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ON INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH FUNCTIONAL ARGUMENTS IN UNBOUNDED DOMAINS

In paper [3] there was proved, among other theorems, a theorem on the existence of a unique solution of the first Fourier problem in a bounded domain for a system of semilinear parabolic integro-differential equations with functional arguments.

In this paper we extend the above result to a domain unbounded in the direction of the time-axis. At first we derive some estimate of Friedman's type for a solution of the first Fourier problem in the considered domain for a single linear parabolic equation. This estimate enable us to apply the Banach fixed point theorem and to prove the existence mentioned. The same results are also obtained for the half-space.

1. An estimate of the solution of a linear problem

Let G be an open domain of the Euclidean space E_{n+1} of the variables $(x,t) = (x_1,\dots,x_n,t)$ whose boundary consists of a closed domain $\bar{R}_0$ of the hyperplane $t=0$ and of a surface S situated in the half-space $t>0$. We assume that for every $\tau > 0$ the domain

$$G^{\tau} = G \cap \left\{(x,t):\ 0<t<\tau\right\}$$

is bounded.

Let $h=h(t)$ be a function defined for $t \geqslant 0$ and possessing continuous, non-negative and non-decreasing derivative $h'(t)$. We introduce the following norms:

$$|u|^G_{h,o} = \sup_{P\in G} |e^{-h(t)}u(P)|, \qquad P = (x,t),$$

$$|u|^G_{h,\alpha} = |u|^G_{h,o} + \sup_{P,P'\in G}\left\{\exp\left[-h\left(\max(t,t')\right)\right] \frac{|u(P)-u(P')|}{[d(P,P')]^\alpha}\right\},$$

$$|u|^G_{h,1+\alpha} = |u|^G_{h,\alpha} + \sum_{i=1}^{n} |u_{x_i}|^G_{h,\alpha} \qquad (0<\alpha<1),$$

where

$$d(P,P') = (|x-x'|^2 + |t-t'|)^{\frac{1}{2}}, \quad |x-x'| = \left[\sum_{i=1}^{n}(x_i-x_i')^2\right]^{\frac{1}{2}}.$$

The set of all functions $u(x,t)$ for which $|u|^G_{h,k+\alpha}<\infty\,(k=0,1)$ will be denoted by $C_{h,k+\alpha}(G)$. In the case $h=h(t)\equiv 0$ we shall omit the superscript 0. Note that $C_{h,k+\alpha}(G)$ is a Banach space.

In this section we deduce an estimate for the norm $|u|^G_{h,1+\beta}$, where $u(x,t)$ is a solution of the problem

$$(1.1)\quad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t =$$

$$= f(x,t), \quad (x,t)\in G,$$

$$(1.2)\qquad u(x,t) = \varphi(x,t), \quad (x,t)\in \Sigma = \bar{R}_0 \cup S^{1)}.$$

The following assumptions will be needed (see section 1 of [2]).

(1.I) For every $\tau>0$ the following conditions are fulfilled:

1° the operator L (with $a_{ij}\equiv a_{ji}$) is uniformly parabolic in the domain $\overline{G^\tau}$;

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1) By a solution of this problem we shall always understand a regular solution, i.e. continuous in the domain $\bar{G}$ and possessing in G continuous derivatives appearing in Lu.

2^o the coefficients of the operator L satisfy in G^τ the uniform Hölder condition with exponent $\alpha \in (0,1)$ independent of τ ;

3^o the coefficients a_{ij} satisfy the uniform Lipschitz condition on the surface

$$S^\tau = S \cap \left\{(x,t):\ 0<t \leqslant \tau\right\}.$$

(1.II) The function $\varphi(x,t)$ defined on $\sum$ possesses an extension $\Phi(x,t)$ which belongs to $C_{1+\beta}(G^\tau) \cap C_{2+\alpha}(G^\tau)$ $(0<\beta<1)$ for every $\tau>0$.

(1.III) For every $\tau>0$ the function $f(x,t)$ satisfies in G^τ the uniform Hölder condition with exponent α and

$$L\,\Phi(x,0) = f(x,0) \quad \text{for} \quad (x,0) \in \partial R_c.$$

(1.IV) For every $\tau>0$ the surface S^τ belongs both to $\bar{\bar{C}}_{2+\alpha}$ and to C_{2-0} (see [2], p. 257).

It follows from assumption (1.I) the existence of a positive non-increasing function $K_0(\tau)$ $(\tau>0)$ and positive non-decreasing functions $K_1(\tau)$ and $K_2(\tau)$ $(\tau>0)$ such that

$$(1.3) \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j \geqslant K_0(\tau)|\xi|^2, \quad (x,t)\in \overline{G^\tau}, \quad \xi \in E_n,$$

$$(1.4) \qquad \sum_{i,j=1}^{n} |a_{ij}|_\alpha^{G^\tau} + \sum_{i=1}^{n} |b_i|_0^{G^\tau} + |c|_0^{G^\tau} \leqslant K_1(\tau),$$

$$(1.5) \qquad \sum_{i,j=1}^{n} |a_{ij}|_{1-0}^{S^\tau} \leqslant K_2(\tau).$$

T h e o r e m 1. If assumptions (1.I) - (1.IV) are satisfied, then problem (1.1), (1.2) has a unique solution $u(x,t)$. Moreover, for every $a \in (0,1)$ there exists a function $h=h(t)$ depending only on $a, \alpha, \beta, K_i(\tau)$ $(i=0,1,2)$ and on the domain G and possessing the following properties:

1° h(t) is defined for $t \geqslant 0$ and it has continuous non-decreasing derivative $h'(t) \geqslant 1$;

2° the norms

$$|f|^{G}_{h,0}\,,\quad |L\Phi|^{G}_{h,0}\,,\quad |\Phi|^{G}_{h,1+\beta}$$

are finite;

3° there holds true the estimate

$$(1.6)\qquad |u|^{G}_{h,1+\beta} \leqslant a\left(|f|^{G}_{h,0} + |L\Phi|^{G}_{h,0}\right) + |\Phi|^{G}_{h,1+\beta}\,.$$

P r o o f. The first part of the theorem follows from the existence of a unique solution of the problem

$$Lu = f(x,t),\quad (x,t) \in \overline{G^{\tau}} \setminus \textstyle\sum^{\tau},$$

$$u(x,t) = \varphi(x,t),\quad (x,t) \in \textstyle\sum^{\tau} = \overline{R}_0 \cup S^{\tau}$$

for every $\tau > 0$ (see Theorem 7 of [1], p. 65).

Now we shall prove the second part of the theorem in the case $\varphi(x,t) \equiv 0$. At first we choose an arbitrary function $h=h(t)$ satisfying the condition 1° of Theorem 1 and such that

$$h(t) \geqslant C_1 + \ln\left(|f(x,t)|^{G_t}_{0}\right) \qquad (C_1=\text{const.}),$$

where $G_t = \left\{x : (x,t) \in \overline{G} \setminus \overline{S}\right\}$. Then $|f|^{G}_{h,0} < \infty$. In the further considerations the functions $h(t)$ and $h'(t)$ will be suitably enlarged (in the case of necessity) in such a manner that their monotonicity and continuity will be preserve.

