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ON CURVATURE TENSORS IN A SPECIAL NORMAL (f, g, u, v, λ)-STRUCTURE MANIFOLD

Many mathematicians including Blair, Ludden, Yano, Okumura, and Goldberg have studied submanifolds of codimension 2 of almost complex manifolds and hypersurfaces of almost contact manifolds. These manifolds admit under certain conditions, what we call an (f,g,u,v,λ)-structure. In 1970 Yano and Okumura defined and studied normal (f,g,u,v,λ)-structure, which is a particular case of an (f,g,u,v,λ)-structure.

In the present paper we define a special normal (f,g,u,v,λ)-structure and study different type of curvature tensors, viz curvature tensor, Weyl (projective) curvature tensor and conformal curvature tensor in a special normal (f,g,u,v,λ)-structure manifold.

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1. Introduction

Let us consider an n (=2m) dimensional differentiable manifold M of class C^∞ which admits an (f,U,V,u,v,λ)-structure (see [6]), that is, there exist on M a tensor field f of type (1.1), vector fields U and V, 1-forms u and v, and a function λ satisfying the following conditions:

$$(1.1)\qquad u(U) = 1 - \lambda^2, \qquad u(V) = 0 ,$$

$$(1.2)\qquad v(U) = 0 , \qquad v(V) = 1 - \lambda^2 ,$$

(1.3) $$\bar{\bar{X}} + X = u(X)U + v(X)V ,$$

(1.4) $$\bar{U} = -\lambda V, \qquad \bar{V} = \lambda U ,$$

and

(1.5a) $$u\cdot f = \lambda v , \qquad v\cdot f = -\lambda u ,$$

where $\bar{X} = fX$ and 1-forms $u\cdot f$ and $v\cdot f$ are respectively defined by

$$(u\cdot f)(X) = u(fX) = u\bar{X} ,$$

and

$$(v\cdot f)(X) = v(fX) = v\bar{X} ,$$

for any vector field X, hence the last condition (1.5a) may be written as

(1.5b) $$u\bar{X} = \lambda v(X) , \qquad v\bar{X} = -\lambda u(X) .$$

Next, we assume that in M with an (f,U,V,u,v,λ)-structure, there exists a positive definite Riemannian metric g such that

(1.6a) $$g(U,X) = u(X) ,$$

(1.6b) $$g(V,X) = v(X) ,$$

and

(1.7) $$g(\bar{X},\bar{Y}) = g(X,Y) - u(X)u(Y) - v(X)v(Y) ,$$

for any vector fields X and Y of M. Such structure is called an (f,g,u,v,λ)-structure and an (f,g,u,v,λ)-structure is said to be normal if the Nijenhuis tensor N of f satisfies (see [6]):

$$S(X,Y) = N(X,Y) + du(X,Y)U + dv(X,Y)V = 0 ,$$

where du and dv are exterior derivatives of 1-forms u and v respectively.

Let F be a tensor field of type (0,2) of M defined by

(1.8) $$F(X,Y) = g(\bar{X},Y) ,$$

for any vector fields X and Y of M, then we have

(1.9) $$F(X,Y) = - F(Y,X) ,$$

that is, F is a 2-form [6].

D e f i n i t i o n. A normal (f,g,u,v,λ)-structure is said to be a <u>special normal</u> (f,g,u,v,λ)- <u>structure</u>, if the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and the following two conditions are satisfied:

(1.10) $$2F(X,Y) = (D_Xv)(Y) - (D_Yv)(X) ,$$

and

(1.11) $$(D_Xu)(Y) - (D_Yu)(X) = 0 ,$$

D_X being the Riemannian connection on the manifold M.

