DEMONSTRATIO MATHEMATICA
Vol. V No 4 1973

Jan Luchter

THE DETERMINATION OF NON-HOMOGENEOUS LINEAR GEOMETRIC OBJECTS OF THE FIRST CLASS FOR WHICH THE NUMBER OF COORDINATES IS NOT GREATER THAN THE DIMENSION OF THE SPACE

In this article we shall determine non-homogeneous linear geometric objects of the type $[m,n,1]$ where $m \leq n$, with the following transformation rule:

$$(*) \qquad \omega' = F(A)\omega + g(A),$$

or

$$(**) \qquad \omega' = F(J)\omega + g(J),$$

where $J = \operatorname{Det} A$.

In the case $(*)$ we assume no regularity condition concerning the matrix functions $F(X)$ and $g(X)$ (Theorem I), but in the case $(**)$ we assume that $F(J)$ is a measurable function and $g(J)$ is a continuous function. (Theorem II). In the proof of Theorem I we apply the results of M. Kucharzewski and A. Zajtz [2] concerning the determination of homogeneous linear geometric objects of the type $[m,n,1]$ where $m \leq n$. In the proof of Theorem II we use the results of M. Kuczma and A. Zajtz [3].

First let us recall some definitions.

An object ω is called a non-homogeneous linear geometric object of the first class if with a transformation of the coordinate system

$$\xi^{\lambda'} = \varphi^{\lambda'}(\xi^{\lambda}) \qquad (\lambda, \lambda' = 1,2,\dots,n)$$

the coordinates of the objects transform according to the rule

(1) $$\omega^{I'} = F^{I'}_{I}(A^{\lambda'}_{\lambda})\,\omega^{I} + g^{I'}(A^{\lambda'}_{\lambda}) \qquad (I, I' = 1,2,\ldots,m),$$

where $\omega(\omega^1,\ldots,\omega^m)$ and $\omega(\omega^{1'},\ldots,\omega^{m'})$ are the coordinates of the object in the systems (ξ^{λ}) and $(\xi^{\lambda'})$, respectively. Moreover, $F^{I'}_{I}$ and $A^{\lambda'}_{\lambda}$ are matrices of dimension $m \times m$ and $n \times n$, respectively, where the latter matrix satisfies the condition

$$A^{\lambda'}_{\lambda} = \frac{\partial \xi^{\lambda'}}{\partial \xi^{\lambda}} \quad \text{and Det}\left[A^{\lambda'}_{\lambda}\right] \neq 0 .$$

The object ω is called an object of the type J if the matrix functions F and g are functions of the determinant of the matrix $\left[A^{\lambda'}_{\lambda}\right]$ only, i.e. if.

(2) $$\omega^{I'} = F^{I'}_{I}(J)\,\omega^{I} + g^{I'}(J),$$

where $J = \text{Det}\left[A^{\lambda}_{\lambda'}\right]$.

To simplify the notation we assume the following matrix notation:

$$\omega = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^m \end{bmatrix}, \quad A = \left[A^{\lambda'}_{\lambda}\right], \quad F(A) = F^{I'}_{I}(A), \quad g(A) = \begin{bmatrix} g^1(A) \\ \vdots \\ g^m(A) \end{bmatrix}.$$

With this notation, formulas (1) and (2) take the form

(3) $$\omega' = F(A)\omega + g(A)$$

(4) $$\omega' = F(J)\omega + g(J).$$

In order to determine all non-homogeneous linear geometric objects of the first class we must solve the following system of matrix functional equations:

(5) $$F(BA) = F(B)F(A),$$

(6) $$g(BA) = F(B)g(A) + g(B);$$

or, in the special case of J-objects, the system

(7) $$F(J_2J_1) = F(J_2)F(J_1),$$

(8) $$g(J_2J_1) = F(J_2)g(J_1)+g(J_2)\ ^{*)}.$$

Now we give a general solution (without any regularity assumption about the functions $F(X)$ and $g(X)$) to the system of equations (5) and (6), where F, A and B are nonsingular matrices of dimension $m \times m$, $n \times b$ and $m \times 1$, respectively, where $2 < m \leqslant n$ **).