Proceeding like in [1] (section 3, chapter VII) we shall estimate the norm $|u|^{G^2}_{h,1+\beta}$. For this purpose we extend the coefficients of L into a closed cylinder $\Omega_0 = \overline{D}_0 \times [0,2]$ containing the domain $\overline{G^2}$ in such a manner that the extended functions satisfy the uniform Hölder condition (with exponent

α) and such that (1.3), (1.4) hold in Ω_0 with $K_0(2)$, $K_1(2)$ possibly replaced by the other constant depending only on $K_0(2)$, $K_1(2)$. Let $\Gamma(x,t;\xi,\tau)$ be the fundamental solution in Ω_0 (for the extension of L). We shall need the following lemma.

L e m m a 1. Let $\Omega = \bar{E} \times [0,\delta]$, where $\bar{E} \subset \bar{D}_0$ and $\delta \in (0,2]$ is an arbitrary constant. Assume that $h(t)$ is a function satisfying condition 1° of Theorem 1. Then for any $\beta \in (0,1)$ there exists a constant $M > 0$ depending only on α, β, $K_0(2)$ and $K_1(2)$ such that for any continuous function $g(x,t)$ in Ω the function

$$v(x,t) = \int_0^t d\tau \int_E \Gamma(x,t;\xi,\tau) g(\xi,\tau) d\xi$$

fulfils the inequality

$$(1.7) \qquad |v|^G_{h,1+\beta} \leq M[h'(o)]^{-r} |g|^\Omega_{h,o} ,$$

where $r = (1-\beta)/(3+\beta)$..

The proof of this lemma is quite similar to that of Theorem 1 of [4].

Now we shall estimate the norm $|u|^{G^\delta}_{h,1+\beta}$ with sufficiently small δ. At first we derive the interior estimates. Let $\Omega = \bar{E} \times [0,\delta] \subset G^2 \setminus S^2$, where the boundary ∂E of E is of class $C_{1+\alpha}$. The solution $u(x,t)$ of problem (1.1), (1.2) (with $\varphi(x,t) \equiv 0$) is given in the domain Ω by the formula

$$(1.8) \qquad u(x,t) = -\int_0^t d\tau \int_E \Gamma(x,t;\xi,\tau) f(\xi,\tau) d\xi + \int_0^t d\tau \int_{\partial E} \Gamma(x,t;\xi,\tau) k(\xi,\tau) d\xi \equiv u_1(x,t) + u_2(x,t),$$

where the function $k(y,t)$ $(y \in \partial E,\ 0 < t \leq \delta)$ is defined as follows

$$(1.9) \quad k(y,t) = 2F(y,t) + 2\sum_{k=1}^{\infty} \int_0^t d\tau \int_{\partial E} M_k(y,t;\xi,\tau) F(\xi,\tau) d\xi$$

and

$$M_1(y,t;\xi,\tau) = \frac{2\partial\Gamma(y,t;\xi,\tau)}{\partial\nu(y,t)},$$

$$M_{k+1}(y,t;\xi,\tau) = \int_\tau^t d\sigma \int_{\partial E} M_1(y,t;z,\sigma)M_k(z,\sigma;\xi,\tau)dz \qquad (k=1,2,\ldots),$$

$$F(y,t) = -\int_0^t d\tau \int_E \frac{\partial\Gamma(y,t;\xi,\tau)}{\partial\nu(y,t)} f(\xi,\tau)d\xi - \frac{\partial u(y,t)}{\partial\nu(y,t)}.$$

It follows from the estimate

$$\left|\frac{\partial\Gamma(x,t;\xi,\tau)}{\partial\nu(x,t)}\right| \leqslant M_1(t-\tau)^{-\mu} |x-\xi|^{-n-1+2\mu+\alpha}, \quad 1-\frac{\alpha}{2}<\mu<1$$

that

$$(1.10) \qquad |F|^{S_0}_{h,0} \leqslant M_2\left[|f|^{\Omega}_{h,0} + \sum_{i=1}^{n}|u_{x_i}|^{\Omega}_{h,0}\right], \qquad S_0 = \partial E\times(0,\delta].$$

Since for the series $\sum_{k=1}^{\infty} M_k(x,t;\xi,\tau)$ holds true the same estimate as for $\frac{\partial\Gamma}{\partial\nu}$ (with other constant M_1) therefore, by (1.9), (1.10), we have

$$(1.11) \qquad |k|^{S_0}_{h,0} \leqslant M_3\left[|f|^{\Omega}_{h,0} + \sum_{i=1}^{n}|u_{x_i}|^{\Omega}_{h,0}\right].$$

Now let $\Omega_1 = \bar{B}\times[0,\delta]$, where $B\subset\bar{B}\subset E$. For every $\xi\in\partial E$ and $\tau\in(0,\delta)$ holds true the estimate (see inequality (2.29) of [1], p. 194)

$$(1.12) \qquad \left|\Gamma(x,t;\xi,\tau)\right|^{\Omega_1}_{h,1+\beta} \leqslant M_4,$$

where the norm is taken with respect to $(x,t)\in\Omega_1$ and $\Gamma(x,t;\xi\tau)$ is defined to be zero for $t<\tau$. Hence we obtain, for $x,x'\in B$ and $t\in(0,\delta)$, the following inequalities

$$(1.13) \qquad e^{-h(t)}|u_2(x,t)| \leqslant e^{-h(t)}\int_0^t e^{h(\tau)}d\tau\int_{\partial E}|\Gamma(x,t;\xi,\tau)|e^{-h(\tau)}|k(\xi,\tau)|d\xi \leqslant$$

$$\leqslant M_5[h'(0)]^{-1}|k|_{h,0}^{S_0},$$

$$(1.14) \qquad e^{-h(t)}|x-x'|^{-\beta}|u_2(x,t)-u_2(x',t)| \leqslant$$

$$\leqslant e^{-h(t)}\int_0^t e^{h(\tau)}d\tau\int_{\partial E}\frac{|\Gamma(x,t;\xi,\tau)-\Gamma(x',t;\xi,\tau)|}{|x-x'|^{\beta}}e^{-h(\tau)}|k(\xi,\tau)|d\xi \leqslant$$

$$\leqslant M_5\left[h'(0)\right]^{-1}|k|_{h,o}^{S_o}.$$

If $x \in B$ and $0<t<t'<\delta$, then

$$(t'-t)^{-\beta/2}|\Gamma(x,t';\xi,\tau)| \leqslant (t'-\tau)^{-\beta/2}|\Gamma(x,t';\xi,\tau)| \leqslant \text{const.}$$

Hence and by (1.12) we have

$$(1.15) \qquad e^{-h(t')}(t'-t)^{-\beta/2}|u_2(x,t)-u_2(x,t')| \leqslant$$

$$\leqslant \int_0^t e^{-h(t)+h(\tau)}d\tau\int_{\partial E}\frac{|\Gamma(x,t;\xi,\tau)-\Gamma(x,t';\xi,\tau)|}{(t'-t)^{\beta/2}}e^{h(\tau)}|k(\xi,\tau)|d\xi +$$

$$+\int_t^{t'} e^{-h(t')+h(\tau)}d\tau\int_{\partial E}\frac{|\Gamma(x,t';\xi,\tau)|}{(t'-t)^{\beta/2}}e^{-h(\tau)}|k(\xi,\tau)|d\xi \leqslant$$

$$\leqslant M_6|h'(0)|^{-1}|k|_{h,o}^{S_o}.$$

Since the estimates (1.13)-(1.15) hold true also for derivatives $\frac{\partial u_2}{\partial x_i}$ therefore, in view of (1.11),

$$|u_2|^{\Omega_1}_{h,1+\beta} \leqslant M_7[h'(0)]^{-1}\Big[|f|^{\Omega}_{h,0} + \sum_{i=1}^{n} |u_{x_i}|^{\Omega}_{h,0}\Big].$$

