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ON MIXED BOUNDARY VALUE PROBLEM FOR HYPERBOLIC SYSTEMS

1. Introduction

In the present paper the conditions of uniqueness and existence of the solutions of mixed boundary value problem for hyperbolic systems (of partial differential equations of second order) are discussed. The obtained results generalize theorems proved by O.A.Ładyzhenskaya in [1] to the case of hyperbolic systems. Notation used throughout the paper and formulation of the problem are given in Section 2. Uniqueness is proved in Section 3, whereas the existence of the solution and a method of its construction are considered in Section 4.

2. Formulation of the problem

We shall use the following notations: $x = (x_0, x_1, \dots, x_m)$ is a point of $(m+1)$ - dimensional space, where x_0 is time-coordinate. The remaining space coordinates are briefly denoted by $\bar{x} = (x_1, \dots, x_m) \in R^m$. Let Ω be an open bounded region in R^m and let $\partial\Omega$ be its boundary. We denote by $Q = \Omega \times (0,a)$ the set of points $x = (x_0, \bar{x})$ and by S its lateral surface.

The system of differential equations considered in this paper is of the form

$$(1)\quad \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij} u_{x_j}\right) + \sum_{i=1}^{m} A_i(x) u_{x_i} + A_0 u_{x_0} + Au - u_{x_0 x_0} = F(x)$$

where $u(x) = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix}$ is a vector function.

The matrix of scalar coefficients $a_{ij}(x)$ is symmetrical, $a_{ij}(x) = a_{ji}(x)$, and the functions $a_{ij}(x)$ are supposed to be continuous and to have continuous and bounded generalized derivatives $\frac{\partial a_{ij}}{\partial x_i}$ $(i,j=0,\dots,m)$. The elements of square $n \times n$ matrices A, A_i, $(i=0,1,\dots,m)$ are composed of functions of x. The elements of A_i are supposed to be bounded and to have measurable and bounded generalized derivatives.

The right hand side of (1)

$$F(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{bmatrix}$$

is a vector function of the point x; $F_i(x)$ are square integrable over Q. We write: $F_i(x) \in L_2(Q)$, $(i = 1,2,\dots n)$.

In further considerations we shall omit the summation sign in (1) and we assume the summation convention over repeated indices. Thus

$$A_i(x)u_{x_i} = \sum_{i=1}^{m} A_i u_{x_i}$$

and so on.

In the sequel we shall consider various functional spaces. Thus $L_2(\Omega)$ and $L_2(Q)$ are spaces of square integrable functions, over Ω and Q, respectively. After defining weak derivatives we shall define the corresponding spaces $W_p^{(k)}$ of differentiable functions. We shall denote by a dot $\dot{D}(\Omega)$, $\dot{D}_1(Q)$, $\dot{D}_2(Q)$ the spaces of functions with compact support and by a circle $\mathring{D}(\Omega)$, $\mathring{D}_1(Q)$, $\mathring{D}_2(Q)$ the spaces obtained from $\dot{D}(\Omega)$, $\dot{D}_1(Q)$ and $\dot{D}_2(Q)$ by completion in the corresponding norm.

The norm in the space $L_2(\Omega)$ is defined by the formula

$$\|f\|_{L_2(\Omega)} = \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}$$

or more generally in $L_p(\Omega)$

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}$$

The function $\omega_{k_1,\dots,k_m}$ integrable over each subregion Ω_1 of Ω is said to be the generalized derivative of a function $\varphi(x)$, of order $k=k_1+\dots+k_m$ and of the form $\frac{\partial^k}{\partial x_1^{k_1}\dots\partial x_m^{k_m}}$, if and only if for each differentiable, in classical meaning, function ψ vanishing together with all derivatives up to order k in close vicinity of the boundary of Ω, there is

$$\int_{\Omega_1}\left[\varphi\frac{\partial^k\psi}{\partial x_1^{k_1}\dots\partial x_m^{k_m}}+(-1)^{k+1}\psi\omega_{k_1,\dots,k_m}\right]d\Omega_1 = 0$$

where $k=k_1+\dots+k_m$.

Generalized derivatives in the region Q are defined in an analogous way.

Let us consider the set of all functions φ which have in $\bar{\Omega}$ continuous generalized derivatives with respect to $x_1,\dots,x_m$ up to order k inclusively, and introduce the norm in this set by the formula

$$\|\varphi\|_{W_p^{(k)}(\Omega)}=\left\{\int_\Omega\left[\sum_{i=0}^{k}\sum_{\alpha_1,\dots,\alpha_n=1}^{n}\left(\frac{\partial^i\varphi}{\partial x_{\alpha_1}\dots\partial x_{\alpha_i}}\right)^2\right]^{\frac{p}{2}}d\Omega\right\}^{\frac{1}{p}}$$

in which $p>1$.