The equation (5) does not involve the function g, hence it can be considered independently of the equation (6). The general solution of equation (5) has been given by M. Kucharzewski and A. Zajtz in [2]. Namely, in order that the function $F(X)$ satisfy equation (5) for non-singular matrices A and B, it must have one of the following forms:

(9) $$F(X) = \phi(J)C\,X\,C^{-1} \qquad (m=n)$$

(10) $$F(X) = \phi(J)C(X^T)^{-1}C^{-1} \qquad (m=n)$$

(11) $$F(X) = G(J), \quad J = \mathrm{Det}\,X \qquad (m<n).$$

In these formulas C is an arbitrary non-singular constant matrix, $\phi(J)$ - an arbitrary scalar matrix satisfying the functional equation $\phi(J_2J_1) = \phi(J_2)\,\phi(J_1)$, and $G(J)$ - an arbitrary matrix of a real variable satisfying the functional equation $G(J_2J_1) = G(J_2)G(J_1)$. Since in (9) and (10) $\phi(J)$ is a scalar matrix function, it has the form $\varphi(J)E$ where $\varphi(J)$ is a scalar multiplicative function satisfying the equation $\varphi(J_2J_1) = \varphi(J_2)\varphi(J_1)$.

Replacing multiplication by the scalar matrix $\phi(J)$ with multiplication by the scalar function $\varphi(J)$, we can write the solutions (9) - (11) in the following form:

---

*) This system has been solved in [4].

**) In the case m=n=2, the system (5) and (6) has been solved in [1].

$$F(X) = \varphi(J)\ C\ X\ C^{-1} \tag{12}$$

$$F(X) = \varphi(J)\ C(X^T)^{-1}\ C^{-1} \tag{13}$$

$$F(X) = G(J). \tag{14}$$

We now interpret equation (6) as follows: given a matrix function of one of the forms (12), (13), (14), we seek a vector function g, about which we make no regularity assumption.

In the course of solving this problem we shall use the following lemmas proved in [1] and [5].

L e m m a 1. If the solution of equation (5) is of the form $\tilde{F} = CFC^{-1}$, then the solution g corresponding to it has the form $\tilde{g} = CE$ (see [1]).

L e m m a 2. If there exists a scalar matrix λE $(\lambda \neq 1)$ such that the matrix $F(\lambda E) - E$ is non-singular, then the general solution of equation (6) has the form

$$g(A) = \left[F(A) - E\right] v\ , \tag{15}$$

where v is an arbitrary constant vector. (see [1]).

L e m m a 3. If the matrix

$$F\left(\begin{bmatrix} q & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right) - E$$

is non-singular, then the general solution of equation (6) has the form (15) (see [5]).

In virtue of Lemma 1 it suffices to consider equation (6) assuming that (12), (13) and (14) have the form

(16) Ia. $F(A) = \varphi(J)A$

(17) Ib. $F(A) = \varphi(J)(A^T)^{-1}$

(18) II. $F(A) = G(J)$,

where $\varphi(J)$ and $G(J)$ are the same as in (12), (13) and (14).

We consider three cases depending on the form of the function $F(X)$ in (16) - (18).

Ia. The function $F(X)$ is of the form $F(A) = \varphi(J)$. We shall show that if $(J) \neq J^{-\frac{1}{n}}$, then the assumption of Lemma 2 holds, so that the general solution of equation (6) has the form (15), i.e.

$$g(A) = \left[F(A)-E\right] v.$$

Indeed, we can confirm this as follows: we examine for what function F there is no scalar matrix λE $(\lambda \neq 1)$ such that the matrix $F(\lambda E)-E$ is non-singular, Accordingly, substitute λE for A, then the matrix $F(\lambda E)-E = \varphi(\lambda^n)\lambda E - E = \left[\varphi(\lambda^n)\lambda-1\right] E$ is singular if and only if $\varphi(\lambda^n)\lambda-1=0$, that is if

$$(*) \qquad \varphi(\lambda^n) = \frac{1}{\lambda}.$$

Let $\lambda > 0$ and put $\lambda^n = x$, i.e. $\lambda = x$. Substituting this into $(*)$ we get $\varphi(x) = x^{-\frac{1}{n}}$. Hence only for $\varphi(J) = J^{-\frac{1}{n}}$ $(J > 0)$ there is no scalar matrix λE such that $F(\lambda E) - E$ is non--singular. This implies that if the function $F(X)$ is of the form (16) and $\varphi(J) \neq J^{-\frac{1}{n}}$ then the assumptions of Lemma 2 hold and the general solution of equation (6) has the form (15).