This inequality together with the estimate (1.7) for the function $u_1(x,t)$ imply (by (1.8)) that

$$(1.16)\qquad |u|^{\Omega_1}_{h,1+\beta} \leqslant M_8[h'(0)]^{-r}\Big(|f|^{\Omega}_{h,0} + \sum_{i=1}^{n} |u_{x_i}|^{\Omega}_{h,0}\Big),$$

where the constant $M_8 > 0$ depends only on $B,E,\alpha,\beta,K_0(2)$ and $K_1(2)$.

In order to obtain the boundary estimates for the function $u(x,t)$ in G^{δ} we use the integral representation of the function $U'(z,t)$ which was established in section 3.2, chapter VII of [1], where $U'(z,t)$ is defined by relations (3.5) - (3.11) (of the above-mentioned section 3.2). Proceeding similarly to the proof of the interior estimates (1.16) one can derive counterparts of inequalities (3.20) and (3.21) (of [1]) in the "h-norm". Therefore, as a counterpart of (3.22) (of [1]) we get

$$|u|^{G^{\delta}}_{h,1+\beta} \leqslant M_9[h'(0)]^{-r}\Big(|f|^{G^{\delta}}_{h,o} + \sum_{i=1}^{n}{}' |u_{x_i}|^{G^{\delta}}_{h,o}\Big),$$

where $\delta \in (0,2]$ is sufficiently small and M_9 is a positive constant depending only on α, β, G^2 and $K_i(2)$ $(i=0,1,2)$. Hence, if $h'(0)$ is such that

$$M_9[h'(0)]^{-r} \leqslant 1/2\ ^{2)},$$

then

$$(1.17)\qquad |u|^{G^{\delta}}_{h,1+\beta} \leqslant M[h'(0)]^{-r}|f|^{G^{\delta}}_{h,o}, \quad M=2M_9.$$

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2) Otherwise we enlarge $h'(0)$ in such a way that this inequality is satisfied.

Let us denote by $G^{p,q}$ $(0\leqslant p<q)$ the domain $G\cap\{(x,t): p<t<q\}$ and by $\Sigma^{p,q}$ its parabolic boundary. It follows from the proof of (1.17) that for any $\rho\in[0,2-\delta]$ and any function $g\in C_\alpha(G^{\rho,\rho+\delta})$ such that $g(x,\rho)=0$ on $\bar{S}$, the solution $v(x,t)$ of the problem

$$(1.18)\qquad Lv = g(x,t),\quad (x,t)\in \overline{G^{\rho,\rho+\delta}}\setminus\Sigma^{\rho,\rho+\delta},$$

$$(1.19)\qquad v(x,t) = 0,\quad (x,t)\in\Sigma^{\rho,\rho+\delta}$$

fulfils the inequality

$$|v|^{G^{\rho,\rho+\delta}}_{h,1+\beta}\leqslant \bar{M}[h'(\rho)]^{-r}|g|^{G^{\rho,\rho+\delta}}_{h,o}$$

provided δ is sufficiently small depending only on α,β, G^2 and $K_i(2)$ $(i=0,1,2)$. Hence, by the monotonicity of $h'(t)$, we have

$$|v|^{G^{\rho,\rho+\delta}}_{h,1+\beta}\leqslant \bar{M}[h'(0)]^{-r}|g|^{G^{\rho,\rho+\delta}}_{h,o}.$$

Further, repeating the argumentation of section 3.3 of [1] (p. 200,201) we obtain the estimate

$$(1.20)\qquad |u|^{G^2}_{h,1+\beta}\leqslant H_o[h'(0)]^{-r}|f|^{G^2}_{h,o},$$

where $H_o>0$ is a constant depending only on α,β, G^2 and $K_i(2)$ $(i=0,1,2)$. Hence, if $h'(0)$ is so large that

$$H_o[h'(0)]^{-r}\leqslant a/2\ ^{3)},$$

3) See the footnote 2)

then

$$|u|^{G^2}_{h,1+\beta} \leqslant \frac{a}{2} |f|^{G^2}_{h,0} , \tag{1.21}$$

where $a \in (0,1)$ is an arbitrary fixed number.

Now we proceed to estimate the norm $|u|^{G}_{h,1+\beta}$. For this purpose it suffices to estimate the norms

$$|u|^{G^{k,k+1}}_{h,1+\beta} \quad (k=2,3,\dots).$$

At first note that from the proof of (1.20) it follows that for any $\varrho \geqslant 0$ and any function $g \in C_\alpha(G^{\varrho,\varrho+2})$ such that $g(x,\varrho)=0$ on $\bar{S}$, the solution $v(x,t)$ of problem (1.18), (1.19) with $\delta=2$ satisfies the inequality

$$|v|^{G^{\varrho,\varrho+2}}_{h,1+\beta} \leqslant H_\varrho \left[h'(\varrho)\right]^{-r} |g|^{G^{\varrho,\varrho+2}}_{h,0} ,$$

where $H_\varrho > 0$ is a constant depending only on α, β, $G^{\varrho,\varrho+2}$ and $K_i(\varrho)$ $(i=0,1,2)$. This inequality will be used for $\varrho = 1,2,\dots$ in the case when

$$H_\varrho\left[h'(\varrho)\right]^{-r} \leqslant a\cdot 2^{-\varrho-3} \text{ 4)} ;$$

then

$$|v|^{G^{\varrho,\varrho+2}}_{h,1+\beta} \leqslant a\cdot 2^{-\varrho-3} |g|^{G^{\varrho,\varrho+2}}_{h,0} \tag{1.22}$$

Now we estimate the norm $|u|^{G^{2,3}}_{h,1+\beta}$. For this purpose let us consider the function $v(x,t) = \xi(t-1)u(x,t)$, where

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4) See the footnote 2)

$$\xi(t) = \begin{cases} 2t^2, & 0 \leqslant t < 1/2, \\ -2t^2+4t-1, & 1/2 \leqslant t \leqslant 1, \\ 1, & t > 1. \end{cases}$$

It is easy to see that

$$Lv = \xi(t-1)f(x,t) - \xi'(t-1)u, \quad (x,t) \in \overline{G^{1,3}} \setminus \Sigma^{1,3},$$

$$v(x,t) = 0, \quad (x,t) \in \Sigma^{1,3}.$$

Since $0 \leqslant \xi(t) \leqslant 1$ and $|\xi'(t)| \leqslant 2$, therefore, by (1.22), we have

$$|v|^{G^{1,3}}_{h,1+\beta} \leqslant 2^{-4}a\left(|f|^{G^{1,3}}_{h,o} + 2|u|^{G^{1,3}}_{h,o}\right).$$