The space derived from this set by the completion in the norm indicated above is denoted $W_p^{(k)}(\Omega)$. Thus $\varphi(\bar{x})\in W_p^{(k)}(\Omega)$ means that $\varphi(\bar{x})$ has generalized derivatives up to order k, integrable in power p. For example $W_2^{(1)}(\Omega)$ is the set of square integrable functions having in Ω square integrable generalized derivatives of the first order, that is

$$\|f\|_{W_2^{(1)}(\Omega)}=\left[\int_\Omega\left(|f|^2+\sum_{i=1}^{m}\left|\frac{\partial f}{\partial x_i}\right|^2\right)d\Omega\right]^{\frac{1}{2}}.$$

The space $W_p^{(1)}(\Omega)$ is defined analogously.

Let $\dot{D}(\Omega)$ be the set of all functions continuous and differentiable in Ω which vanish in close vicinity of $\partial\Omega$, that is, which vanish in a ring-like set of points, the distance of which from $\partial\Omega$ is less than $\varepsilon > 0$. By completing $\dot{D}(\Omega)$ in the norm of $W_2^{(1)}(\Omega)$ we shall derive $\mathring{D}(\Omega)$, or a class. Since the norm used is that of $W_2^{(1)}(\Omega)$, there is the inclusion $\mathring{D}(\Omega) \subset W_2^{(1)}(\Omega)$ and hence $\mathring{D}(\Omega)$ is a subspace of $W_2^{(1)}(\Omega)$. In an analogous manner we introduce the space $D_1(Q)$ of all functions continuous and differentiable in Q, vanishing for $x \in \overline{Q_\delta} = \Omega_\delta \times [0,a]$, where Ω_δ is the set of points of Ω, the distance of which from $\partial\Omega$ does not exceed δ. Completing $\dot{D}_1(Q)$ in the norm of $W_2^{(1)}(Q)$ we obtain $\mathring{D}_1(Q)$. Now let us consider the set $\dot{D}_2(Q)$ of such functions in $\mathring{D}_1(Q)$, which vanish for each $x_0 \in [a-\varepsilon, a]$, where ε is an arbitrary positive number not exceeding a. Thus $\dot{D}_2(Q)$ is the set of all functions having first derivatives and vanishing in a close vicinity of upper and lateral sides of the cylindrical region Q. The completion of $\dot{D}_2(Q)$ in the norm of $W_2^{(1)}$ leads to $\mathring{D}_2(Q)$.

We say that a vector of a matrix belongs to a space, if their components are elements of this space.

With the notation introduced above, the problem considered in this paper may be formulated as follows:

A solution of the system (1) is required which satisfies the following mixed boundary value conditions:

$$\left[\begin{array}{l} \left.\begin{array}{l} u(0,\bar{x}) = \varphi(\bar{x}) \\ u_{x_0}(0,\bar{x}) = \psi(\bar{x}) \end{array}\right\} \text{ for } \bar{x} \in \Omega \\ u(x) = 0 \quad \text{for } x \in S \end{array}\right. \tag{2}$$

where

$$\varphi(\bar{x}) = \begin{bmatrix} \varphi_1(\bar{x}) \\ \vdots \\ \varphi_n(\bar{x}) \end{bmatrix} \in \mathring{D}(\Omega) \qquad \text{and} \qquad \psi(\bar{x}) = \begin{bmatrix} \psi_1(\bar{x}) \\ \vdots \\ \psi_n(\bar{x}) \end{bmatrix} \in L_2(\Omega).$$

As we shall prove in the paper, the considered problem has a weak solution. A weak solution of (1), (2) is defined as follows. First observe, that introducing the function

$$(3) \qquad \Phi(x) \overset{\text{def}}{=} \begin{Bmatrix} 0 & \text{for } b \leqslant x_0 \leqslant a \\ \int_b^{x_0} u(z,\bar{x})dz & \text{for } 0 \leqslant x_0 \leqslant b \end{Bmatrix} \in \mathring{D}_2(Q)$$

and taking the scalar product of (1) and (3), after performing integration over Q according to (2), we arrive at the equality

$$(4) \qquad \begin{aligned} &\int_Q \left[\Phi^T_{x_0} u_{x_0} - a_{ij}\Phi^T_{x_j} u_{x_i} + \Phi^T\left(A_i u_{x_i} + Au - F\right)\right]dx + \\ &+ \int_\Omega \Phi^T(0,\bar{x})\,\psi(\bar{x})d\bar{x} = 0 \end{aligned}$$

where

$$(5) \qquad \Phi^2 \overset{\text{def}}{=} \Phi^T \cdot \Phi = \Phi_1^2 + \ldots + \Phi_n^2 .$$