As a consequence of the equation (1.10) we can obtain

(1.12) $$(D_XF)(Y,Z) + (D_YF)(Z,X) + (D_ZF)(X,Y) = 0 .$$

Let M be a special normal (f,g,u,v,λ)-structure manifold, then we can verify the following identities [6]

(1.13) $$D_X\lambda = u(X) ,$$

(1.14) $$(D_Xu)(Y) = - \lambda g(X,Y) ,$$

i.e., $$\operatorname{div} U = - \lambda n ,$$

(1.15) $$(D_Xv)(Y) + (D_Yv)(X) = 0 ,$$

i.e., V is a Killing vector field,

(1.16) $$(D_ZF)(X,Y) = g(Z,Y)v(X) - g(Z,X)v(Y) ,$$

and

$$(1.17) \qquad F(X,Y) = (D_X v)(Y) = -(D_Y v)(X).$$

2. Curvature tensors in a special normal (f,g,u,v,λ)-structure manifold

The curvature tensor K of a special normal (f,g,u,v,λ)-manifold is a tensor field of type (1,3) defined by (see [7])

$$(2.1) \qquad K(X,Y,Z) \equiv D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where $[X,Y] \equiv D_X Y - D_Y X$. The following Ricci identities are also well known (see [7])

$$(2.2) \quad (D_X D_Y w)(Z) - (D_Y D_X w)(Z) - (D_{[X,Y]} w)(Z) = -w(K(X,Y,Z)),$$

for a 1-form w and

$$(2.3) \quad (D_X D_Y S)(Z,T) - (D_Y D_X S)(Z,T) - (D_{[X,Y]} S)(Z,T) =$$

$$= K(X,Y,S(Z,T)) - S(K(X,Y,Z),T) - S(Z,K(X,Y,T)),$$

for a tensor field S of type (1,2).

The tensor field $'K$ of type (0,4) defined by

$$(2.4) \qquad 'K(X,Y,Z,T) \equiv g(K(X,Y,Z),T)$$

is called the covariant curvature tensor. Now we will prove the following theorems

Theorem 2.1. In a special normal (f,g,u,v,λ)-structure manifold, we have

$$(2.5a) \quad (D_Z F)(X,Y) = 'K(X,Y,Z,V) = g(Z,Y)v(X) - g(Z,X)v(Y),$$

and consequently

$$(2.5b) \qquad (D_V F)(X,Y) = 0.$$

P r o o f. From the equation (1.17), we have

$$(D_XF)(Y,Z) = (D_XD_Yv)(Z) + (D_Yv)(D_XZ) .$$

Consequently

$$(D_XF)(Y,Z) - (D_YF)(X,Z) =$$

$$= (D_XD_Yv)(Z) - (D_YD_Xv)(Z) - (D_{[X,Y]}v)(Z) .$$

Solving the above equation with the help of the equations (1.6b), (1.12), (1.16), (2.2) and (2.4), we get

$$(D_ZF)(X,Y) = v(K(X,Y,Z)) = g(K(X,Y,Z),V) = {}'K(X,Y,Z,V) =$$

$$= g(Z,Y)v(X) - g(Z,X)v(Y) ,$$

which completes the proof of (2.5a) and (2.5b).

T h e o r e m 2.2. In a special normal (f,g,u,v,λ)--structure manifold, we have

$$(2.6) \qquad {}'K(X,Y,Z,U) = g(Z,Y)u(X) - g(Z,X)u(Y) .$$

P r o o f. From the equations (1.13) and (1.14), we have

$$(D_XD_Yu)(Z) + (D_Yu)(D_XZ) = - u(X)g(Y,Z) .$$

Interchanging X and Y and subtracting, we get

$$u(K(X,Y,Z)) = u(X)g(Y,Z) - u(Y)g(X,Z) .$$

Using (1.6a) and (2.4), we obtain (2.6).