Hence, in view of the relation

$$v(x,t) \equiv u(x,t), \quad (x,t) \in \overline{G^{2,3}}$$

and by (1.21), there is satisfied the inequality

$$|u|^{G^{2,3}}_{h,1+\beta} \leqslant 2^{-2}a|f|^{G^{3}}_{h,o}.$$

Using the previous method with $G^{1,3}$, $\Sigma^{1,3}$, $\xi(t-1)$ and $\xi'(t-1)$ replaced by $G^{2,4}$, $\Sigma^{2,4}$, $\xi(t-2)$ and $\xi'(t-2)$, respectively, we get the estimate

$$|u|^{G^{3,4}}_{h,1+\beta} \leqslant 2^{-3}a|f|^{G^{4}}_{h,o}.$$

Proceeding in the above manner one can easy to check, by induction, that

$$(1.23) \qquad |u|^{G^{k,k+1}}_{h,1+\beta} \leqslant 2^{-k}a|f|^{G^{k+1}}_{h,o} \qquad (k=2,3,\ldots).$$

Inequalities (1.21) and (1.23) imply the estimate

(1.24) $$|u|^{G}_{h,1+\beta} \leqslant a|f|^{G}_{h,o} .$$

Now we are going to the case when $\varphi(x,t)\not\equiv 0$. Then the function $v(x,t)=u(x,t)-\Phi(x,t)$ is a solution of the problem

$$Lv = f(x,t) - L\Phi, \quad (x,t)\in G,$$

$$v(x,t) = 0, \quad (x,t)\in \sum .$$

We impose the following conditions on the function $h(t)$:

$$\exp h(t) \geqslant C_2 |L\Phi(x,t)|^{G_t}_{o} ,$$

$$\exp h(t) \geqslant C_2 |\Phi(x,t)|^{G_t}_{o} ,$$

$$\exp h(t) \geqslant C_2 \max_i |\Phi_{x_i}(x,t)|^{G_t}_{o} ,$$

$$\exp h(t) \geqslant C_2 \sup_{P,P'\in\overline{G}^t} \frac{|\Phi(x',t') - \Phi(x,t)|}{[d(P,P')]^{\beta}} ,$$

$$\exp h(t) \geqslant C_2 \max_i \left\{ \sup_{P,P'\in\overline{G}^t} \frac{|\Phi_{x_i}(x',t') - \Phi_{x_i}(x,t)|}{[d(P,P')]^{\beta}} \right\} ,$$

C_2 being a positive constant. These conditions yield the relation

$$|L\Phi|^{G}_{h,0} , \quad |\Phi|^{G}_{h,1+\beta} < \infty .$$

Hence, by (1.24), we have

$$|v|^{G}_{h,1+\beta} \leqslant a(|f|^{G}_{h,o} + |L\Phi|^{G}_{h,o})$$

which immediately implies (1.6). Thus Theorem 1 is completely proved.

R e m a r k. It follows from the above proof that the estimate (1.6) remains valid in each of the following two cases:

1° if we replace the function $h(t)$ by a function $h_1(t)$ satisfying condition 1° of Theorem 1 and such that $h_1(t) \geqslant h(t)$ and $h_1'(t) \geqslant h'(t)$;

2° if we replace the functions $f(x,t)$, $\Phi(x,t)$ by other functions satisfying assumptions (1.II), (1.III) and condition 2° of Theorem 1.

2. On the first Fourier problem for a system of integro-differential equations with functional arguments

Let G^{T_0} (T_0=const.<0) be a bounded open domain of the space E_{n+1}, enclosed by domains $\overline{R_{T_0}}$ and $\overline{R}_0$[5] lying on the planes $t=T_0$ and $t=0$ respectively, and by a surface S^{T_0} situated in the strip $T_0 < t < 0$.

In this section we are dealing with the existence and uniqueness of solutions of the problem:

$$\begin{aligned} L^k u^k \equiv \sum_{i,j=1}^{n} a_{ij}^k(x,t) u^k_{x_i x_j} + \sum_{i=1}^{n} b_i^k(x,t) u^k_{x_i} + c^k(x,t) u^k - u^k_t = \\ (2.1) \quad = F^k\Big(x,t,\Psi_1^k\big(u(x,t)\big),\Psi_2^k\big(u(x,t)\big),\Psi_3^k\big(u(x,t)\big)\Big),\ (x,t)\in G, \end{aligned}$$

$$(2.2) \quad u^k(x,t) = \varphi^k(x,t), \quad (x,t)\in \overline{G^{T_0}} \cup S \quad (k=1,\dots,N),$$

where

$$\Psi_1^k\big(u(x,t)\big) = \Big(\big\{u^i(x,t)\big\},\ \big\{u^i_{x_j}(x,t)\big\},\ \Big[\int_{G_t} u^i(y,t)\mu_1^{ki}(x,t;dy)\Big\},$$

5) We shall use the notation of the previous section.

$$\left\{ \int_{G_t} u^i(y,\tau)\nu_1^{ki}(x,t;dy,d\tau) \right\} \Bigg),$$

$$\Psi_m^k\Big(u(x,t)\Big) = \Bigg(\left\{ u^i\Big(w_{3m-5}^{ki}(x,t), q_{3m-5}^{ki}(t)\Big) \right\} ,$$

$$\left\{ u_{x_j}^i\Big(z_{m-1,j}^{ki}(x,t), s_{m-1,j}^{ki}(t)\Big) \right\} ,$$

$$\left\{ \int_{G_t} u^i\Big(w_{3m-4}^{ki}(y,t), q_{3m-4}^{ki}(t)\Big) \; \mu_m^{ki}(x,t;dy) \right\} ,$$

$$\left\{ \int_{G_t} u^i\Big(w_{3m-3}^{ki}(y,\tau), q_{3m-3}^{ki}(\tau)\Big)\nu_m^{ki}(x,t;dy,d\tau) \right\} \Bigg)$$

where $k,i=1,\dots,N;\ j=1,\dots,n;\ m=2,3,$

$$G_t = \left\{ x : (x,t) \in G^{T_0} \cup R_0 \cup G \right\}, \quad t > T_0 .$$

Functions q, s, transformations w, z and measures μ, ν (occurring in the definitions of symbols Ψ_1^k, Ψ_m^k) will be defined in assumptions (2.IV), (2.V) and (2.VI), respectively.

The following assumptions are introduced:

(2.I) The operators L^k $(k=1,\dots,N)$ satisfy assumption (1.I), i.e. the inequalities (1.3)-(1.5) hold true with a_{ij}, b_i and c replaced by a_{ij}^k, b_i^k and c^k, respectively.

(2.II) The functions $F^k\Big(x,t,p_1^1,\dots,p_1^{N_0},p_2^1,\dots,p_2^{N_0},p_3^1,\dots$ $\dots,p_3^{N_0}\Big)$ $(k=1,\dots,N;\ N_0=3N+nN)$ are defined in the set $\bar{G} \times E_{3N_0}$ and fulfil the following conditions:

1° For any $\tau > 0$ and any bounded domain $H \subset E_{3N_0}$ functions $F^k(x,t,p_1,p_2,p_3)$ satisfy the uniform Hölder condition with exponent α in $(x,t) \in \overline{G^\tau}$, uniformly with respect to $(p_1,p_2,p_3) \in H$.