According to this equality, a function $u \in D_1(Q)$ is said to be a weak solution of (1), (2), if for an arbitrary vector function

$$\Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} \in \mathring{D}_2(Q);$$

the identity

$$\int_Q \left[\Phi^T_{x_0} u_{x_0} - a_{ij}\Phi^T_{x_j} u_{x_i} + \Phi^T\left(A_i u_{x_i} + Au - F\right)\right] dx + \int_\Omega \Phi^T(0,\bar{x})\,\psi(\bar{x})d\bar{x} = 0$$

holds true and the condition $u(0,\bar{x}) = \varphi(\bar{x})$ is satisfied almost everywhere.

3. Uniqueness of weak solution

We shall prove the following theorem.

T h e o r e m o n u n i q u e n e s s. A weak solution of the problem (1), (2) is unique.

P r o o f. Let us assume that there exist two different solutions $u_{(1)}(x)$ and $u_{(2)}(x)$ and let $u(x) = u_{(1)}(x)-u_{(2)}(x)$. The vector function $u(x)$ satisfies the equality

$$\int_Q \left[\Phi^T_{x_0} u_{x_0} - a_{ij}\Phi^T_{x_j} u_{x_i} + \Phi^T\left(A_i u_{x_i} + Au\right)\right] dx = 0$$

and the condition $u(0,\bar{x}) = 0$.
If we take

$$\Phi(x) = \left.\begin{cases} 0 & \text{for } b \leqslant x_0 \leqslant a \\ \int_b^{x_0} u(z,\bar{x})dz & \text{for } 0 \leqslant x_0 \leqslant b \end{cases}\right\} \in \mathring{D}_2(Q)$$

then

$$\int_Q \left[\Phi^T_{x_0} \Phi_{x_0 x_0} - a_{ij}\Phi^T_{x_j} \Phi_{x_0 x_i} + \Phi^T\left(A_i \Phi_{x_0 x_i} + A\Phi_{x_0}\right)\right] dx = 0.$$

Making use of the formulas

$$\frac{\partial}{\partial x}\Phi^2_{x_0} = \frac{\partial}{\partial x_0}\left(\Phi^T_{x_0}\Phi_{x_0}\right) =$$

(6)

$$= \Phi^T_{x_0 x_0}\Phi_{x_0} + \Phi^T_{x_0}\Phi_{x_0 x_0} = 2\Phi^T_{x_0}\Phi_{x_0 x_0}$$

and

$$(7)\quad \frac{\partial}{\partial x_0}\left(a_{ij}\Phi^T_{x_i}\Phi_{x_j}\right) = 2\left[a_{ij}\Phi^T_{x_i}\Phi_{x_0x_j} + \left(a_{ij}\right)_{x_0}\Phi^T_{x_i}\Phi_{x_j}\right]$$

we obtain

$$\int_Q\left[\frac{1}{2}\left(\Phi_{x_0}\right)^2_{x_0} - \frac{1}{2}\left(a_{ij}\Phi^T_{x_i}\Phi_{x_j}\right)_{x_0} + \frac{1}{2}\frac{\partial a_{ij}}{\partial x_0}\Phi^T_{x_i}\Phi_{x_j} + \right.$$

(8)

$$\left. + \Phi^T\left(A_i\Phi_{x_0x_i} + A\Phi_{x_0}\right)\right]dx = 0.$$

Taking into account the definition of $\Phi(x)$ we can write

$$\int_Q\frac{1}{2}\left(\Phi^2_{x_0}\right)_{x_0}dx = \frac{1}{2}\int_\Omega d\bar{x}\int_0^b\left(\Phi^2_{x_0}\right)_{x_0}dx_0 = \frac{1}{2}\int_\Omega\Phi^2_{x_0}\Big|_0^b d\bar{x} =$$

(9)

$$= \frac{1}{2}\int_\Omega\Phi_{x_0}(b,\bar{x})d\bar{x}.$$

Moreover, we have

$$\frac{1}{2}\int_Q\left(a_{ij}\Phi^T_{x_i}\Phi_{x_j}\right)_{x_0}dx = \frac{1}{2}\int_\Omega d\bar{x}\int_0^b\left(a_{ij}\Phi^T_{x_i}\Phi_{x_j}\right)_{x_0}dx_0 =$$