T h e o r e m 2.3. In a special normal (f,g,u,v,λ)--structure manifold, the following relations hold

$$(2.7) \qquad v(D_Xf)(Y) = v(X)v(Y) - (1 - \lambda^2)g(X,Y) ,$$

$$(2.8) \qquad u(D_Xf)(Y) = u(X)v(Y) ,$$

$$(2.9) \qquad (D_ZF)(X,Y) = g((D_Zf)(X),Y) .$$

P r o o f o f (2.7). From the second part of the equation (1.5b), we get

$$v(\bar{Y}) = -\lambda u(Y), \qquad \text{i.e.,} \qquad v(fY) = -\lambda u(Y).$$

Taking covariant derivative of the above equation in the direction of X and using (1.7), (1.8), (1.13) and (1.14), we obtain

$$v(D_X\bar{Y}) = -(D_Xv)(\bar{Y}) - (D_X\lambda)u(Y) - \lambda(D_Xu)(Y) - \lambda u(D_XY)$$

$$= -F(X,\bar{Y}) - u(X)u(Y) - \lambda(D_Xu)(Y) - \lambda u(D_XY)$$

$$= -g(\bar{X},\bar{Y}) - u(X)u(Y) + \lambda^2 g(X,Y) - \lambda u(D_XY)$$

$$= -(1-\lambda^2)g(X,Y) + v(X)v(Y) - \lambda u(D_XY),$$

while on the other hand,

$$v(D_X\bar{Y}) = v\left[(D_Xf)(Y) + \overline{D_XY}\right] = v(D_Xf)(Y) - \lambda u(D_XY),$$

which completes the proof of (2.7).

P r o o f o f (2.8). Since,

$$u(\bar{Y}) = \lambda v(Y), \qquad \text{i.e.,} \qquad u(fY) = \lambda v(Y),$$

therefore as a consequence of (1.5b), (1.8), (1.13), (1.14) and (1.17), we obtain

$$u(D_X\bar{Y}) = -(D_Xu)(\bar{Y}) + u(X)v(Y) + \lambda F(X,Y) + \lambda v(D_XY)$$

$$= \lambda g(X,\bar{Y}) + u(X)v(Y) - \lambda g(X,\bar{Y}) + \lambda v(D_XY)$$

$$= u(X)v(Y) + \lambda v(D_XY),$$

while,

$$u(D_X\bar{Y}) = u\left[(D_Xf)(Y) + \overline{D_XY}\right] =$$

$$= u(D_Xf)(Y) + \lambda v(D_XY)$$

and (2.8) is proved. The relation (2.9) follows directly from (1.8).

In terms of the n local orthonormal vector fields $X_1, X_2, \ldots, X_n$ we can define a global tensor field R of type (0,2)

$$R(Y,Z) = \sum_{i=1}^{n} {}'K(X_i,Y,Z,X_i)$$

and a global scalar field

$$R' = \sum_{i=1}^{n} R(X_i,X_i) .$$

The tensor field R and the scalar R' are called the Ricci tensor and the scalar curvature respectively.

We will now establish the following

T h e o r e m 2.4. In a special normal (f,g,u,v,λ) - -structure manifold, we have

$$(2.10) \qquad {}'K(V,Y,Z,V) = (1-\lambda^2)g(Z,Y) - v(Y)v(Z) ,$$

$$(2.11) \qquad {}'K(V,Z,Y,X) = v(X)g(Z,Y) - v(Y)g(Z,X) ,$$

$$(2.12) \qquad K(V,Z,Y) = g(Z,Y)V - v(Y)Z ,$$

$$(2.13) \qquad K(X,Y,V) = v(Y)X - v(X)Y ,$$

$$(2.14) \qquad R(Y,V) = (n-1)v(Y) ,$$

$$(2.15) \qquad r(V) = (n-1)V ,$$

where

$$g(r(Y),Z) \equiv R(Y,Z) .$$

P r o o f. Proof of (2.10) is trivial. Using symmetric and skew-symmetric properties of the curvature tensor in the equation (2.5a), (2.11) can be obtained. Equation (2.11) may be written as

$$g(K(V,Z,Y,),X) = v(X)g(Z,Y) - v(Y)g(Z,X) .$$

Contraction of the last equation by X gives (2.12).

From the equation (2.5a), we have

$$'K(X,Y,V,Z) = v(Y)g(Z,X) - v(X)g(Z,Y)$$

or

$$g(K(X,Y,V),Z) = v(Y)g(Z,X) - v(X)g(Z,Y) .$$

Contracting the above equation by Z, we obtain (2.13).