2^0 There exist a positive constant M_1 and a positive function $M_2(t)$ $(t \geqslant 0)$ such that for any $(x,t) \in \bar{G}$ and $p_i, \bar{p}_i \in E_{N_0}$ $(i=1,2,3)$ we have

$$(2.3)\quad |F^k(x,t,p_1,p_2,p_3) - F^k(x,t,\bar{p}_1,\bar{p}_2,\bar{p}_3)| \leqslant M_1(|p_1-\bar{p}_1| + |p_2-\bar{p}_2|) + M_2(t)|p_3-\bar{p}_3| ,$$

where

$$|p_i-\bar{p}_i| = \sum_{j=1}^{N_0} \left|p_i^j - \overline{p_i^j}\right| .$$

(2.III) The vector-function $\varphi = (\varphi^1, \ldots, \varphi^N)$, defined on $\overline{G^{T_0}} \cup S$, belongs to $C^N_{1+\beta}(G^{T_0})$ $(0<\beta<1)$ and possesses such an extension $\Phi = (\Phi^1, \ldots, \Phi^N)$ that $\Phi \in C^N_{1+\beta}(G^\tau) \cap C^N_{2+\alpha}(G^\tau)$ [6) for every $\tau>0$. Moreover, if a function $\bar{\Phi}$ (belonging to $C^N_{1+\beta}(G^\tau) \cap C^N_{2+\gamma}(G^\tau)$ $(0<\gamma<1)$ for every $\tau>0$) is an extension of φ, then

$$L^k \overline{\Phi^k} = F^k\left(x,0, \Psi^k_1(\bar{\Phi}), \Psi^k_2(\bar{\Phi}), \Psi^k_3(\bar{\Phi})\right), \quad (x,0)\in \partial R_0 .$$

(2.IV) The functions $q_i^{kj}(t)$, $s_{lm}^{kj}(t)$ $(k,j=1,\ldots,N;\ i=1,\ldots,6;\ l=1,2;\ m=1,\ldots,n)$ map the interval $(0,\infty)$ into (T_0,∞) and satisfy the uniform Hölder condition with exponent $\alpha_0/2$ $(0<\alpha_0 \leqslant 1)$ in every interval $(0,\tau)$, $0<\tau<\infty$. Moreover

$$q_i^{kj}(t),\ s_{2m}^{kj}(t) \leqslant t, \quad k,j=1,\ldots,N; \quad i=4,5,6; \quad m=1,\ldots,n.$$

(2.V) For every $t>0$ the transformations w_i^{kj} and z_{lm}^{kj} map G_t into $G_{q_i^{kj}(t)}$ and $G_{s_{lm}^{kj}(t)}$, respectively. These

6) For the definitions of these symbols see section 2 of [2].

transformations satisfy the uniform Hölder condition with exponent α_0 in every domain G^{τ} $(\tau > 0)$, i.e. for any $P(x,t)$, $P'(x',t') \in G^{\tau}$ we have

$$\left|w_i^{kj}(P) - w_i^{kj}(P')\right|, \quad \left|z_{lm}^{kj}(P) - z_{lm}^{kj}(P')\right| \leqslant M(\tau)\left[d(P,P')\right]^{\alpha_0},$$

$M(\tau)$ being a positive constant depending on τ.

(2.VI) Let us denote by $\mathfrak{m}_t$ $(t \geqslant 0)$ the σ-field of all Borel's subsets of $\bar{G}_t$ and by $\mathfrak{n}$ the σ-field of all Borel's subsets of $\bar{G}$. By $\mu_i^{kj}(x,t;D)$ and $\nu_i^{kj}(x,t;D)$ $(i=1,2,3;\ j,k=1,\dots,N)$ we will denote finite non-negative measures (depending on $x \in \bar{G}_t$ and $(x,t) \in \bar{G}$, respectively) defined on $\mathfrak{m}_t$ and $\mathfrak{n}$, respectively. The following conditions are imposed:

1° There is a positive constant $N_1 > 0$ such that for any $(x,t) \in \bar{G}$

$$\mu_i^{kj}(x,t;\bar{G}_t), \quad \nu_i^{kj}(x,t;\bar{G}) \leqslant N_1 . \tag{2.4}$$

2° For every $\tau > 0$ there exists finite non-negative measure $\bar{\mu}$ (resp. $\bar{\nu}$) defined on the Borel's subsets of the domain $\bigcup_{0 \leq t \leq \tau} G_t$ (resp. G^{τ}) such that for any $P(x,t)$, $P'(x',t') \in \overline{G^{\tau}}$ we have

$$\left|\mu_i^{kj}(x,t;D) - \mu_i^{kj}(x',t';D)\right| \leqslant \bar{\mu}(D)\left[d(P,P')\right]^{\alpha_1}$$

if $\bar{G}_t \cap \bar{G}_{t'} \neq 0$ and $D \in \mathfrak{m}_t \cap \mathfrak{m}_{t'}$

$$\left(\text{resp. } \left|\nu_i^{kj}(x,t;D) - \nu_i^{kj}(x',t';D)\right| \leqslant \bar{\nu}(D)\left[d(P,P')\right]^{\alpha_1}\right.$$

$$\left.\text{if } D \in \mathfrak{n} \text{ and } D \subset \overline{G^{\tau}}\right),$$

where $\alpha_1 \in (0,1)$ is a constant independent of τ.

3° For every $\tau > 0$ there is a constant $N_2(\tau) > 0$ such that for any $(x,t) \in \overline{G^{\tau}}$

$$\mu_i^{kj}(x,t;D) \leqslant N_2(\tau) m_1(D), \quad D \in \mathfrak{M}_t,$$

$$\left(\text{resp. } \nu_i^{kj}(x,t;D) \leqslant N_2(\tau) m_2(D), \quad D \in \mathfrak{N}, \quad D \subset \overline{G^{\tau}}\right),$$

$m_1(D)$ (resp. $m_2(D)$) being the n-dimensional (resp.(n+1)-dimensional) Lebesgue measure of D [7].

In order to formulate the existence and uniqueness theorem for the considered problem let us put

$$q(t) = \sup_{\tau \leqslant t} \max_{k,j,i} q_i^{kj}(\tau), \qquad s(t) = \max_{k,j,m} s_{1m}^{kj}(t),$$

where $k,j=1,\dots,N$; $i=1,2,3$; $m=1,\dots,n$.

T h e o r e m 2. Under assumptions (1.IV),(2.I)-(2.VI) there exists a function $h(t)$ depending only on α, β, M_1, N_1, $K_i(t)$ ($i=0,1,2$) and on the domain G and possessing the following properties:

1° $h(t)$ is defined for $t \geqslant T_0$ and it has continuous derivative $h'(t) \geqslant 1$;

2° the norms

$$\left| F^k(x,t,0,0,0)\right|^G_{h,0}, \quad \left| L^k \phi^k\right|^G_{h,0}, \quad \left|\phi^k\right|^G_{h,1+\beta} \qquad (k=1,\dots,N)$$

are finite;

3° if

$$(2.5) \qquad M_2(t) \leqslant \exp\left\{ h(t) - h\Big[\max\Big(q(t), s(t)\Big)\Big]\right\},$$

then problem (2.1),(2.2) has a unique solution $u=(u^1,\dots,u^N)$ in the space $C^N_{h,1+\beta}(\Omega)$ [8], where $\Omega = \overline{G^{T_0}} \cup \overline{G}$.

