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COEFFICIENT INEQUALITIES FOR SHAH'S FUNCTIONS

1. The role of the Grunsky inequalities for the coefficient problem in the theory of univalent functions is well known. These inequalities have been sharpened for some subclasses of univalent functions, for example for the case of bounded univalent functions by Nehari [6] and by Schiffer and Tammi [8] and for the case of Bieberbach-Eilenberg functions by Hummel and Schiffer [2]. The purpose of the present paper is to sharpen the Grunsky inequalities for the univalent functions introduced by Shah [9]. We will first arrive at the generalized area theorem for Shah's functions and, as a direct consequence, we will obtain a set of inequalities between a quadratic and Hermitean form which are typical for Grunsky estimates. Secondly we give some application for these inequalities to extremal problems in the family of Shah's functions.

2. We begin with the following definition

D e f i n i t i o n. The class K of all functions f which are regular and univalent in the unit disk $K(0,1)$, vanish at the origin, and have the property that $f(z_1)\,\overline{f(z_2)} \neq -1$ for all pairs of points z_1, z_2 in $K(0,1)$, is called the class of Shah's functions.

Let $f \in K$ and

$$f(z) = b_1 z + b_2 z^2 + \dots, \qquad |z| < 1. \tag{1}$$

If $f \in K$, and C_r and C_r' denote, respectively, the curve described by the points $w = f(re^{i\theta})$ and $w = -\left[f(re^{i\theta})\right]^{-1}$, $0 < r < 1$, if θ varies from 0 to 2π, then it is easy to see that C_r is contained in the interior of the finite domain bounded by C_r'. Hence, C_r and C_r' bound a doubly-connected domain D_r which does not contain the origin.

Now we consider the function

$$g(w) = \sum_{n=-N}^{N} c_n w^n + \beta \log w, \qquad \beta - \text{real},$$

which is regular analytic in D_r and has there a single valued real part. D_r has a positive area in the metric

$$\left| \sum_{n=-N}^{N} n\, c_n w^{n-1} + \frac{\beta}{w} \right| |dw| .$$

Thus

$$\iint_{D_r} \left| \sum_{n=-N}^{N} n\, c_n w^{n-1} + \frac{\beta}{w} \right|^2 d\sigma > 0 ,$$

where $d\sigma$ denotes the element of Euclidean area in the w--plane. Using the method of complex integration by parts, we transform this into

$$\frac{1}{i} \int_{-C_r + C_r'} \operatorname{Re}\left\{g(w)\right\} dg(w)$$

and moreover into

$$(2) \qquad -\frac{1}{i} \int_0^{2\pi} \operatorname{Re}\left\{ g\left(f(re^{i\theta})\right)\right\} \frac{d\left\{ g\left(f(re^{i\theta})\right)\right\}}{d\theta}$$

$$+ \frac{1}{i} \int_0^{2\pi} \operatorname{Re}\left\{ g\left(-\left[\overline{f(re^{i\theta})}\right]^{-1}\right)\right\} \frac{d\left\{ g\left(-\left[\overline{f(re^{i\theta})}\right]^{-1}\right)\right\}}{d\theta} .$$

To simplify the calculation of the integrals in (2), we shall choose the function g(w) in an appropriate way. Let

$$\log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{n,m=0}^{\infty} A_{nm} z^n \zeta^m , \tag{3}$$

and

$$-\log\left(1+\overline{f(\zeta)}\, f(z)\right) = \sum_{n,m=1}^{\infty} B_{nm} z^n \bar{\zeta}^m . \tag{4}$$

Since f is of the class K, both power series converge in $K(0,1) \times K(0,1)$.

Next, we define polynomials of degree n in the variable t by means of the generating function

$$\log \frac{1}{1-tf(z)} = \sum_{n=1}^{\infty} \frac{1}{n} \Phi_n (t) z^n . \tag{5}$$

For $0<|z|<1$ fixed and ζ sufficiently near to the origin, we can write the identity (3) in the form

$$\sum_{n,m=0}^{\infty} A_{nm} z^n \zeta^m = \log \frac{f(z)}{z} + \sum_{m=1}^{\infty} \frac{\zeta^m}{z^m} - \sum_{m=1}^{\infty} \frac{1}{m} \Phi_m \left(\frac{1}{f(z)}\right) \zeta^m . \tag{6}$$

Since for $\zeta = 0$, (3) yields

$$\log \frac{f(z)}{z} = \sum_{n=0}^{\infty} A_{n0}\, z^n , \tag{7}$$

a comparison of equal powers of ζ in (6) leads to the identities

$$\Phi_m\left(\frac{1}{f(z)}\right) = \frac{1}{z^m} - m \sum_{n=0}^{\infty} A_{nm} z^n \tag{8}$$

for $m>0$, while no new information is contained in (6) for $m = 0$.

We recognize that

$$\Phi_m(w) = F_m(w) + \text{const} ,$$

where $F_m(w)$ are the Faber polynomials for the function $\frac{1}{f(z)}$. The matrix (A_{nm}) is symmetric and occurs in the inequalities of the Grunsky type.

From (5) and (4) we have the identities

$$\Phi_m\left(\overline{-f(z)}\right) = m \sum_{n=1}^{\infty} B_{nm}\bar{z}^n , \tag{9}$$

for $m > 0$, and observe that the matrix (B_{nm}) is Hermitean.