(10)

$$= \frac{1}{2}\int_\Omega a_{ij}\Phi^T_{x_i}\Phi_{x_j}\Big|_0^b d\bar{x} = -\frac{1}{2}\int_\Omega a_{ij}(0,\bar{x})\Phi^T_{x_i}(0,\bar{x})\Phi_{x_j}(\ ,\bar{x})d\bar{x}.$$

By the theorem of Gauss it follows that

$$\int_Q\Phi^T A_i\Phi_{x_0x_i}dx =$$

$$= -\int_Q \left(\overset{T}{\Phi} A_i\right)_{x_i} \Phi_{x_0} dx - \int_{\partial Q} \overset{T}{\Phi} A_i \Phi_{x_0} \cos(n, x_i) d\sigma =$$

(11)

$$= -\int_Q \left(\overset{T}{\Phi}_{x_i} A_i \Phi_{x_0} + \overset{T}{\Phi}\left(A_i\right)_{x_i} \Phi_{x_0}\right) dx ,$$

since the integral over ∂Q vanishes for $x_0 = 0$.
Hence we obtain

$$\frac{1}{2}\int_\Omega \left[\Phi^2(b,\bar{x}) + a_{ij}(0,\bar{x}) \overset{T}{\Phi}_{x_i}(0,\bar{x}) \Phi_{x_j}(0,\bar{x})\right] d\bar{x} =$$

$$= \int_Q \left[-\frac{1}{2}(a_{ij})_{x_0} \overset{T}{\Phi}_{x_i} \Phi_{x_j} + \sum_{i=0}^{m}\left(\overset{T}{\Phi}_{x_i} A_i \Phi_{x_0} + \overset{T}{\Phi}(A_i)_{x_i} \Phi_{x_0}\right) - \overset{T}{\Phi} A \Phi_{x_0}\right] dx.$$

Next, from the inequality

(12) $$\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \geqslant \nu \sum_{i=1}^{m} \xi_i^2 , \qquad \nu > 0$$

and from the assumption that the matrices $(A_i)_{x_j}, A_i$ $(ij= =0,1,\dots,m)$, A, and derivatives $(a_{ij})_{x_0}$ for $(i,j=1,\dots,m)$ are bounded it follows that under the proper choice of constant C we have

$$\int_\Omega \left[\Phi_{x_0}(b,\bar{x}) + \nu \Phi_{x_i}^2(0,\bar{x})\right] d\bar{x} = C\int_\Omega \left[\Phi^2 + \sum_{i=0}^{m} \Phi_{x_i}^2\right] d\bar{x} .$$

From the definition of the vector function Φ we infer that

$$\int_\Omega \left[u^2(b,\bar{x}) + \nu\left(\int_b^{x_0} u_{x_i} dz\right)^2\right] d\bar{x} \leqslant$$

$$\leqslant \int_\Omega d\bar{x} \int_0^b \left[\left(\int_b^{x_0} u dz\right)^2 + u^2 + \sum_{i=1}^{m}\left(\int_b^{x_0} u_{x_i} dz\right)^2\right] dx_0 .$$

Denoting $v_i(x) \overset{\text{def}}{=} \int_{x_0}^{0} \frac{\partial u}{\partial x_i}\, dz$, we see, that

(13) $$v_i(b,\bar{x}) = \int_b^0 \frac{\partial u}{\partial x_i}\, dz, \int_b^{x_0} \frac{\partial u}{\partial x_i}\, dz = v_i(b,\bar{x}) - v_i(x)$$

and taking into account the estimate

$$\int_0^b \left(\int_b^{x_0} u dz\right)^2 dx_0 \leqslant \int_0^b \left[(b-x_0)\int_{x_0}^b u^2 dx_0\right] dx_0 \leqslant b^2 \int_0^b u^2 dx_0$$

we find

$$\int_\Omega \left[u^2(b,\bar{x}) + (\nu - c_1 b)\sum_{i=1}^m v_i^2(b,\bar{x})\right] d\bar{x} \leqslant$$

$$\leqslant c_1 \int_\Omega d\bar{x} \int_0^b \left[u^2(x) + \sum_{i=1}^m v_i^2(x)\right] dx_0 .$$

If we take b sufficiently small such that $\nu - c_1 b \geqslant \frac{\nu}{2}$ and if $c_2 = \max\left(c_1, \frac{2c_1}{\nu}\right)$, then