Equations (2.14) and (2.15) are direct consequence of the equations (2.13) and (2.14) respectively.

Similarly we can prove

T h e o r e m 2.5. In a special normal (f,g,u,v,λ)-structure manifold

$$'K(U,Y,Z,U) = (1 - \lambda^2)g(Z,Y) - u(Y)u(Z) ,$$

$$K(U,Z,Y) = g(Z,Y)U - u(Y)Z ,$$

$$K(X,Y,U) = u(Y)X - u(X)Y ,$$

$$R(Y,U) = (n-1)u(Y) ,$$

and

$$r(U) = (n-1)U .$$

T h e o r e m 2.6. A special normal (f,g,u,v,λ)-structure manifold can not be flat.

P r o o f. Let us assume that a special normal (f,g,u,v,λ)-structure manifold M is flat, that is,

$$'K(X,Y,Z,S) = 0 ,$$

which implies

$$'K(V,Y,Z,V) = {}'K(U,Y,Z,U) = 0 ,$$

i.e.,

$$(1 - \lambda^2)g(Z,Y) = u(Z)u(Y) = v(Z)v(Y) ,$$

which is not possible.

T h e o r e m 2.7. If a special normal (f,g,u,v,λ)-structure manifold is symmetric, then it is an Einstein manifold of constant scalar curvature.

P r o o f. Let manifold be symmetric, that is,

$$(D_Y K)(Z,S,T) = 0 .$$

Consequently,

$$(D_X D_Y K)(Z,S,T) - (D_Y D_X K)(Z,S,T) - (D_{[X,Y]} K)(Z,S,T) = 0 .$$

Using the Ricci identity, we get

$$K(X,Y,K(Z,S,T)) - K(K(X,Y,Z),S,T) -$$
$$- K(Z,K(X,Y,S),T) - K(Z,S,K(X,Y,T)) = 0 .$$

Now putting V for both X and Z in the last equation and applying Theorem 2.4, we have

$$K(V,Y,K(V,S,T)) - K(K(V,Y,V),S,T) -$$
$$- K(V,K(V,Y,S),T) - K(V,S,K(V,Y,T)) = 0$$

or

$$g(Y,K(V,S,T))V - v(K(V,S,T))Y - v(Y)K(V,S,T) +$$
$$+ (1 - \lambda^2)K(Y,S,T) - g(K(V,Y,S),T)V +$$
$$+ v(T)K(V,Y,S) - g(S,K(V,Y,T))V + v(K(V,Y,T))S = 0$$

or

$$g(Y,g(S,T)V)V - g(Y,v(T)S)V - (1 - \lambda^2)g(S,T)Y + v(S)v(T)Y -$$
$$- v(Y)g(S,T)V + v(Y)v(T)S + (1 - \lambda^2)K(Y,S,T) - g(g(Y,S)V,T V +$$
$$+ g(v(S)Y,T)V + v(T)g(Y,S)V - v(T)v(S)Y - g(S,g(Y,T)V)V +$$
$$+ g(S,v(T)Y)V + (1 - \lambda^2)g(Y,T)S - v(Y)v(T)S = 0$$

or

$$(1-\lambda^2)\left[K(Y,S,T)+g(Y,T)S-g(S,T)Y\right]=0\ .$$

Since $\lambda(1-\lambda^2)$ is almost everywhere non-zero,

$$K(Y,S,T)=g(S,T)Y-g(Y,T)S\ .$$

Contracting the above equation by Y, we get

$$R(S,T)=(n-1)g(S,T)\ .$$

Transvecting with respect to S and T, we obtain

$$R'=n(n-1)\,.$$

Hence the theorem is proved.

T h e o r e m 2.8. For a recurrent special normal (f,g,u,v,λ)-structure manifold

$$(D_V b)T=(D_T b)V\ ,$$

where b is the parameter of recurrence.