Now we can define the function $g(w)$ as follows. Let x_n, $y_n (n = 1,\dots,N)$ be 2N complex numbers and x_0 a real number. Let

$$g(w) = x_0 \log w + \sum_{m=1}^{N} \left[x_m \bar{\Phi}_m(-w) + y_m \Phi_m(\tfrac{1}{w})\right] . \tag{10}$$

Setting $w = f(z)$ into (10), we find from (8) and (9)

$$g(f(z)) = x_0 \log z + \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=1}^{N} \frac{y_n}{z^n} ,$$

with

$$\alpha_0 = x_0 A_{00} - \sum_{m=1}^{N} m\, y_m A_{0m} , \tag{11}$$

$$\alpha_n = x_0 A_{n0} + \sum_{m=1}^{N} m(x_m B_{nm} - y_m A_{nm}), \quad n = 1,2,\dots \tag{12}$$

Similarly, setting $w = -\left[\overline{f(z)}\right]^{-1}$ into (10), we calculate

$$g\left(-\left[\overline{f(z)}\right]^{-1}\right) = -x_0 \log \bar{z} + x_0(2k + 1)\pi i + \sum_{n=0}^{\infty} \beta_n \bar{z}^n + \sum_{n=1}^{N} \frac{x_n}{\bar{z}^n} ,$$

with

$$(13)\qquad \beta_0 = - x_0\bar{A}_{00} - \sum_{m=1}^{N} m\, x_m\bar{A}_{0m} + x_0(2k+1)\pi i \ ,$$

$$(14)\qquad \beta_n = - x_0\bar{A}_{n0} - \sum_{m=1}^{N} m(x_m\bar{A}_{nm} - y_m\bar{B}_{nm}) \ , \qquad n = 1,2,\dots$$

We can now evaluate the integrals

$$(15)\qquad I_1 = -\frac{1}{i}\int_0^{2\pi} \mathrm{Re}\left\{ g\Big(f(re^{i\theta})\Big) \right\} \frac{d\left\{ g\Big(f(re^{i\theta})\Big)\right\}}{d\theta}$$

and

$$(16)\qquad I_2 = \frac{1}{i}\int_0^{2\pi} \mathrm{Re}\left\{ g\Big(-\Big[\overline{f(re^{i\theta})}\Big]^{-1}\Big) \right\} \frac{d\left\{ g\Big(-\Big[\overline{f(re^{i\theta})}\Big]^{-1}\Big)\right\}}{d\theta} \ .$$

We write

$$g\big(f(z)\big) = x_0 \log z + P(z) \ ,$$

with

$$(17)\qquad P(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=1}^{N} \frac{y_n}{z^n} ,$$

and since P(z) is single valued in K(0,1), we find

$$I_1 = -\frac{1}{i}\int_0^{2\pi} \Big[x_0 \log r + \mathrm{Re}\big\{P(z)\big\}\Big]\Big[x_0 \frac{dz}{z} + d\,P(z)\Big]$$

$$= -2\pi x_0^2 \log r - \frac{1}{2i}\int_0^{2\pi} \bar{P}\, d\,P - x_0 \int_0^{2\pi} \mathrm{Re}\{P\}\, d\theta \ .$$

Using the series development (17) for P(z) we obtain

$$(18)\qquad I_1 = -2\pi x_0^2 \log r - \pi\sum_{n=1}^{\infty} n|\alpha_n|^2 r^{2n} +$$

$$+\pi\sum_{n=1}^{N} n|y_n|^2 r^{-2n} - x_0 \mathrm{Re}\left\{\alpha_0\right\}\cdot 2\pi .$$

Similarly, we evaluate the integral (16). Applying now the decomposition

$$g\left(-\left[\overline{f(z)}\right]^{-1}\right) = -x_0 \log \bar{z} + Q(\bar{z})$$

with

(19) $$Q(z) = \sum_{n=0}^{\infty} \beta_n z^n + \sum_{n=1}^{N} \frac{x_n}{z^n},$$

and since $Q(z)$ is single valued in $K(0,1)$, we obtain

$$I_2 = \frac{1}{i}\int_0^{2\pi}\left[-x_0 \log r + \mathrm{Re}\left\{Q(\bar{z})\right\}\right]\left[-x_0\frac{d\bar{z}}{\bar{z}} + d\left\{Q(\bar{z})\right\}\right]$$

$$= -2\pi x_0^2 \log r + \frac{1}{2i}\int_0^{2\pi}\overline{Q(\bar{z})}\, d\left\{Q(\bar{z})\right\} + x_0\int_0^{2\pi}\mathrm{Re}\left\{Q(\bar{z})\right\} d\theta.$$

Using the series development (19) for $Q(z)$, we find

(20) $$I_2 = -2\pi x_0^2 \log r - \pi\sum_{n=1}^{\infty} n|\beta_n|^2 r^{2n} +$$

$$+\pi\sum_{n=1}^{N} n|x_n|^2 r^{-2n} + x_0 \mathrm{Re}\left\{\beta_0\right\}\cdot 2\pi .$$