(14) $$\int_\Omega \left[u^2(b,\bar{x}) + \sum_{i=1}^m v_i^2(b,\bar{x})\right] d\bar{x} \leqslant c_2 \int_\Omega d\bar{x} \int_0^b \left(u^2 + \sum_{i=1}^m v_i^2\right) dx_0 .$$

Introducing the function

(15) $$y(b) = \int_\Omega d\bar{x} \int_0^b \left(u^2 + \sum_{i=1}^m v_i^2\right) dx_0$$

we obtain

(16) $$\frac{dy(b)}{db} \leqslant c_2 y(b) .$$

After multiplying both sides of (16) by $e^{-c_2 b}$ we obtain

(17) $$\frac{d}{db}\left[e^{-c_2 b}\, y(b)\right] \leqslant 0 .$$

Thus the function $e^{-c_2 b}\,y(b)$ does not increase and (17) implies that $y(b)e^{-c_2 b} \leqslant y(0)$; and consequently $y(b) = 0$ since we have $y(0) = 0$.

From the definition of $y(b)$ we obtain

$$u^2(x) + \sum_{i=1}^{m} v_i^2(x) = 0 \tag{18}$$

and consequently $u(x) \equiv 0$ for $0 \leqslant x_0 \leqslant \frac{\nu}{2c_1}$.

Continuing this procedure we shall prove that

$$u(x) = \begin{bmatrix} u_1(x) \\ \cdot \\ \vdots \\ \cdot \\ u_n(x) \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \cdot \\ \vdots \\ \cdot \\ 0 \end{bmatrix} \qquad \text{for} \qquad \frac{\nu}{2c_1} \leqslant x_0 \leqslant \frac{\nu}{c_1}$$

and so forth up to exhaustion of the set $[0,a]$, q.e.d.

4. Existence of weak solutions

Now we are going to present a method of construction of weak solutions of the problem (1),(2) under the following assumptions: vector functions $\varphi(\bar{x}) - \psi(\bar{x})$ are supposed to be continuous in $\bar{\Omega}$ and vanishing near the boundary $\partial\Omega$ of the region Ω. We assume continuity of $F(x)$, $A_i(x)$ $(i=0,1,\ldots,m)$, $A(x)$ in Q. The coefficients $a_{ij}(x)$ and their derivatives $\frac{\partial a_{ij}}{\partial x_j}$ $(i,j=1,\ldots,m)$ should be continuos in Q.

Our construction is based on the corresponding finite-difference scheme. This scheme is formed by taking in $\bar{x} = (x_1,\ldots,x_n)$ - space hyperplanes $x_i = k_i \Delta x_i$, where k_i are integers $(i=0,1,\ldots,m)$.

The intersection points $(k_0\Delta x_0,\ k_1\Delta x_1,\ldots,k_m\Delta x_m)$ of these hyperplanes will be taken as nodal points of a mesh. We denote by Ω_h the set of all nodal points which belong to the open set Ω ; $Q_h = \left[0, r\Delta x_0\right] \times \Omega_h$, where r is the entire part of $\frac{a}{\Delta x_0}$, and a denotes the step of the mesh. The nodal points of the boundary of Ω_h will be denoted by Ω_h, and we write $S_h = \left[0, r\Delta x_0\right]$. From now on nodal points will be written briefly $x = (x_0, \bar{x}) = (x_0 x_1, \ldots, x_m)$.

Let $v(x), w(x)$ be two vector functions of the nodal points of the mesh. We shall introduce the following notation (see [2]):

$$
\begin{aligned}
l^i v(x) = \frac{1}{\Delta x_i} \Big[& v(x_0, x_1, \ldots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \ldots, x_m) - \\
& - v(x_0, x_1, \ldots, x_m) \Big]
\end{aligned}
\tag{19}
$$

or in a more concise form $l^i v = \left[v(x_i + \Delta x_i) - v(x_i) \right] : \Delta x_i$. In an analogous way

$$
l_i v = \frac{1}{\Delta x_i} \left[v(x_i) - v(x_i - \Delta x_i) \right] .
$$

The following identities may be verified easily

$$
\begin{aligned}
& l^i(w^T v) \equiv (l^i w^T) v + w^T (x_i + \Delta x_i) l^i v \\
& l_i(w^T v) \equiv (l_i w^T) v + w^T (x_i - \Delta x_i) l_i v .
\end{aligned}
\tag{20}
$$

Let us verify the first one

$$
l^i(w^T v) = \frac{1}{\Delta x_i} \left(w^T(x_i + \Delta x_i) v(x_i + \Delta x_i) - w^T v \right) =
$$

$$
= \frac{1}{\Delta x_i} \left[w^T(x_i + \Delta x_i) v(x_i + \Delta x_i) - w^T v + w^T(x_i + \Delta x_i) v - w^T(x_i + \Delta x_i) v \right] =
$$

$$= w^T(x_i+\Delta x_i)\ \underbrace{\frac{v(x_i+\Delta x_i)-v(x_i)}{\Delta x_i}}_{l^i v} + \underbrace{\frac{w^T(x_i+\Delta x_i)-w^T(x_i)}{\Delta x_i}}_{l^i w^T}\ v(x_i) =$$

$$= (l^i w^T)v + w^T l^i v ,$$

q.e.d. The proof of the second identity is similar.