Ib. The function $F(X)$ is of the form (17), i.e. $F(A) = \varphi(J)(A^T)^{-1}$ By a reasoning similar to that in case Ia, we conclude that only for $\varphi(J) = J$ there is no scalar matrix λE such that the matrix $F(\lambda E)-E$ is non-singular. Consequently, if $\varphi(J) \neq J^{-\frac{1}{n}}$, then the general solution of equation (6) has the form (15).

Hence it remains to solve equation (6) in case Ia and Ib, when we have, respectively:

$$\varphi(J) = J^{-\frac{1}{n}} \quad \text{or} \quad \varphi(J) = J^{\frac{1}{n}}.$$

T h e o r e m I. All non-homogeneous linear geometric objects of the type $[m,n,1]$, where $m=n$ have the form (3), where F is defined by formula (12) or (13), and g is of the form (15).

II. The function $F(X)$ has the form (18), i.e.

$$F(A) = G(J).$$

Although the solution (18) applies to the case $m=n$, this assumption will play no role in the sequel and the results to be given now concern arbitrary non-homogeneous linear J-objects of the first class without any limitation on the number of coordinates.

To determine all objects of the type J we must solve the system of equations (7) and (8).
The general solution for the multiplicative equation (7), under the assumption that $F(J)$ is measurable, has been given by M. Kuczma and A. Zajtz in paper [3].
Following [3], let us introduce the notation

$$M = \begin{bmatrix} |J|^a & \frac{1}{1!}|J|^a \ln|J| & \frac{1}{2!}|J|^a \ln^2|J| & \dots & \frac{1}{(p-1)!}|J|^a (\ln|J|)^{p-1} \\ & |J|^a & |J|^a \ln|J| & \dots & \frac{1}{(p-2)!}|J|^a (\ln|J|)^{p-2} \\ & & & & \dots\dots\dots \\ & & & & \dots\dots \\ & & & & \dots \\ & & & & |J|^a \end{bmatrix}$$

$$N = \begin{bmatrix} A & A(b \ln|J|) & \frac{1}{2!}A(b \ln|J|)^2 & \dots & \frac{1}{(s-1)!} A(b \ln|J|)^{s-1} \\ & & A(b \ln|J|) & \dots & \frac{1}{(s-2)!} A(b \ln|J|)^{s-2} \\ & & & & \dots\dots\dots \\ & & & & \dots\dots \\ & & & & \dots \\ & & & & A \end{bmatrix}$$

where A denotes a matrix of the form

$$A = \begin{bmatrix} |J|^a \cos(b \ln|J|) & |J|^a \sin(b \ln|J|) \\ |J|^a \sin(b \ln|J|) & |J|^a \cos(b \ln|J|) \end{bmatrix}$$

and p,s are natural numbers equal to the dimensions of the square matrices M and N, respectively, and a,b are real parameters.

With this notation, the function $F(J)$ being the solution of equation (7) can be written as follows:

$$(24) \qquad F(J) = C^{-1}\,\widetilde{F(J)}\,C, \quad \text{where} \quad \widetilde{F(J)} = \begin{bmatrix} B_1(J) & & \\ & B_2(J) & \\ & & \ddots \end{bmatrix}.$$

In the formula above C is a constant non-singular matrix and $F(J)$ is a quasi-diagonal matrix in which each block $B_j(J)$ on the diagonal has the form M,N, or $(\operatorname{sgn} J)\, M$, $(\operatorname{sgn} J)\, N$ (compare [3]). According to Lemma 1 if $C=E$ it suffices to determine the solution of equation (8) for the function $F(J)$ only.

Now we prove the following lemmas.