Applying now formulas (18) and (20) to the inequality

$$I_1 + I_2 > 0 ,$$

we finally find the estimate

$$-4 x_0^2 \log r + \sum_{n=1}^{N} n\left(|x_n|^2 + |y_n|^2\right) r^{-2n} \geqslant 2 x_0 \mathrm{Re}\left\{\alpha_0 - \beta_0\right\}$$

$$+\sum_{n=1}^{\infty} n\left(|\alpha_n|^2 + |\beta_n|^2\right) .$$

Letting r tend to 1 we obtain the best possible estimate

$$(21) \qquad 2x_0 \operatorname{Re}\left\{\alpha_0 - \beta_0\right\} + \sum_{n=1}^{\infty} n(|\alpha_n|^2 + |\beta_n|^2) \leqslant$$

$$\leqslant \sum_{n=1}^{N} n(|x_n|^2 + |y_n|^2) .$$

Hence by (11) and (13)

$$\alpha_0 - \beta_0 = 2x_0A_{00} + \sum_{n=1}^{N} n(x_n\bar{A}_{0n} - y_nA_{0n}) ,$$

we arrive at the generalized area theorem for Shah's functions.

T h e o r e m 1. Let $f \in K$; then

$$(22) \qquad 4x_0^2 \operatorname{Re}\left\{A_{00}\right\} + 2x_0 \operatorname{Re}\left\{\sum_{n=1}^{N} n(x_n\bar{A}_{0n} - y_nA_{0n})\right\}$$

$$+ \sum_{n=1}^{\infty} n(|\alpha_n|^2 + |\beta_n|^2) \leqslant \sum_{n=1}^{N} n(|x_n|^2 + |y_n|^2)$$

with

$$\alpha_n = x_0A_{n0} + \sum_{m=1}^{N} m(x_mB_{nm} - y_mA_{nm}), \qquad n = 1,2,\dots,$$

$$\beta_n = -x_0\bar{A}_{n0} - \sum_{m=1}^{N} m(x_m\bar{A}_{nm} - y_m\bar{B}_{nm}), \qquad n = 1,2,\dots,$$

where A_{nm} and B_{nm} are given by expansions (3) and (4), are satisfied for any set of complex numbers x_m, $y_m (m = 1,\dots,N)$ and for any x_0-real.

3. We obtain a better insight into inequalities (22) by a change of variables. Let

$$\xi_n = \frac{1}{2}(\bar{x}_n - y_n), \qquad \eta_n = \frac{1}{2}(\bar{x}_n + y_n), \qquad n = 1,2,\dots$$

We thus find

$$\frac{1}{2}(\bar{\alpha}_n - \beta_n) = x_0\bar{A}_{n0} + \sum_{m=1}^{N} m(\bar{\xi}_m\bar{A}_{nm} + \xi_m\bar{B}_{nm}) = U_n(x_0, \xi_m),$$

$$\frac{1}{2}(\bar{\alpha}_n + \beta_n) = \sum_{m=1}^{N} m(\eta_m\bar{B}_{nm} - \bar{\eta}_m\bar{A}_{nm}) = V_n(\eta_m).$$

Using the decomposition

$$\left|\frac{\bar{a}+b}{2}\right|^2 + \left|\frac{\bar{a}-b}{2}\right|^2 = \frac{1}{2}(|a|^2 + |b|^2),$$

we can transform (22) into

$$(23)\quad 2x_0^2 \operatorname{Re} A_{00} + 2x_0 \operatorname{Re}\sum_{n=1}^{N}(n\,\xi_n A_{0n}) + \sum_{n=1}^{\infty} n\left(|U_n(x_0,\xi_m)|^2 + |V_n(\eta_m)|^2\right) \leqslant \sum_{n=1}^{N} n(|\xi_n|^2 + |\eta_n|^2).$$

Substituting now $\eta_n = 0$ $(n = 1,2,\ldots)$ into (23) we find the estimate

$$(24)\quad 2x_0^2 \operatorname{Re} A_{00} + 2x_0 \operatorname{Re}\sum_{n=1}^{N}(n\,\xi_n A_{0n}) + \sum_{n=1}^{\infty}\left[n\left|x_0 A_{n0} + \sum_{m=1}^{N} m(\xi_m A_{nm} + \bar{\xi}_m B_{nm}\right|^2\right] \leqslant \sum_{n=1}^{N} n|\xi_n|^2,$$

while for x_0 and ξ_n vanishing, we obtain the estimate

$$(25)\quad \sum_{n=1}^{\infty}\left[n\left|\sum_{m=1}^{\infty} m(\eta_m A_{nm} - \bar{\eta}_m B_{nm})\right|^2\right] \leqslant \sum_{n=1}^{N} n|\eta_n|^2.$$

We may specialize (24) by choosing all ξ_n to be zero and find the useful estimate

$$(26)\qquad 2\operatorname{Re}\{A_{00}\} + \sum_{n=1}^{\infty} n|A_{n0}|^2 \leqslant 0.$$