The application of the notation introduced above makes it possible to replace (1) and (2) by the following finite-difference scheme

$$(1') \quad l^i a_{ij}(x) l_j u(x) + \sum_{i=0}^{m} A_i(x) l_i u(x) + A(x)u(x) - l^o l_o u(x) = F(x)$$

$$(2') \quad u(\bar{x},0) = \varphi(\bar{x}),\ u(\bar{x},\Delta x_o) = \varphi(\bar{x}) + \Delta x_o\,\psi(\bar{x}) \quad \text{for} \quad \bar{x}\in\Omega$$

$$u(x) = 0 \qquad \text{for} \quad x \in S_h .$$

First we shall prove that the solution $u(x)$ of a finite-difference scheme, defined at nodal points Q_h, (xQ_h-S) is unique.

The values of coefficients of the system (1′) at the nodes placed outside of Q_h and neighbouring with S_h will be defined through their values taken at the closed points lying on S. Further we assume that $u(x) = 0$ at all nodes x of a layer $0\leqslant x_o \leqslant r\Delta x_o$, which does not belong to Q_h. Taking scalar product of vector finite-difference equation (1′) with $l_o u^T(x) + l^o u^T(x)$ and taking into account that

$$(l_o u^T + l^o u^T) l^o l_o u \equiv l^o (l_o u)^2$$

and

$$(l_o u^T + l^o u^T) l^i a_{ij} l_i u = l^i (l_o u^T + l^o u^T) a_{ij} l_i u - l^i (l_o u^T) \left\{ a_{ij} l_i u \right\}_{(x_i+\Delta x_i)}$$

we obtain

$$
\begin{aligned}
&l^i(l_o u^T + l^o u^T) a_{ij} l_j u - l^i(l_o u^T + l_o u^T)\cdot \Big\{ a_{ij} l_j u \Big\}_{(x_i+\Delta x_i)} + \\
(21)\qquad & + (l_o + l^o) u^T \left(\sum_{i=0}^{n} A_i l_i u + Au - F \right) - l^o (l_o u)^2 = 0
\end{aligned}
$$

where $\{\cdot\}_{(x_i+\Delta x_i)}$ denoting the value of $\{\cdot\}$ is calculated at the point $(x_o, x_1, \ldots, x_{i-1}, x_i+h, x_{i+1}, \ldots, x_m)$. Assuming $h=\Delta x_1=\Delta x_2=\ldots=\Delta x_m$ and multiplying both sides of (21) by $h^m \Delta x_o$ and performing summation we arrive at

$$
\begin{aligned}
&\sum_{j=1}^{p-1} \Delta x_o \sum_{\Omega_h} h^m \Big[l^i(l_o u^T + l^o u^T) a_{ij} l_i u) - l^i(l_o u^T + l_o u^T) \Big\{ a_{ij} l_j u \Big\}_{(x_i+h)} + \\
(22)\qquad & + (l_o u^T + l^o u^T) \left(\sum_{i=0}^{m} A_i l_i u + Au - F \right) - l^i (l_o u)^2 \Big] = 0
\end{aligned}
$$

where $\sum_{\Omega_h}(\cdot)$ – denotes the sum of values of $(\cdot)$ at the nodal points of Ω_h, $\sum_{j=1}^{p-1}(\cdot)$ – the sum of values of $(\cdot)$ for $x_o = \Delta x_o,\ 2\Delta x_o, \ldots, (p-1)\Delta x_o$, $(p \leqslant r)$.
Making use of the equalities

$$
\sum_{\Omega_h} \Big[l^i(l_o u^T + l^o u^T) a_{ij} l_j u \Big] = 0
$$

$$
\sum_{\Omega_h} l^i(l_o u^T + l^o u^T)\cdot \Big\{ a_{jj} l_j u \Big\}_{(x_i+h)} =
$$

$$
= \sum_{\Omega_h} \Big\{ l_i(l_o u^T + l^o u^T) a_{ij} l_j u \Big\}_{(x_i+h)} = \sum_{\Omega_h} l_i(l_o u^T + l^o u^T) a_{ij} l_j u
$$