P r o o f. For a recurrent manifold,

$$(D_Y K)(Z,S,T)=b(Y)K(Z,S,T)\,,$$

that is,

$$(D_X D_Y K)(Z,S,T)-(D_Y D_X K)(Z,S,T)-(D_{[X,Y]}K)(Z,S,T)=$$
$$=\big((D_X b)Y\big)K(Z,S,T)-\big((D_Y b)X\big)K(Z,S,T)\,.$$

Again using the Ricci identity in the last equation and putting V for both X and Z, we get

$$(1-\lambda^2)\left[K(Y,S,T)-g(S,T)Y+g(Y,T)S\right]=$$

$$=\left\{(D_V b)Y-(D_Y b)V\right\}\left\{g(S,T)V-v(T)S\right\}\,.$$

Taking the skew-symmetric part of this equation, we have

$$(2.16) \qquad 2(1-\lambda^2)\left[K(Y,S,T) - g(S,T)Y + g(Y,T)S\right] =$$
$$= \left\{(D_V b)Y - (D_Y b)V\right\}\left\{g(S,T)V - v(T)S\right\} -$$
$$- \left\{(D_V b)S - (D_S b)V\right\}\left\{g(S,T)V - v(T)Y\right\} .$$

Contraction of this equation yields

$$2(1-\lambda^2)\left[R(S,T) - (n-1)g(S,T)\right] =$$
$$= (n-2)v(T)\left\{(D_V b)S - (D_S b)V\right\} .$$

Taking skew-symmetric part of the above equation, we get

$$4(1-\lambda^2)\left[R(S,T) - (n-1)g(S,T)\right] =$$
$$= (n-2)v(T)\left\{(D_V b)S - (D_S b)V\right\} + (n-2)v(S)\left\{(D_V b)T - (D_T b)V\right\} .$$

Putting V for S and applying Theorem 2.4, we get

$$(n-2)(1-\lambda^2)\left\{(D_V b)T - (D_T b)V\right\} = 0 .$$

Since $n > 2$ and $\lambda(1-\lambda^2)$ is almost everywhere non-zero,

$$(D_V b)T = (D_T b)V .$$

Hence proved.

T h e o r e m 2.9. A recurrent special normal (f,g,u,v,λ)-structure manifold is a manifold of constant Riemannian curvature.

P r o o f. Using Theorem 2.8 in the equation (2.16), we get

$$K(Y,S,T) = g(S,T)Y - g(Y,T)S .$$

Contraction of this equation by Y yields

$$R(S,T) = (n-1)g(S,T) ,$$

i.e.,

$$R' = n(n-1) ,$$

which completes the proof of the theorem.

3. Weyl (projective) curvature tensor in a special normal (f,g,u,v,λ)-structure manifold

We will now consider Weyl (projective) curvature tensor, which is a tensor of type (1,3) given by

$$W(X,Y,Z) \equiv K(X,Y,Z) - \frac{1}{n-1}\Big[R(Y,Z)X - R(X,Z)Y\Big] .$$

We put $'W(X,Y,Z,S) \equiv g(W(X,Y,Z),S)$. Then we have

$$'W(X,Y,Z,S) = 'K(X,Y,Z,S) - \frac{1}{n-1}\Big[R(Y,Z)g(X,S) - R(X,Z)g(Y,S)\Big].$$

Theorem 3.1. A special normal (f,g,u,v,λ)-structure manifold satisfies the following relations

$$W(V,Y,Z) = \Big[g(Y,Z) - \frac{1}{n-1} R(Y,Z)\Big]V ,$$

$$W(X,Y,V) = 0 ,$$

$$'W(X,Y,Z,V) = v(X)\Big[g(Y,Z) - \frac{1}{n-1} R(Y,Z\Big] -$$
$$- v(Y)\Big[g(X,Z) - \frac{1}{n-1} R(X,Z)\Big] ,$$

$$'W(V,Y,Z,S) = v(S)\Big[g(Y,Z) - \frac{1}{n-1} R(Y,Z)\Big] ,$$

$$'W(V,Y,Z,V) = (1 - \lambda^2)\Big[g(Y,Z) - \frac{1}{n-1} R(Y,Z)\Big] ,$$

$$'W(X,Y,V,S) = 0 .$$

The proof follows trivially from the definition of the Weyl curvature tensor.