4. Now we will study the quadratic forms

$$\sum_{n,m=1}^{N} nm\, \xi_n(\xi_m A_{nm} + \bar{\xi}_m B_{nm})$$

and

$$\sum_{n,m=1}^{N} nm\, \eta_n(\eta_m A_{nm} - \bar{\eta}_m B_{nm}) .$$

By Schwarz inequality and in view of (24) with $x_0 = 0$ and in view of (25), we obtain

$$(27)\quad \Big| \sum_{n,m=1}^{N} nm\, \xi_n(\xi_m A_{nm} + \bar{\xi}_m B_{nm}) \Big| = \Big| \sum_{n=1}^{N} \sqrt{n} \cdot \xi_n\Big(\sqrt{n} \sum_{m=1}^{N} m\, (\xi_m A_{nm} + \bar{\xi}_m B_{nm}\Big)\Big| \leqslant \Big(\sum_{n=1}^{N} n|\xi_n|^2\Big)^{1/2}\Big(\sum_{n=1}^{N} n\Big|\sum_{m=1}^{N} m(\xi_m A_{nm} + \bar{\xi}_m B_{nm})\Big|^2\Big)^{1/2} \leqslant \sum_{n=1}^{N} n\,|\xi_n|^2 .$$

$$(28)\quad \Big| \sum_{n,m=1}^{N} nm\, \eta_n(\eta_m A_{nm} - \bar{\eta}_m B_{nm}) \Big| = \Big| \sum_{n=1}^{N} \sqrt{n}\, \eta_n\Big(\sqrt{n} \sum_{m=1}^{N} m\Big(\eta_m A_{nm} - \bar{\eta}_m B_{nm}\Big)\Big| \leqslant \Big(\sum_{n=1}^{N} n\,|\eta_n|^2\Big)^{1/2} \cdot \Big(\sum_{n=1}^{N} n\Big|\sum_{m=1}^{N} m(\eta_m A_{nm} - \bar{\eta}_m B_{nm})\Big|^2\Big)^{1/2} \leqslant \sum_{n=1}^{N} n\,|\eta_n|^2.$$

By denoting

$$\lambda_n = n\,\xi_n = n\,\eta_n ,$$

one obtains from (27) and (28)

$$(29)\qquad \Big| \sum_{n,m=1}^{N} (A_{nm}\, \lambda_n \lambda_m \pm B_{nm}\, \lambda_n \bar{\lambda}_m) \Big| \leqslant \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 .$$

Since the form $\sum_{n,m=1}^{N} B_{nm}\lambda_n\bar{\lambda}_m$ is real and since this inequality holds for any complex numbers $\lambda_1,\ldots,\lambda_N$, we may pass to the inequality

$$(30)\qquad \Big|\sum_{n,m=1}^{N} A_{nm}\lambda_n\lambda_m\Big| \leqslant \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 \pm \sum_{n,m=1}^{N} B_{nm}\lambda_n\bar{\lambda}_m ,$$

or, equivalently,

$$(31)\qquad \Big|\operatorname{Re}\sum_{n,m=1}^{N} A_{nm}\lambda_n\lambda_m\Big| \leqslant \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 \pm \sum_{n,m=1}^{N} B_{nm}\lambda_n\bar{\lambda}_m .$$

5. We thus found an infinite set of necessary conditions for a function f to belong to the class K. Observe, however, that estimates (29) taken together are already sufficient to guarantee the convergence of series (3) and (4) in $K(0,1)\times K(0,1)$. Indeed in view of (29) we obtain the inequalities

$$(32)\qquad \Big|\sum_{n,m=1}^{N} A_{nm}\lambda_n\lambda_m\Big| \leqslant \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 ,$$

and

$$(33)\qquad \Big|\sum_{n,m=1}^{N} B_{nm}\lambda_n\bar{\lambda}_m\Big| \leqslant \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 .$$

By a proper choice of λ_n in both inequalities (32) and (33) we may prove that the coefficients A_{nm} and B_{nm} $(n, m = 1,2,\ldots)$ are bounded and as an immediate consequence of this fact and of the fact that the function f is regular in a neighborhood of $z = 0$ we derive the convergence of series (3) and (4) in $K(0,1)\times K(0,1)$. Hence we have proved

T h e o r e m 2. Let

$$f(z) = b_1z + b_2z^2 + \ldots$$

be regular in a neighborhood of $z = 0$ and let A_{nm} and B_{nm} denote the polynomials in $b_1, b_2,\ldots$ given by the expansions

$$\log \frac{f(z) - f(\xi)}{z - \xi} = \sum_{n,m=0}^{\infty} A_{nm} z^n \xi^m , \qquad A_{nm} = A_{mn} ,$$

$$-\log\left(1+f(\bar{\xi})f(z)\right) = \sum_{n,m=1}^{\infty} B_{nm} z^n \bar{\xi}^m , \qquad B_{nm} = \bar{B}_{mn} .$$

In order that $b_1, b_2, \dots$ be the coefficients of a Shah's function it is necessary and sufficient that the inequalities

$$\left| \sum_{n,m=1}^{N} A_{nm} \lambda_n \lambda_m \right| \leqslant \sum_{n=1}^{N} \frac{1}{n} |\lambda_n|^2 \pm \sum_{n,m=1}^{N} B_{nm} \lambda_n \bar{\lambda}_m$$

be satisfied for any set of complex numbers $\lambda_1, \lambda_2, \dots, \lambda_N$.