and

$$
\sum_{s=1}^{p-1} l^o (l_o u)^2 = \frac{1}{\Delta x_o} (l_o u)^2 \Bigg|_{x_o=\Delta x_o}^{x_o=p\Delta x_o}
$$

we obtain

$$\sum_{s=1}^{p-1} \Delta x_o \sum_{\Omega_h} h^m l_i (l_o u^T + l^o u^T) a_{ij} l_j u - \sum_{s=1}^{p-1} \Delta x_o \sum_{\Omega_h} h^m (l_o u^T + \tag{23}$$

$$+ l^o u^T)\left(\sum_{i=0}^{n} A_i l_i u + Au - F\right) + \sum_{\Omega_h} h^m (l_o u)^2 \Bigg|_{x_o = \Delta x_o}^{x_o = p\Delta x_o} = 0.$$

Now, making use of identity (20) we arrive at two equalities

$$\text{a)} \qquad h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left\{ l_o \left[a_{ij} (l_i u^T) \cdot l_j u \right] \right\}_{s\Delta x_0} =$$

$$= h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p-1} \left\{ a_{ij} (l_i l_o u^T + l_i l_o u^T) l_j u \right\}_{s\Delta x_0} + \tag{24}$$

$$+ \left\{ l^o a_{ij} \right\}_{(s-1)\Delta x_0} \left\{ l_i u^T \right\}_{s\Delta x_0} \left\{ l_j u \right\}_{(s-1)\Delta x_0} +$$

$$+ h^m \sum_{\Omega_h} \Delta x_o \left[\left\{ a_{ij} l_i u^T l_j u \right\}_{p\Delta x_0} + \left\{ a_{ij} l^o l^i u^T l_j u \right\}_{x_0 = 0} \right]$$

$$\text{b)} \qquad h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left\{ l_o (a_{ij} l_i u^T \cdot l_j u \right\}_{s\Delta x_0} =$$

$$= h^m \sum_{\Omega_h} \left[\left\{ a_{ij} l_i u^T l_j u \right\}_{p\Delta x_0} - \left\{ a_{jj} l_i u^T l_j u \right\}_{x_0 = 0} \right].$$

Substraction of both sides of these identities leads to an expression which vanishes identically. Addition of this expression to the left side of (23) yields

$$h^m \sum_{\Omega_h} \left\{ (l_o u)^2 + a_{ij} l_i u^T l_j u \right\}_{p\Delta x_0} = h^m \sum_{\Omega_h} \left\{ (l^o u)^2 + a_{ij} l_i u^T l_j u \right\}_{x_0 = 0} +$$

$$+ h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p-1} \left(l_o u^T + l^o u^T \right) \cdot \left(\sum_{i=0}^{m} A_i l_i u + Au - F \right) +$$

$$+\Delta x_o h^m \sum_{\Omega_h}\left\{a_{ij}l_iu^Tl_ol_ju\right\}_{p\Delta x_0} + h^m\Delta x_o\sum_{\Omega_h}\left\{a_{ij}l^ol_iu^Tl_ju\right\}_{x_0=0} +$$

(25)

$$+ h^m\sum_{\Omega_h}\Delta x_o\sum_{s=1}^{p-1}\left\{l^oa_{ij}\right\}_{(s-1)\Delta x_0}\left\{l_iu^T\right\}_{s\Delta x_0}\cdot\left\{l^ju\right\}_{(s-1)\Delta x_0}.$$

Denoting $\gamma_1 = \max\limits_{ij}\ \max\limits_{Q}\ |a_{ij}|$

$$\gamma = \max_{ijk}\ \max_{Q}\left(\left|a_{ij}^{(k)}\right|,\ \left|a_{ij}^{o}\right|,\left|\frac{\partial a_{ij}}{\partial x_o}\right|\right)$$

where $a_{ij}^{(k)}$ are elements of matrix A_k, and a_{ij}^{o} are elements of matrix A_o, $\varkappa = \dfrac{\Delta x_o}{h}$ and applying the estimate

$$\Delta x_o h^m \sum_{\Omega_h}\left\{a_{ij}l_iu^Tl_ol_ju\right\}_{p\Delta x_0} \leqslant$$

$$\leqslant h^m\gamma_1\varkappa\sum_{\Omega_h}\left\{\sum_{j=1}^{m}a_{ij}l_iu^T\left(l_ou-l_ou(x_j-h)\right)\right\}_{p\Delta x_0} \leqslant$$

$$\leqslant\ h^m\gamma_1 m\sum_{\Omega_h}\left\{\sum_{j=1}^{m}(l_iu)^2 + m(l_ou)^2\right\}_{p\Delta x_0} \leqslant$$

$$\leqslant\ h^m\gamma_1\varkappa^2\ m\sum_{\Omega_h}\left\{\sum_{i=0}^{m}(l_iu)^2\right\}_{p\Delta x_0};$$