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7) For this condition remains valid Remark 2 of section 2 of [3].

8) $C^N_{h,1+\beta}(\Omega)$ denotes the Banach space of all vector-functions $u = (u^1,\dots,u^N)$ with finite norm

$$|u|^{\Omega}_{h,1+\beta} = \sum_{k=1}^{N} |u^k|^{\Omega}_{h,1+\beta}.$$

P r o o f. Let us consider the problem

$$(2.6)\qquad L^k u^k = f^k(x,t) \equiv F^k(x,t,0,0,0) - F^k(x,0,0,0,0) + $$
$$+ L^k \Phi^k(x,0), \quad (x,t)\in G \quad (k=1,\dots,N),$$

$$(2.7)\qquad u^k(x,t) = \varphi^k(x,t), \ (x,t)\in \sum .$$

For

$$(2.8)\qquad a = \left[2N(1+M_1)(1+2N_1)\right]^{-1}$$

there exist (by Theorem 1) functions $h^k(t)$ $(k=1,\dots,N)$ satisfying condition 1^o of Theorem 1 and such that

$$(2.9)\qquad |f^k|^G_{h_k,0}, \ |L^k \Phi^k|^G_{h_k,0}, \ |\Phi^k|^G_{h_k,1+\beta} < \infty ,$$

$$(2.10)\qquad |u^k|^G_{h_k,1+\beta} \leqslant a\left(|f^k|^G_{h_k,0} + |L^k\Phi^k|^G_{h_k,0}\right) + |\Phi^k|^G_{h_k,1+\beta} ,$$

where $u^k(x,t)$ $(k=1,\dots,N)$ is a solution of problem (2.6), (2.7). Let us write

$$(2.11)\qquad h(t) = \max_{1\leqslant k\leqslant N} h^k(0) + \int_0^t \max_k \left[h^k(t)\right]' dt .$$

It is easy to see that the function $h(t)$ satisfies condition 1^o of Theorem 1 and the inequalities

$$h(t) \geqslant h^k(t), \quad h'(t) \geqslant \left[h^k(t)\right]' \quad (k=1,\dots,N).$$

Hence, in view of the relations (2.9), (2.10) and Remark of section 1, we have

$$(2.12)\qquad |f^k|^G_{h,0}, \ |L^k\Phi^k|^G_{h,0}, \ |\Phi^k|^G_{h,1+\beta} < \infty \qquad (k=1,\dots,N),$$

$$(2.13)\qquad |u^k|^G_{h,1+\beta} \leqslant a\left(|f^k|^G_{h,o} + |L^k\Phi^k|^G_{h,o}\right) + |\Phi^k|^G_{h,1+\beta} \ .$$

It follows from (2.12) that

$$(2.14)\qquad \left|F^k(x,t,0,0,0)\right|^G_{h,o} < \infty \qquad (k=1,\dots,N).$$

Now we extend the function $h(t)$ into the interval $[T_o,\infty)$ setting

$$(2.15)\qquad h(t) = h'(0)t + h(0) \quad \text{for} \quad T_o \leqslant t < 0 \ .$$

According to the above considerations this extended function $h(t)$ fulfils conditions 1^o and 2^o of Theorem 2.

Let us denote by Λ the set of all functions $u(x,t)\in C^N_{h,1+\beta}(\Omega)$ such that

$$u^k(x,t) = \varphi^k(x,t), \quad (x,t)\in \overline{G^{T_o}} \cup S \quad (k=1,\dots,N).$$

Obviously Λ is a closed set of the space $C^N_{h,1+\beta}(\Omega)$. Now for $u\in\Lambda$ consider the problem

$$(2.16)\qquad L^k v^k = F^k\left(x,t,\Psi^k_1(u),\Psi^k_2(u),\Psi^k_3(u)\right) \equiv f^k(x,t), \quad (x,t)\in G \quad (k=1,\dots,N),$$

$$(2.17)\qquad v^k(x,t) = \varphi^k(x,t), \ (x,t)\in \overline{G^{T_o}} \cup S \ .$$

Assumptions (2.II), (2.IV)-(2.VI) imply, by Lemma 4 of [2] and Lemma 2 of [3], that $f^k\in C_{\alpha_2}(G^{\tau})$ for every $\tau>0$, where $\alpha_2 = \beta\alpha_o\alpha_1$. Therefore, in virtue of Theorem 1, problem (2.16), (2.17) has a unique solution $v=(v^1,\dots,v^N)$. This enables us to define a transformation Z by formula $Zu=v$.

Now we shall show that Z maps Λ into itself. It follows from (2.5), (2.IV), (2.V) and condition 1^o of (2.VI) that

$$\left|\Psi^k_i\left(u(x,t)\right)\right|^G_{h,o} < \infty \quad (i=1,2) \ ,$$

$$\left| M_2(t)\, \Psi_3^k\big(u(x,t)\big)\right|_{h,o}^{G} < \infty$$ 9) .

Hence, taking into considerations relation (2.14) and the inequality

$$|F^k(x,t,p_1,p_2,p_3)| \leqslant |F^k(x,t,0,0,0)| + M_1(|p_1| + |p_2|) + M_3(t)|p_3|$$

(which is a consequence of (2.3)), we have $|f^k|_{h,o}^{G} < \infty$. This implies, by (2.12), (2.13) and Remark of section 1, that $v^k \in C_{h,1+\beta}(G)$, whence, owing to (2.III) and (2.17), v=Zu belongs to Λ .

Now we shall prove that Z is a contraction. So let $u, \bar{u} \in \Lambda$ and $v=Zu$, $\bar{v}=Z\bar{u}$. Then

(2.18) $$L^k(v^k - \overline{v^k}) = f^k(x,t) - \overline{f^k}(x,t), \quad (x,t) \in G, \quad (k=1,\dots,N)$$

(2.19) $$v^k(x,t) - \overline{v^k}(x,t) = 0, \quad (x,t) \in G^{\overline{T}_o} \cup S ,$$

where $$\overline{f^k}(x,t) = F^k\big(x,t, \Psi_1^k(\bar{u}), \Psi_2^k(\bar{u}), \Psi_3^k(\bar{u})\big) .$$

Using assumptions (2.IV), (2.V), condition 1^o of (2.VI) and inequality (2.5) we find that

$$\left|\Psi_i^k(u) - \Psi_i^k(\bar{u})\right|_{h,o}^{G} \leqslant (1+2N_1)\left|u - \bar{u}\right|_{h,1+\beta}^{\Omega} \qquad (i=1,2),$$

$$\left|M_2(t)\left[\Psi_3^k(u) - \Psi_3^k(\bar{u})\right]\right|_{h,o}^{G} \leqslant (1+2N_1)\left|u - \bar{u}\right|_{h,1+\beta}^{\Omega} ,$$