6. The inequality (31) is analogous to the Nehari condition for univalent bounded functions. We shall now drop the assumption $x_0 = 0$ in inequality (24). The development of this paper shows clearly that the introduction of the additional variable x_0 is very important in the discussion of the coefficient problem.

To make a further application of inequality (24), we denote

$$\lambda_n = \begin{cases} x_0 & \text{for} \quad n = 0, \\ n\,\xi_n & \text{for} \quad n = 1, 2, \dots, \end{cases}$$

and instead of (24) we can write the inequality

$$(34) \quad 2\,\mathrm{Re} \sum_{n=0}^{N} A_{0n} \lambda_0 \lambda_n + \sum_{n=1}^{\infty} n\left(\left| \sum_{m=0}^{N} A_{nm} \lambda_m + \sum_{m=1}^{N} B_{nm} \bar{\lambda}_m \right|^2\right) \leqslant$$

$$\leqslant \sum_{n=1}^{N} \frac{1}{n} |\lambda_n|^2 .$$

We then consider the expression

$$P(\lambda) = \mathrm{Re}\left\{ \sum_{n,m=0} A_{nm} \lambda_n \lambda_m + \sum_{n,m=1}^{N} B_{nm} \lambda_n \bar{\lambda}_m \right\}$$

and ask for the maximum value of the real function $P(\lambda)$ under the additional conditions

$$(35)\qquad \operatorname{Im}\lambda_0 = 0\,, \qquad \sum_{n=1}^{N}\frac{1}{n}\,|\lambda_n|^2 = 1\,.$$

We are sure that such a maximum does exist under conditions (35) since, in view of (26), $\operatorname{Re}\left\{A_{00}\right\}\leqslant 0$.

Using Lagrange multipliers in the complex domain [1] we obtain the maximum conditions

$$(36)\qquad 2\operatorname{Re}\left\{\sum_{n=0}^{N} A_{n0}\,\lambda_n^*\right\} = -i\sigma\,,$$

$$(37)\qquad \sum_{m=0}^{N} A_{nm}\lambda_m^* + \sum_{m=1}^{N} B_{nm}\bar{\lambda}_m^* = \tau\frac{\bar{\lambda}_n^*}{n}\,, \qquad n = 1,\dots,N,$$

with real multipliers σ and τ and with the extremal point $(\lambda_0^*, \lambda_1^*, \dots, \lambda_N^*)$. In view of (36) we find

$$(38)\qquad \operatorname{Re}\left\{\sum_{n=0}^{N} A_{n0}\,\lambda_n^*\right\} = 0\,.$$

Using (37) and (38) we obtain from (34)

$$(39)\qquad |\tau|^2\sum_{n=1}^{N}\frac{1}{n}\,|\lambda_n^*|^2 = \sum_{n=1}^{N} n\,\Big|\sum_{m=0}^{N} A_{nm}\,\lambda_m^* + \sum_{m=1}^{N} B_{nm}\,\bar{\lambda}_m^*\Big|^2 \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} n\,\Big|\sum_{n=0}^{N} A_{nm}\,\lambda_m^* + \sum_{m=1}^{N} B_{nm}\,\bar{\lambda}_m^*\Big|^2 \leqslant \sum_{m=1}^{N}\frac{1}{n}\,|\lambda_n^*|^2\,,$$

which implies

$$(39')\qquad |\tau| \leqslant 1\,.$$

On the other hand

$$(40)\qquad P(\lambda) \leqslant \operatorname{Re}\left\{\sum_{n,m=0}^{N} A_{nm}\,\lambda_n^*\lambda_m^* + \sum_{n,m=1}^{N} B_{nm}\,\lambda_n^*\bar{\lambda}_m^*\right\}$$

under the restrictions (35). Rewriting (40) in the form

$$P(\lambda)\leqslant\lambda_o^* \operatorname{Re}\left\{\sum_{n=0}^{N} A_{no}\,\lambda_n^*\right\}+\operatorname{Re}\left\{\sum_{n=1}^{N}\left[\lambda_n^*\left(\sum_{m=0}^{N}A_{nm}\,\lambda_m^* + \sum_{m=1}^{N}B_{nm}\,\bar{\lambda}_m^*\right)\right]\right\}$$

and using (38), (37) and (39'), we easily find

$$(40')\qquad P(\lambda)\leqslant \operatorname{Re}\left\{\sum_{n=1}^{N}\left(\lambda_n^*\cdot\tau\,\frac{\bar{\lambda}_n^*}{n}\right)\right\}=\tau\sum_{n=1}^{N}\frac{1}{n}\,|\lambda_n^*|^2\leqslant 1\,.$$

Finally, if $(\lambda_o, \lambda_1,\dots, \lambda_N)$ is an arbitrary point, λ_o -real, one easily confirms that

$$(40'')\ P(\lambda) = \operatorname{Re}\left\{\sum_{n,m=0}^{N} A_{nm}\,\lambda_n\,\lambda_m + \sum_{n,m=1}^{N} B_{nm}\,\lambda_n\,\bar{\lambda}_m\right\}\leqslant\sum_{n=1}^{N}\frac{1}{n}\,|\lambda_n|^2\,.$$