Similarly we can obtain

T h e o r e m 3.2. In a special normal (f,g,u,v,λ)--structure manifold, the following relations hold

$$W(U,Y,Z) = \left[g(Y,Z) - \frac{1}{n-1} R(Y,Z) \right] U ,$$

$$W(X,Y,U) = 0 ,$$

$$'W(X,Y,Z,U) = u(X) \left[g(Y,Z) - \frac{1}{n-1} R(Y,Z) \right] - $$

$$- u(Y) \left[g(X,Z) - \frac{1}{n-1} R(X,Z) \right] ,$$

$$'W(U,Y,Z,S) = u(S) \left[g(Y,Z) - \frac{1}{n-1} R(Y,Z) \right] ,$$

$$'W(U,Y,Z,U) = (1 - \lambda^2) \left[g(Y,Z) - \frac{1}{n-1} R(Y,Z) \right] ,$$

$$'W(X,Y,U,S) = 0 .$$

T h e o r e m 3.3. A projectively flat special normal $(f,g,u,v\lambda)$-structure manifold is an Einstein manifold and is of constant Riemannian curvature.

P r o o f. Suppose that a special normal (f,g,u,v,λ)--structure manifold is projectively flat, that is,

$$'W(X,Y,Z,S) = 0,$$

i.e.,

$$'W(U,Y,Z,U) = {}'W(V,Y,Z,V) = 0 ,$$

i.e.,

$$(1 - \lambda^2) \left[g(Y,Z) - \frac{1}{n-1} R(Y,Z) \right] = 0 .$$

Since $\lambda(1 - \lambda^2)$ is almost everywhere non-zero,

$$R(Y,Z) = (n-1)g(Y,Z)$$

and

$$R' = n(n-1) .$$

Hence the theorem is proved.

Theorem 3.4. If a special normal (f,g,u,v,λ)-structure manifold is projectively symmetric then it is projectively flat.

Proof. Let us assume that the manifold with a special normal (f,g,u,v,λ)-structure is projectively symmetric, that is,

$$(D_Y W)(Z,S,T) = 0 ,$$

which yields

$$(D_X D_Y W)(Z,S,T) - (D_Y D_X W)(Z,S,T) - (D_{[X,Y]} W)(Z,S,T) = 0 .$$

Using the Ricci identity, we get

$$K(X,Y,W(Z,S,T)) - W(K(X,Y,Z),S,T) -$$

$$- W(Z,K(X,Y,S),T) - W(Z,S,K(X,Y,T)) = 0 .$$

Putting V for both X and Z and using Theorems 2.4 and 3.1, we obtain

$$K(V,Y,W(V,S,T)) - W(K(V,Y,V),S,T) -$$

$$- W(V,K(V,Y,S),T) - W(V,S,K(V,Y,T)) = 0 ,$$

or

$$g\Big(Y,(g(S,T) - \frac{1}{n-1} R(S,T))\, V\Big)V - (1 - \lambda^2)\Big(g(S,T) - \frac{1}{n-1} R(S,T)\Big)Y -$$

$$- v(Y)\Big(g(S,T) - \frac{1}{n-1} R(S,T)\Big)V + (1 - \lambda^2)W(Y,S,T) +$$

$$+ v(S)\Big(g(Y,T) - \frac{1}{n-1} R(Y,T)\Big)V + v(T)\Big(g(S,Y) - \frac{1}{n-1} R(S,Y)\Big)V = 0 ,$$

or

$$(3.1)\quad (1-\lambda^2)W(Y,S,T) - (1-\lambda^2)\Big(g(S,T) - \frac{1}{n-1}R(S,T)\Big)Y +$$

$$+ v(S)\Big(g(Y,T) - \frac{1}{n-1}R(Y,T)\Big)V + v(T)\Big(g(S,Y) - \frac{1}{n-1}R(S,Y)\Big)V = 0.$$

