reduced by methods of potential theory to a system of n+2 integral equations (see [2], p.76, formulas (18) - (20)). This system *), after introducing additional notations

$$\tilde{\Gamma}_\nu = \begin{cases} \tilde{\Gamma} & \text{for } \nu = 0 \\ \dfrac{\partial \tilde{\Gamma}}{\partial x_\nu}, & \text{for } \nu = 1,2,\dots,n \end{cases}$$

$$\tilde{N}^{(u_0)}(P,Q) = 2N^{(u_0)}(P,Q)\left[\lambda_n^{(u_0)}(P)\right]^{-1}$$

$$\tilde{G}(\dots) = 2\,G(\dots)\left[\lambda_n^{(u_0)}(P)\right]^{-1},$$

can be written in the form

$$(24)\quad u_\nu(X) = -\iiint\limits_{\Omega} \tilde{\Gamma}_\nu^{(u_0)}(X,Y)\left[\lambda_n^{(u_0)}(Y)\right]^{-1} H\left[Y,u_0(Y),u_1(Y),\dots\right.$$

$$\left.\dots,u_n(Y)\right] dY + \iint\limits_{S} \tilde{\Gamma}_\nu^{(u_0)}(X,Q)\varphi(Q)dQ \qquad \nu = 0,1,2,\dots,n.$$

$$(25)\quad \varphi(P) = \iiint\limits_{\Omega} \tilde{N}^{(u_0)}(P,Y)\left[\lambda_n^{(u_0)}(Y)\right]^{-1} H\left[Y,u_0(Y),u_1(Y),\dots\right.$$

$$\left.\dots,u_n(Y)\right] dY + \iint\limits_{S} N^{(u_0)}(P,Q)\varphi(Q)dQ - \tilde{G}\left[P,u_0(P),\bar{u}_{s_p^{(1)}}(P),\dots,\bar{u}_{s_p^{(q)}}(P)\right].$$

*) The reader is referred to [2] for an explanation of all symbols used in the system of equations under discussion.

In the above system the functions $u_0(X), u_1(X), \ldots, u_n(X), \varphi(P)$, are unknown. All the remaining quantities appearing in this system are assumed to be given.

We will seek a suitable function $\varphi(P)$ in the class of Hölder functions. Consequently, in view of all assumptions accepted in [2], and since the kernel $\widetilde{N}^{(u_0)}(P,Q)$ of equation (25) has a weakly-singular estimation with respect to the surface integral (see [2], formula (16')), it follows that equation (25) can be represented in the following equivalent form (cf. [2], eq. (34)):

$$\varphi(P) = \tilde{\Xi}(P) + \iint_S \tilde{\mathcal{H}}^{(u_0)}(P,Q)\,\tilde{\Xi}(Q)\,dQ\,, \tag{26}$$

where $\tilde{\mathcal{H}}^{(u_0)}(P,Q)$ denotes the solving kernel for the kernel $\widetilde{N}^{(u_0)}(P,Q)$, and the function $\tilde{\Xi}(P)$ is expressed by the formula

$$\Xi(P) = \iiint_{\Omega} \widetilde{N}^{(u_0)}(P,Y)\left[\lambda_n^{(u_0)}(Y)\right]^{-1} H\left[Y, u_0(Y), u_1(Y), \ldots, u_n(Y)\right] dY -$$
$$-\widetilde{G}\left[P, u_0(P), \bar{u}_{s_p^{(1)}}(P), \ldots, \bar{u}_{s_p^{(q)}}(P)\right].$$

It is not difficult to notice that the system of integral equations (24) and (26) is a special case of the functional system (1)

In addition, it can be shown that the assumptions accepted in [2] imply (after slight modification) assumptions $1^\circ - 4^\circ$ formulated in section 2 of the present paper.

Therefore by Theorem 1 the system of integral equations (24) and (26), hence as well (24) and (25), possesses at least one solution. On the other hand, from the properties of potentials appearing in system (24) and (25) it follows that there exists at least one function $u(X)$ being a solution of the boundary-value problem posed in paper [2].

As the second example for application of the system (1) we take a problem considered by Z.Rojek in paper [6]. This is the Hilbert boundary-value problem consisting in determining a system of functions $\Phi_1(z), \Phi_2(z), \ldots, \Phi_n(z)$. These functions are to be holomorphic separately in each of the regions S^+, $S_1^-, S_2^-, \ldots, S_p^-$ bounded by a finite system of $p+1$ non-intersecting smooth Jordan curves $L_0, L_1, \ldots, L_p$.

Moreover, we require that they satisfy, at each point of the set $L = \sum_{\nu=0}^{p} L_\nu$, some strongly-singular boundary condition given by formula (1) in paper [6].

The problem formulated above has been reduced in [6] to a system of 2n integral equations of the form (see [6], formula (12)) *)

$$\varphi_\nu(t) = \frac{\lambda}{2} \gamma_\nu(t) \int_L \frac{F_\nu[t, \tau, \varphi_1(\tau), \ldots, \varphi_{2n}(\tau)]}{\tau - t} d\tau +$$

$$+ \frac{\lambda}{2\pi i} X_\nu^+(t) \tilde{Y}_\nu(t) \int_L \frac{1}{X_\nu^+(\tau)(\tau - t)} \left[\int_L \frac{F_\nu[\tau, \tau_1, \varphi_1(\tau_1), \ldots, \varphi_{2n}(\tau_1)]}{\tau_1 - \tau} d\tau_1 \right] d\tau + \tag{27}$$

$$+ X_\nu^+(t) \tilde{Y}_\nu(t) P_\nu(t) \qquad \nu = 1, 2, \ldots, 2n$$

with unknown functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_{2n}(t)$.

The system of integral equations (27) constitutes a special case of system (1). Since it can be shown that the assumptions in [6] imply assumptions $1^\circ - 4^\circ$ formulated in section 2, by Theorem 1 there exists at least one solution of system (27); that is, there exists at least one solution of the stated Hilbert problem. We could give many other applications of the functional system (1) to the investigation of various continuous and discontinuous boundary-value problems for analytic functions and for partial differential equations of elliptic and parabolic type. However, the application to the investigation of boundary-value problems for partial differential equations of parabolic type requires some modi-

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*) All symbols used here are explained in [6].

fications as well as more detailed analysis and will be postponed to another paper.

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