We may next raise the question, for which functions $f\in K$ can equality be achieved in (40''). The extremum of $P(\lambda)$ can only be achieved for such functions for which all signs in (39), (40) and (40') are equality signs. In order that equalities hold in the estimate (39), we have to demand

$$\sum_{m=0}^{N} A_{nm}\,\bar{\lambda}_m + \sum_{m=1}^{N} B_{nm}\,\bar{\bar{\lambda}}_m = 0 \qquad\text{for}\quad n > N$$

and

$$|\tau| = 1\,.$$

In view of (37) and (38) equality is true in the estimate (40) if and only if

$$\operatorname{Re}\left\{\sum_{n=0}^{N} A_{no}\,\lambda_n\right\} = 0$$

and

$$\sum_{m=0}^{N} A_{nm}\,\lambda_m + \sum_{m=1}^{N} B_{nm}\,\bar{\lambda}_m = \tau\,\frac{\bar{\lambda}_n}{n} \qquad\text{for}\quad n=1,\dots,N\,.$$

Finally, in order that equalities hold in (40'), we have to demand

$$\tau = 1 .$$

Hence, for a given vector $(\lambda_0, \lambda_1, \ldots, \lambda_N)$, the extremum function $f \in K$ must satisfy the conditions

$$\sum_{m=0}^{N} A_{nm} \lambda_m + \sum_{m=1}^{N} B_{nm} \bar{\lambda}_m = \begin{cases} (\frac{1}{n}) \bar{\lambda}_n & \text{for} \quad 0 < n \leqslant N, \\ 0 & \text{for} \quad n > N \end{cases}$$

and

$$\operatorname{Re} \left\{ \sum_{n=0}^{N} A_{no} \lambda_n \right\} = 0 .$$

Thus we have proved the following theorem.

T h e o r e m 3. Let $f(z) = b_1 z + b_2 z^2 + \ldots$, be regular in a neighborhood of $z = 0$ and let A_{nm} and B_{nm} denote the polynomials in $b_1, b_2, \ldots$, given by expansions (3) and (4). In order that $b_1, b_2, \ldots$ be the coefficients of a Shah's function is necessary and sufficient that the inequalities

$$(41) \quad \operatorname{Re} \left\{ \sum_{n,m=0}^{N} A_{nm} \lambda_n \lambda_m \right\} \leqslant \sum_{n=1}^{N} \frac{1}{n} |\lambda_n|^2 - \sum_{n,m=1}^{N} B_{nm} \lambda_n \bar{\lambda}_m$$

be satisfied for any set of complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ and λ_0 -real. Equality in (41) is true if and only if

$$\operatorname{Re} \left\{ \sum_{n=0}^{N} A_{no} \lambda_n \right\} = 0$$

and

$$\sum_{m=0}^{N} A_{nm} \lambda_m + \sum_{m=1}^{N} B_{nm} \bar{\lambda}_m = \begin{cases} (\frac{1}{n}) \bar{\lambda}_n & \text{for} \quad 0 < n \leqslant N, \\ 0 & \text{for} \quad n > N . \end{cases}$$

7. We will now give applications of inequalities (22) and (41). We begin with an application of inequality (26) which follows from (22).

Since by (26) we clearly have

$$\text{(42)} \qquad \operatorname{Re}\left\{A_{00}\right\} \leqslant 0$$

and since $A_{00} = \log f'(0) = \log b_1$, we arrive at theorem

T h e o r e m 4. For any $f \in K$ there holds

$$\text{(43)} \qquad \left|f'(0)\right| \leqslant 1 ,$$

or

$$\text{(44)} \qquad |b_1| \leqslant 1 .$$

Next, we can use the Schwarz inequality to find

$$\log\left|\frac{f(z)}{z}\right| = \operatorname{Re}\left\{\sum_{n=0}^{\infty} A_{n0} z^n\right\} \leqslant \operatorname{Re}\left\{A_{00}\right\} + \sum_{n=1}^{\infty} n|A_{n0}|^2 \cdot \log\frac{1}{1-|z|^2},$$

and hence by (26)

$$\sum_{n=1}^{\infty} n|A_{n0}|^2 \leqslant -2 \operatorname{Re}\left\{A_{00}\right\},$$

we have theorem

T h e o r e m 5. For any $f \in K$ there holds

$$\log\left|\frac{f(z)}{z}\right| \leqslant \log|b_1|\left(1 + 2\log(1 - |z|^2)\right) ,$$

or, equivalently,

$$|f(z)| \leqslant \frac{|b_1|\,|z|}{(1-|z|^2)^{\log|b_1|^{-2}}} .$$

8. Let

(45) $$\mu_\lambda(f) = \lim_{r\to 1}\left[\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^\lambda d\theta\right]^{\frac{1}{\lambda}}, \qquad \lambda > 0 .$$

The following result also follows from the estimate (26).