Contracting (3.1) by Y, we get .

$$n(1-\lambda^2)\Big[g(S,T) - \frac{1}{n-1}R(S,T)\Big] = 0 .$$

Since $n \neq 0$, and $\lambda(1-\lambda^2)$ is almost everywhere non-zero,

$$(3.2)\qquad R(S,T) = (n-1)g(S,T) .$$

By virtue of (3.2), the equation (3.1) reduces to

$$(1-\lambda^2)W(Y,S,T) = 0 ,$$

i.e.,

$$W(Y,S,T) = 0 .$$

Hence, proof of the theorem is completed.

Next let us assume that a special normal (f,g,u,v,λ)-structure manifold is projectively recurrent. In this case, working on the lines similar to that of Theorem 2.8, the integrability conditions of

$$(D_Y W)(Z,S,T) = b(Y)W(Z,S,T)$$

yield

$$W(Z,S,T) = 0 ,$$

which provides the proof of the following theorem.

Theorem 3.5. If a special normal (f,g,u,v,λ)-structure manifold is projectively recurrent, then its Weyl (projective) curvature tensor is zero.

4. Conformal curvature tensor in a special normal (f,g,u,v,λ)-structure manifold

In this section we will study conformal curvature tensor, which is a tensor of type (1,3) and is given by

$$C(X,Y,Z) \equiv K(X,Y,Z) - \frac{1}{n-2}\Big[R(Y,Z)X - R(X,Z)Y - g(X,Z)\dot{r}(Y) - g(Y,Z)r(X)\Big] + \frac{R'}{(n-1)(n-2)}\Big[g(Y,Z)X - g(X,Z)Y\Big].$$

Putting $'C(X,Y,Z,S) \equiv g(C(X,Y,Z),S)$, we get

$$'C(X,Y,Z,S) = {}'K(X,Y,Z,S) - \frac{1}{n-2}\Big[R(Y,Z)g(X,S) - R(X,Z)g(Y,S) - g(X,Z)R(Y,S) + g(Y,Z)R(X,S)\Big] + \frac{R'}{(n-1)(n-2)}\Big[g(Y,Z)g(X,S) - g(X,Z)g(Y,S)\Big].$$

T h e o r e m 4.1. In a special normal (f,g,u,v,λ)-structure manifold, we have

$$(n-2)\,'C(X,Y,Z,V) = \alpha\big(v(Y)g(X,Z) - v(X)g(Z,Y)\big) - v(X)R(Y,Z) + v(Y)R(X,Z),$$

$$(n-2)\,'C(V,Y,Z,V) = (\alpha+n-1)v(Y)v(Z) - \alpha(1-\lambda^2)g(Y,Z) - (1-\lambda^2)R(Y,Z),$$

$$(n-2)\,'C(V,Y,Z,S) = \alpha\big(v(Z)g(Y,S) - v(S)g(Y,Z)\big) - v(S)R(Y,Z) + v(Z)R(Y,S),$$

$$(n-2)C(X,Y,V) = \alpha\big(v(X)Y - v(Y)X\big) + v(X)r(Y) - v(Y)r(X),$$

$$(n-2)C(V,Y,Z) = \alpha\big(v(Z)Y - g(Y,Z)V\big) - R(Y,Z)V + v(Z)r(Y),$$

$$(n-2)C(V,Y,V) = (1-\alpha-n)v(Y)V + (1-\lambda^2)\left[\alpha Y + r(Y)\right] ,$$

where

$$\alpha \equiv (1 - \frac{R'}{n-1}) .$$

P r o o f. The proof follows trivially from the definition of the conformal curvature tensor.