9) We recall that $\Psi_i^k(u)$ is a vector-function with N_o components. For the vector-function $w = \left(w^1,\dots,w^{N_o}\right)$ the norm $|w|_{h,o}^{G}$ is defined by formula

$$|w|_{h,o}^{G} = \sum_{i=1}^{N_o} |w^i|_{h,o}^{G} .$$

whence, by (2.3), we have

(2.20) $$\left|f^k(x,t) - \overline{f^k}(x,t)\right| \leqslant (1+2M_1)(1+2N_1)\,|u-\bar{u}|^{\Omega}_{h,1+\beta} < \infty .$$

According to the Remark of section 1 relations (2.18) - (2.20) imply, by (2.13) (with $\phi^k \equiv 0$), the inequality

$$\left|v^k - \overline{v^k}\right|^G_{h,1+\beta} \leqslant a(1+2M_1)(1+2N_1)\,|u-\bar{u}|^{\Omega}_{h,1+\beta} \qquad (k=1,\dots,N) ,$$

whence (in view of (2.19))

$$|Zu - Z\bar{u}|^{\Omega}_{h,1+\beta} \leqslant aN(1+2M_1)(1+2N_1)\,|u-\bar{u}|^{\Omega}_{h,1+\beta} .$$

From the last inequality and from (2.8) it immediately follows that Z is a contraction. Therefore, by the Banach fixed point theorem, Z has a unique fixed point u which is obviously a (unique) solution of the problem (2.1), (2.2) in the space $C^N_{h,1+\beta}(\Omega)$. This completes the proof.

3. On the Cauchy problem

Now let $G = E_n \times (0,\infty)$ and $G^{\tau} = E_n \times (0,\tau)\,(\tau>0)$. We preserve the meaning of symbols L and L^k and notation concerning norms and functional spaces, which were used in the previous sections.

In this section we derive an estimate for the norm $|u|^G_{h,1+\beta}$, where $u(x,t)$ is a solution of the problem

(3.1) $$Lu = f(x,t), \qquad (x,t) \in G ,$$

(3.2) $$u(x,0) = \varphi(x), \quad x \in E_n .$$

The above estimate will be applied to prove the existence and uniqueness of solutions of the problem

(3.3) $$L^k u^k = F^k\left(x,t,\Psi_1^k(u),\Psi_2^k(u),\Psi_3^k(u)\right), \quad (x,t) \in G ,$$

(3.4) $$u^k(x,t) = \varphi^k(x,t),\ (x,t) \in G^{T_o} = E_n \times [T_o,0] \quad (T_o=\text{const.}<0,\ k=1,\dots,N),$$

where

$$\psi_1^k\Big(u(x,t)\Big)=\Bigg(\Big\{u^i(x,t)\Big\}\ ,\ \Big\{u^i_{x_j}(x,t)\Big\},\Big\{\int_{D_1}u^i(y,t)\,\mu_1^{ki}(x,t;dy)\Big\}\ ,$$

$$\Big\{\int_0^t\omega_1^{ki}(x,t;d\tau)\int_{D_2}u^i(y,\tau)\,\nu_1^{ki}(x,t;dy)\Big\}\Bigg)\ ,$$

$$\psi_m^k\Big(u(x,t)\Big)=\Bigg(\Big\{u^i\Big(w^{ki}_{3m-5}(x),q^{ki}_{3m-5}(t)\Big)\Big\}\ ,\ \Big\{u^i_{x_j}\Big(z^{ki}_{m-1,j}(x),s^{ki}_{m-1,j}(t)\Big)\Big\},$$

$$\Big\{\int_{D_1}u^i\Big(w^{ki}_{3m-4}(y),q^{ki}_{3m-4}(t)\Big)\mu_m^{ki}(x,t;dy)\Big\}\ ,$$

$$\Big\{\int_0^t\omega_m^{ki}(x,t;d\tau)\int_{D_2}u^i\Big(w^{ki}_{3m-3}(y),q^{ki}_{3m-3}(\tau)\Big)\nu_m^{ki}(x,t;dy)\Big\}\Bigg)$$

$(k,i=1,\dots,N;\ j=1,\dots,n;\ m=2,3)$,

D_1, D_2 being arbitrary fixed closed domains of the space E_n.

For the problem (3.1), (3.2) we introduce the following assumptions:

(3.I) For every $\tau>0$ the operator L is uniformly parabolic in G^τ and its coefficients belong to $C_\alpha(G^\tau)$, where $\alpha\in(0,1)$ is independent of τ. Thus the inequalities (1.3) and (1.4) remain valid.

(3.II) The function $f(x,t)$ is bounded in every domain $\overline{G^\tau}$ and satisfy the uniform Hölder condition with exponent α in every bounded domain $H\times[0,\tau]$ $(H\subset E_n)$.

(3.III) The function $\varphi(x)$ together with its first and second order derivatives are bounded in E_n. Moreover, φ and φ_{x_i} are uniformly Hölder continuous with exponent $\beta\in(0,1)$ in E_n, while the derivatives $\varphi_{x_ix_j}$ are locally Hölder continuous with exponent α in E_n.

T h e o r e m 3. If assumptions (3.I) - (3.III) are fulfilled, then problem (3.1), (3.2) has a unique solution $u(x,t)$ in the class of all functions bounded in every domain $\overline{G^\tau}$. Moreover, for every $a\in(0,1]$ there exists a function

$h=h(t)$, depending only on a, α, β, $K_0(\tau)$ and $K_1(\tau)$, satisfying condition 1^o of Theorem 1 and such that

$$|f|^G_{h,o}, \quad |L\varphi|^G_{h,o} < \infty,$$

(3.5) $$|u|^G_{h,1+\beta} \leqslant a(|f|^G_{h,o} + |L\varphi|^G_{h,o}) + e^{-h(0)}|\varphi|^{E_n}_{1+\beta}.$$

P r o o f. The first part of the theorem follows from the existence and uniqueness of solutions of the problem

$$Lu = f(x,t), \quad (x,t) \in E_n \times (0,\tau],$$

$$u(x,0) = \varphi(x), \quad x \in E_n$$

for every $\tau > 0$ (see Theorem 12 of [1], p. 25 and Theorem 10 of [1], p. 44).

Now we outline the proof of the second part of the theorem. In the case $\varphi(x) \equiv 0$ we choose the same function $h=h(t)$ as at the beginning of the proof of Theorem 1 (obviously with $G_t = E_n$). Proceeding like in the proof of Theorem 1 of [4], we can derive the inequality

$$|u|^{G^2}_{h,1+\beta} \leqslant H_o\left[h'(0)\right]^{-r}|f|^{G^2}_{h,o}, \qquad r = (1-\beta)/(3+\beta),$$

where H_a is a positive constant depending only on $\alpha, \beta, K_o(2)$ and $K_1(2)$. Next, repeating the argumentation used in the proof of Theorem 1 following after the inequality (1.20), we obtain the estimate

(3.6) $$|u|^G_{h,1+\beta} \leqslant a|f|^G_{h,o}.$$

In the case $\varphi(x) \not\equiv 0$ observe that the function $v(x,t) = u(x,t) - \varphi(x)$ is a solution of the problem

$$Lv = f(x,t) - L\varphi, \quad (x,t) \in G,$$

$$v(x,0) = 0, \quad x \in E_n .$$

Let us assume that $h(t) \geqslant C_1 + \ln K_1(t)$. Then $|L\varphi|^G_{h,o} < \infty$, whence, by (3.6), we have

$$|v|^G_{h,1+\beta} \leqslant a(|f|^G_{h,o} + |L\varphi|^G_{h,o})$$

which implies (3.5). This completes the proof.