Theorem 6. If $f \in K$ and $f(z) = b_1 z + b_2 z^2 + \ldots$ and if $\mu_\lambda(f)$ denotes the mean value (45), then

(46) $$\mu_2(f) \leqslant 1 ,$$

or, equivalently,

$$\sum_{n=1}^{\infty} |b_n|^2 \leqslant 1 .$$

Inequality (46) is sharp in two respects: if $0 < M < 1$ and $2 < \lambda < \infty$, then in general neither the inequality $\mu_2(f) \leqslant M$ nor the inequality $\mu_\lambda(f) \leqslant 1$ is true. The equality in (46) holds only for the function

(47) $$f(z) = \frac{e^{i\varphi}\sqrt{1-|c|^2}\, z}{1 - cz}, \qquad |c| < 1, \qquad \varphi\text{-real}.$$

It is well known that $\mu_\lambda(f)$ is a non-decreasing function of λ and that, therefore, (46) is stronger than the inequality $\mu_1(f) \leqslant 1$.

Proof. The following theorem was shown by Lebedev and Milin [4], [5].

Theorem. For any sequence $a_1, a_2, \ldots$ such that

$$\sum_{n=1}^{\infty} n|a_n|^2 < \infty ,$$

the numbers $c_1, c_2, \ldots$ defined by the power series expansion

$$\sum_{n=0}^{\infty} c_n z^n = \exp\left[\sum_{n=1}^{\infty} a_n z^n\right]$$

are subject to the inequality

$$(48) \qquad \sum_{n=0}^{\infty} |c_n|^2 \leqslant \exp\left[\sum_{n=1}^{\infty} n|a_n|^2 \right],$$

where the sign of equality is possible only if $a_n = \frac{c^n}{n}$, $(n=1,2,\ldots)$ $|c|<1$. For the coefficient c_n itself, these authors obtain the inequality

$$(49) \qquad |c_n| \leqslant \exp\left\{ \frac{1}{2}\left[\sum_{k=1}^{n} k|a_k|^2 - \sum_{k=1}^{n} \frac{1}{k} \right]\right\},$$

with the equality only for $a_n = \frac{c^n}{n}$ $(n=1,2,\ldots)$ $|c| = 1$.

To prove our theorem, we apply (49) to the case where the coefficients

$$a_n = A_{n0}, \qquad n = 1, 2, \ldots,$$

and

$$c_n = \frac{b_{n+1}}{b_1}, \qquad n = 0, 1, \ldots .$$

Indeed, in view of inequality (26) and identity (7) we are in a position to apply the above theorem. At first in view of (48) we find

$$(50) \qquad \sum_{n=1}^{\infty} |b_n|^2 \leqslant \exp\left[\sum_{n=1}^{\infty} n|A_{n0}|^2 \right] \cdot |b_1|^2$$

and, secondly, in view of

$$(51) \qquad \sum_{n=1}^{\infty} n|A_{n0}|^2 \leqslant -2\log|b_1|,$$

we have the inequality

$$(52) \qquad \sum_{n=1}^{\infty} |b_n|^2 \leqslant 1 .$$

The equality in (50) is possible only if

$$(53)\qquad A_{n0} = \frac{c^n}{n}, \qquad n = 1, 2, \dots, \qquad |c| < 1,$$

which means that

$$f(z) = \frac{b_1 z}{1-cz}, \qquad |c| < 1 .$$

Conditions (53) under which there is equality in (50) imply that the equality in (51) is possible only if

$$|b_1| = \sqrt{1 - |c|^2} .$$

Thus we have proved the following: up to a factor of modulus 1, the only functions to be considered are

$$(54)\qquad f_0(z) = \frac{z\sqrt{1-|c|^2}}{1 - cz} .$$

It is easy to prove that these functions are in K. Indeed, suppose the existence of two points $z^*, \xi^* \in K(0,1)$ such that $f_0(z^*)\,\overline{f_0(\xi^*)} = -1$. We easily arrive at the inequality

$$\left| \frac{cz^* - 1}{z^* - \bar{c}} \right| < 1$$

which is impossible for $|c| < 1$ and for $|z^*| < 1$.

This establishes the statement regarding equality in (46). We are now in a position to obtain the last conclusion of our theorem, i. e. that we cannot have $\mu_\lambda(f) \leqslant 1$ if $\lambda > 2$ for each $f \in K$. Indeed, $\mu_{\lambda_1}(f) = \mu_{\lambda_2}(f)$, $\lambda_1 < \lambda_2$, is possible only if f is constant on $|z| = 1$ except for a set of measure zero. Since f_0 do not have this property and since

$\mu_2(f_o) = 1$, we must have $\mu_\lambda(f_o) > 1$ for $\lambda > 2$. This completes the proof of our theorem.

In view of (52) one obtains

$$|b_n| \leqslant 1, \qquad n = 1, 2, \dots, \tag{55}$$

for $f \in K$. This estimate cannot possibly be sharp for $n > 1$ if f is restricted to K. The following theorem provides bounds for these coefficients which, while not sharp, are of the correct order.

Theorem 7. Let $f \in K$ and

$$f(z) = b_1 z + b_2 z^2 + \dots .$$

Denote

$$B_n = \max_{f \in K} |b_n| .$$

Then

$$\frac{e^{-\frac{1}{2}}}{\sqrt{n}} \leqslant B_n < \frac{e^{-\frac{1}{2}\gamma}}{\sqrt{n-1}}, \qquad n = 2, 3, \dots, \tag{56}$$

where $\gamma = 0{,}577\dots$ is Euler's constant.