L e m m a 4. If the matrix $F(J)$ is quasi-diagonal (it splits into at least two cells on the diagonal), then each solution $g(J)$ has the form

$$g(J) = \begin{bmatrix} g^{(p_1)}(J) \\ \vdots \\ g^{(p_n)}(J) \end{bmatrix},$$

where $g^{(p_j)}(J)$ is a vector function with the number of coordinates equal to the order of the block $B_j(J)$ corresponding to it, and $g^{(p_j)}(J)$ satisfies equation (8) with $F(J)$ replaced by $B_j(J)$.

P r o o f. It suffices to prove the lemma when the matrix F consists of two cells, since by induction it is easy to extend the proof for an arbitrary number of cells. Accordingly, assume that

$$F(J) = \begin{bmatrix} B_{p_1}(J) & 0 \\ 0 & B_{p_2}(J) \end{bmatrix}$$

and

$$g(J) = \begin{bmatrix} g^{(p_1)}(J) \\ g^{(p_2)}(J) \end{bmatrix}.$$

Introducing the matrices F and g into (8) we obtain

$$\begin{bmatrix} g^{(p_1)}(J_2J_1) \\ g^{(p_2)}(J_2J_1) \end{bmatrix} = \begin{bmatrix} B_{p_1}(J_2) & 0 \\ 0 & B_{p_2}(J_2) \end{bmatrix} \begin{bmatrix} g^{(p_1)}(J_1) \\ g^{(p_2)}(J_1) \end{bmatrix} + \begin{bmatrix} g^{(p_1)}(J_2) \\ g^{(p_2)}(J_2) \end{bmatrix} =$$

$$= \begin{bmatrix} B_{p_1}(J_2)\, g^{(p_1)}(J_1) \\ B_{p_2}(J_2)\, g^{(p_2)}(J_1) \end{bmatrix} + \begin{bmatrix} g^{(p_1)}(J_2) \\ g^{(p_2)}(J_2) \end{bmatrix}.$$

This implies

$$g^{(p_1)}(J_2J_1) = B_{p_1}(J_2)\, g^{(p_1)}(J_1) + g^{(p_1)}(J_2)$$

and

$$g^{(p_2)}(J_2J_1) = B_{p_2}(J_2)g^{(p_2)}(J_1) + g^{(p_2)}(J_2).$$

L e m m a 5. The set of all solutions of equation (8) is closed with respect to the operation of addition and multiplication by a number (that is,any linear combination, with constant coefficients, of solutions is again a solution).

P r o o f. Let $g_1(J)$, and $g_2(J)$ be solutions of equation (8), and let λ_1, λ_2 by any constant numbers. By direct verification we show that the function $h(J) = \lambda_1 g_1(J) + \lambda_2 g_2(J)$ is also a solution of equation (8). In fact,

$$\begin{aligned} h(J_2J_1) &= \lambda_1 g_1(J_2J_1)+\lambda_2 g_2(J_2J_1) = \\ &= \lambda_1\left[F(J_1)g_1(J_2)+g_1(J_1)+\lambda_2\left[F(J_1)g_2(J_2)+g_2(J_1)\right] = \right. \\ &= F(J_1)\left[\lambda_1 g_1(J_2+\lambda_2 g_2(J_2)\right] + \left[\lambda_1 g_1(J_1)+\lambda_2 g_2(J_1)\right] = \\ &= F(J_1)h(J_2)+ h(J_1). \end{aligned}$$

This author in [4] has proved a theorem on the solution of equation (8).

T h e o r e m 2. Assume that in equation (8) the matrix function $F(J)$ is measurable and the sought vector function $g(J)$ is continuous. The general solution of equation (8), in which the function $F(J)$ has a canonical quasi-diagonal form, is a vector function $g(J)$ of the form

$$g(J) = \begin{bmatrix} g^{(p_1)}(J) \\ \vdots \\ g^{(p_n)}(J) \end{bmatrix},$$

where $g^{(p_j)}(J)$ is a vector function with the number of coordinates p_j equal to the dimension of the respective cell $B_j(J)$, satisfying equation (8) with $F(J)$ replaced by $B_j(J)$.