R e m a r k. Note that Remark of section 1 holds true (with obvious modifications) for problem (3.1), (3.2).

Now we shall consider problem (3.3), (3.4). The following assumptions will be needed.

(3.IV) Operators L^k $(k=1,\dots,N)$ satisfy assumption (3.I).

(3.V) Assumption (2.II) with condition 1^o replaced by the following one:

For any $\tau > 0$ and any bounded domains $H_1 \subset E_n$ and $H_2 \subset E_{3N_o}$ the functions $F^k(x,t,p_1,p_2,p_3)$ $(k=1,\dots,N)$ satisfy the uniform Hölder condition with exponent α in $(x,t) \in H_1 \times [0,\tau]$, uniformly with respect to $(p_1,p_2,p_3) \in H_2$.

(3.VI) The functions $\varphi^k(x,t)$ $(k=1,\dots,N)$ belong to $C_{1+\beta}(\overline{G^{T_o}})$ while the derivatives $\varphi^k_{x_ix_j}$ and φ^k_t are bounded in G^{T_o} and satisfy the uniform Hölder condition with exponent α in every bounded domain contained in $\overline{G^{T_o}}$.

(3.VII) The transformations w^{kj}_i and z^{kj}_{lm} $(k,j=1,\dots,N;$ $i=1,\dots,6;$ $l=1,2;$ $m=1,\dots,n)$ map the space E_n into itself and satisfy the local Hölder condition with exponent $\alpha_o \in (0,1)$; i.e. for any bounded domain $H \subset E_n$ and any $x,x' \in H$ we have

$$|w^{kj}_i(x) - w^{kj}_i(x')|, \quad |z^{kj}_{lm}(x) - z^{kj}_{lm}(x')| \leqslant M(H)\,|x-x'|^{\alpha_o} ,$$

$M(H)$ being a positive constant depending on H.

(3.VII) Denote by $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{N}$ the σ-field of all Borel's subsets of the domains D_1, D_2 and of the interval $[0, \infty)$, respectively. By $\mu_i^{kj}(x,t;D)$, $\nu_i^{kj}(x,t;D)$ and $\omega_i^{kj}(x,t;D)$ $(k, j=1,\dots,N;\ i=1,2,3)$ we denote finite non-negative measures (depending on $(x,t) \in \bar{G}$) defined on $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{N}$, respectively. The following conditions are imposed:

1° There is a constant $N_1 > 0$ such that for any $(x,t) \in \bar{G}$ we have

$$\mu_i^{kj}(x,t;D_1), \quad \nu_i^{kj}(x,t;D_2), \quad \omega_i^{kj}\big(x,t;(0,\infty)\big) \leqslant N_1 .$$

2° For any bounded domain $H \subset \bar{G}$ there exist finite non-negative measures $\bar{\mu}$, $\bar{\nu}$ and $\bar{\omega}$ defined on $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{N}$ respectively, such that for any points $P(x,t)$, $P'(x',t') \in H$ hold the inequalities

$$\left|\mu_i^{kj}(x,t;D) - \mu_i^{kj}(x',t';D)\right| \leqslant \bar{\mu}(D)\left[d(P,P')\right]^{\alpha_1}, \quad D \in \mathcal{M}_1 ,$$

$$\left|\nu_i^{kj}(x,t;D) - \nu_i^{kj}(x',t';D)\right| \leqslant \bar{\nu}(D)\left[d(P,P')\right]^{\alpha_1}, \quad D \in \mathcal{M}_2 ,$$

$$\left|\omega_i^{kj}(x,t;D) - \omega_i^{kj}(x',t';D)\right| \leqslant \bar{\omega}(D)\left[d(P,P')\right]^{\alpha_1}, \quad D \in \mathcal{N} ,$$

$$D \subset \left[0, \max(t,t')\right] ,$$

where $\alpha_1 \in (0,1)$ is independent of H.

3° For every bounded domain $H^{\tau} = H \times [0,\tau]$ $(H \subset E_n)$ there is a constant $N_2 = N_2(H)$ such that for any $(x,t) \in H^{\tau}$ we have

$$\omega_i^{kj}(x,t;D) \leqslant N_2\, m(D), \quad D \in \mathcal{N} , \quad D \subset [0,\tau] ,$$

$m(D)$ being the Lebesgue measure of D.

T h e o r e m 4. Under assumptions (2.IV), (3.IV) - - (3.VIII) there exists a function $h(t)$ depending only on $\alpha, \beta, M_1, N_1, K_0(t)$ and $K_1(t)$ such that:

1^o there is fulfilled condition 1^o of Theorem 1;

2^o the norms

$$\left|F^k(x,t,0,0,0)\right|^G_{h,o}, \quad \left|L^k\varphi^k\right|^G_{h,o} \quad (k=1,\dots,N)$$

are finite;

3^o if condition (2.5) is satisfied, then problem (3.3), (3.4) has a unique solution $u=(u^1,\dots,u^N)$ in the space $C^N_{h,1+\beta}(\Omega)$, where $\Omega = E_n \times (T_0,\infty)$.

P r o o f. We proceed similarly as in the proof of Theorem 2. Namely, let us consider the problem

$$(3.7) \qquad L^k u^k = F^k(x,t,0,0,0) \equiv f^k(x,t), \quad (x,t)\in G ,$$

$$(3.8) \qquad u^k(x,0) = \varphi^k(x,0), \quad x\in E_n \qquad (k=1,\dots,N) .$$

For

$$a = \left[2N(1+M_1)(1+N_1+N_1^2)\right]^{-1}$$

there exist (by Theorem 3) functions $h^k(t)$ $(k=1,\dots,N)$ satisfying condition 1^o of Theorem 1 and such that

$$\left|f^k\right|^G_{h_k,0}, \quad \left|L^k\varphi^k\right|^G_{h,k,0} < \infty ,$$

$$\left|u^k\right|^G_{h_k,1+\beta} \leqslant a\left(\left|f^k\right|^G_{h_k,0} + \left|L^k\varphi^k\right|^G_{h_k,0}\right) + e^{-h_k(0)}\left|\varphi^k(x,0)\right|^{E_n}_{1+\beta} ,$$

where $u^k(x,t)$ $(k=1,\dots,N)$ is a solution of (3.7), (3.8). Hence, by Remark to Theorem 3, we have

$$\left|f^k\right|^G_{h,o}, \quad \left|L^k\varphi^k\right|^G_{h,o} < \infty ,$$

$$\left|u^k\right|^G_{h,1+\beta} \leqslant a\left(\left|f^k\right|^G_{h,o} + \left|L^k\varphi^k\right|^G_{h,o}\right) + e^{-h(0)}\left|\varphi^k(x,0)\right|^{E_n}_{1+\beta} ,$$

where the function h(t) is defined by formula (2.11). The further argumentation is the same as in the proof of Theorem 2 after relation (2.14). Thus the proof is completed.

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