Proof. Inequality (49) for $a_n = A_{n0}$ $(n = 1, 2, \dots)$ and $c_n = \frac{b_{n+1}}{b_1}$ $(n = 0, 1, \dots)$ assumes the form

$$\left|\frac{b_{n+1}}{b_1}\right| \leqslant \exp\left\{\frac{1}{2}\left[\sum_{k=1}^{n} k|A_{k0}|^2 - \sum_{k=1}^{n} \frac{1}{k}\right]\right\}, \qquad n = 1, 2, \dots,$$

with the equality only for $A_{k0} = \frac{c^k}{k}$ $(k = 1, \dots, n)$, $|c| = 1$. Applying (51) we find

$$|b_n| \leqslant \exp\left\{-\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k}\right\}, \qquad n = 2, 3, \dots$$

In view of

$$-\sum_{k=1}^{n-1}\frac{1}{k} + \log(n-1) < -\gamma,$$

where γ is Euler's constant, we prove the right-hand inequality in (56).

In order to obtain the left-hand inequality we observe that for the function (54) with $c = (1 - \frac{1}{n})^{\frac{1}{2}}$, we have

$$b_n^2 = \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} > \frac{1}{en} .$$

This completes the proof of (56).

Nehari in [7] gave exactly the same inequalities for the coefficients of Bieberbach-Eilenberg functions.

9. Now we will give some applications of inequality (41). We begin with an application of inequality (33). By setting in (33) $\lambda_n = z^n$ $(n = 1,2,\ldots)$ in view of (4) one obtains

T h e o r e m 8. Let $f \in K$. Then for $|z|<1$

$$|f(z)| \leqslant \frac{|z|}{\left(1 - |z|^2\right)^{\frac{1}{2}}} .$$

This result is the well known distortion theorem for the class K. By setting in (41) $\lambda_n = z^n$ $(n = 0, 1,\ldots)$ and using limiting value where appropriate, we obtain

T h e o r e m 9. Let $f \in K$. Then for $|z| <1$

$$\left|\log f'(z) + \log\left(1 + |f(z)|^2\right)\right| \leqslant - \log(1 - |z|^2) ,$$

hence

$$|f'(z)| \leqslant \left[(1 - |z|^2)(1 + |f(z)|^2)\right]^{-1} .$$

Similarly, if in equality (31) we take $\lambda_n=z^n$ $(n=1,2,\ldots)$ we obtain

T h e o r e m 10. Let $f \in K$. Then for $|z| < 1$

$$\left|\log \frac{f'(z)b_1z^2}{f^2(z)} \pm \log\left(1 + |f(z)|^2\right)\right| \leqslant - \log(1 - |z|^2) ,$$

hence

$$|f(z)|^2\left(1+|f(z)|^2\right) \frac{1-|z|^2}{|z|^2} \leqslant |b_1 f'(z)| \leqslant \frac{|f(z)|^2}{1+|f(z)|^2} \frac{|z|^2}{1-|z|^2} .$$

This result follows also from the inequality obtained by Jerkins [3], p. 201.

10. We will now give some application of (31) to the problem of maximizing of $|a_3 - a_2^2|$ in the expansion

$$f(z) = b_1 z + b_2 z^2 + \dots, \qquad a_n = \frac{b_n}{b_1} ,$$

$f \in K$, when b_1 is given. We see that $A_{11} = a_3 - a_2^2$. Putting now in (31) $N = 1$ and $\lambda_1 = 1$ we get

Theorem 11. If $f \in K$, then

$$|a_3 - a_2^2| \leqslant 1 - |b_1|^2 .$$

This inequality is sharp, the equality holding for the function

$$\frac{f(z)}{\left(1 + f(z)\right)^2} = \frac{b_1 z}{(1 + z)^2} .$$

The above inequality is exactly the same as for the class S_1 of univalent bounded functions, $|f(z)| < 1$, which is the subclass of K.

BIBLIOGRAPHY

[1] Z. Charzyński: Sur les functions univalentes algébriques bornées. Rozprawy Mat. 10. Warszawa 1955.

[2] J.A. Humel, M. Schiffer: Coefficient inequalites for Bieberbach - Eilenberg functions. Arch. Rational. Mech. Anal. 32 (1969) 87-99.

[3] J.A. J e n k i n s: On Bieberbach - Eilenberg functions. Trans. Amer. Math.Soc. 119 (1965) 195 - 215.

[4] I.M. M i l i n: Ocenka koefficientow odnolistnych funkcji.Dokł. Akad. Nauk SSSR 160 (1965) 769 - 771.

[5] I.M. M i l i n: O koefficientach odnolistnych funkcij. Dokł. Akad. Nauk SSSR 167 (1967) 1255 - 1258.

[6] Z. N e h a r i: Some inequalities in the theory of functions. Trans. Amer.Math.Soc. 75 (1953) 256 - 286.

[7] Z. N e h a r i: On the coefficients of Bieberbach - Eilenberg functions, J.Analyse Math. 23 (1970) 297 - 303.

[8] M. S c h i f f e r, O. T a m m i: On the coefficient problem for bounded univalent functions. Trans. Amer. Math. Soc. 140 (1969) 461 - 474.

[9] T a o - S h i n g - S h a h: On the moduli of some classes of analytic functions. Acta Math. Sinica 5 (1955) 439 - 454.

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