T h e o r e m 4.2. If a special normal (f,g,u,v,λ)-structure manifold is conformally flat, then the following relations hold

$$R(Y,Z) = \beta v(Y)v(Z) - \alpha g(Y,Z) , \tag{4.1}$$

$$r(Y) = \beta v(Y)V - \alpha Y . \tag{4.2}$$

Consequently,

$$(n-2)K(X,Y,Z) = \beta v(Z)\left[v(Y)X - v(X)Y\right] - \tag{4.3}$$

$$-(\alpha+1)\left[g(Y,Z)X - g(X,Z)Y\right] + \beta\left[g(Y,Z)v(X) - g(X,Z)v(Y)\right] V ,$$

where

$$\beta \equiv \frac{\alpha+n-1}{1-\lambda^2} .$$

P r o o f o f (4.1). Let us assume that a special normal (f,g,u,v,λ)-structure manifold is conformally flat, that is,

$$'C(X,Y,Z,S) = 0 ,$$

and as a particular case

$$'C(V,Y,Z,V) = 0 ,$$

i.e.,

$$(1-\lambda^2)R(Y,Z) = (\alpha+n-1)v(Y)v(Z) - \alpha(1-\lambda^2)g(Y,Z) .$$

Putting $\beta = \frac{\alpha+n-1}{1-\lambda^2}$, we get

$$R(Y,Z) = \beta v(Y)v(Z) - \alpha g(Y,Z) .$$

P r o o f o f (4.2). The equation (4.1) may also be written as

$$g\big(r(Y),Z\big) = \beta v(Y)v(Z) - \alpha g(Y,Z) .$$

Contracting by Z, we get

$$r(Y) = \beta v(Y)V - \alpha Y .$$

Proof of (4.3) is trivial.

Similarly we can prove the following two theorems

T h e o r e m 4.3. A special normal (f,g,u,v,λ)-structure manifold satisfies the following relations:

$$(n-2)\,'C(X,Y,Z,U) = \alpha\big(u(Y)g(X,Z) - u(X)g(Z,Y)\big) - u(X)R(Y,Z) + R(X,Z)U(Y) ,$$

$$(n-2)'C(U,Y,Z,U) = (\alpha+n-1)u(Y)u(Z) - (1-\lambda^2)\left[\alpha g(Z,Y)+R(Y,Z)\right],$$

$$(n-2)\,'C(U,Y,Z,S) = \big(u(Z)g(Y,S) - u(S)g(Y,Z)\big) - u(S)R(Y,Z) - u(Z)R(Y,S) ,$$

$$(n-2)C(X,Y,U) = \alpha\big(u(X)Y - u(Y)X\big) + u(X)r(Y) - u(Y)r(X) ,$$

$$(n-2)C(U,Y,Z) = \alpha\big(u(Z)Y - g(Y,Z)U\big) - R(Y,Z)U + u(Z)r(Y),$$

$$(n-2)C(U,Y,U) = (1-\alpha-n)u(Y)U + (1-\lambda^2)\big(\alpha Y + r(Y)\big) .$$

T h e o r e m 4.4. If a special normal (f,g,u,v,λ)--structure manifold is conformally flat, then we have

$$R(Y,Z) = \beta u(Y)u(Z) - \alpha g(Y,Z) ,$$

$$r(Y) = \beta u(Y)U - \alpha Y$$

and consequently,

$$(n-2) K(X,Y,Z) = \beta u(Z) \left[u(Y)X - u(X)Y \right] -$$
$$-(\alpha+\beta) \left[g(Y,Z)X - g(X,Z)Y \right] + \beta \left[g(Y,Z)u(X) - g(X,Z)u(Y) \right] U .$$

It is well known that conformally symmetric and recurrent manifolds respectively satisfy the following conditions

$$(D_Y C)(Z,S,T) = 0 ,$$

and

$$(D_Y C)(Z,S,T) = b(Y)C(Z,S,T) .$$

T h e o r e m 4.5. If a special normal (f,g,u,v,λ)--structure manifold is conformally symmetric or recurrent then its conformal curvature tensor vanishes.

Proof is similar to that of Theorems 3.4 and 3.5.

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