(26)

$$h^m\Delta x_o\sum_{\Omega_h}\left\{a_{ij}l^ol_iu^Tl_ju\right\}_{x_0=0} +$$

$$- h^m\sum_{\Omega_h}\sum_{j=1}^{m}\left\{a_{ij}\left(l^ou^T - l^ou^T(x_o-h)\right)l_ju\right\}_{x_0=0} \leqslant$$

$$\leqslant\ h^m\gamma_1\varkappa\, m\sum_{\Omega_h}\left\{\sum_{i=1}^{m}(l_iu)^2 + m(l^ou)^2\right\}_{x_0=0} \leqslant$$

$$\leqslant h^m\sum_{\Omega_h}\Delta x_o\sum_{s=1}^{p-1}\left\{l^oa_{ij}\right\}_{(s-1)\Delta x_0}\left\{l_iu^T\right\}_{s\Delta x_0}\left\{l^ju\right\}_{(s-1)\Delta x_0} \leqslant$$

$$\leqslant h^m \gamma m \sum_{\Omega_h} \sum_{s=1}^{p-1} \Delta x_o \sum_{i=1}^{m} \left\{(l_i u)^2\right\}_{s\Delta x_0} +$$

$$+ \frac{m}{2} \gamma h^m \Delta x_o \sum_{\Omega_h} \sum_{i=1}^{m} \left\{(l_j u)^2\right\}_{x_0=0} \leqslant$$

$$\leqslant h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p-1} \cdot (A_i l_i u + A_o l_o u + Au - F) \leqslant$$

$$\leqslant c_1 h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{n} \left\{ \sum_{i=0}^{m} (l_i u)^2 + u^2 + F^2 \right\}_{s\Delta x_0}$$

where constant c_1 depends on the dimension of the space and on the elements of the matrices A_i and A, we arrive at

$$h^m \sum_{\Omega_h} \left\{(l_o u)^2 + a_{ij} l_i u^T \cdot l_j u\right\}_{p\Delta x_0} \leqslant$$

$$\leqslant h^m \sum_{\Omega_n} \left\{ (l_o u)^2 + a_{ij} l_i u^T l_j u \right\}_{x_0=0} +$$

$$+ c_1 h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left\{ \sum_{i=0}^{m} (l_i u)^2 + u^2 + F^2 \right\}_{s\Delta x_0} +$$

(27) $$+ h^m \gamma_1 \varkappa m \sum_{\Omega_h} \left\{ \sum_{i=1}^{m} (l_i u)^2 + m(l_o u)^2 \right\}_{p\Delta x_0} +$$

$$+ h^m \gamma_1 \varkappa m \sum_{\Omega_h} \left\{ \sum_{i=1}^{m} (l_i u^2) + m(l^o u^2)_{x_0=0} \right. +$$

$$\left. + h^m \cdot \gamma m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p-1} \sum_{s=1}^{m} (l_i u)^2 \right\}_{s\Delta x_0} + \frac{m}{2} \gamma h^m \Delta x_o \sum_{\Omega_h} \sum_{s=1}^{m} \left\{(l_i u)^2\right\}_{x_0=0} .$$

Applying the inequality $a_{ij}\, \xi_i\, \xi_j \leqslant \nu \xi^2$ and transferring the third term to the left side, we obtain

$$h^m \sum_{\Omega_h} \left\{(l_o u)^2 (1-\gamma_1 \varkappa m^2) + \sum_{i=1}^{m} (l_i u)^2 (\nu - \gamma_1 \varkappa m)\right\}_{p\Delta x_0} \leqslant$$

$$h^m \sum_{\Omega_h} \left\{ (l^o u)^2 + a_{ij} l_i u^T l_j u + m\gamma \Delta x_o h^m \sum_{i=1}^{m} (l_i u)^2 \right\}_{x_0=0} +$$
$$+ c_2 h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left\{ \sum_{i=0}^{m} (l_i u)^2 + u^2 + F^2 \right\}_{x_0 = s\Delta x_0} .$$

If we choose $\varkappa = \frac{\Delta x_o}{h}$ in such a way that $1 - \gamma_1 \varkappa m^2 \geq \nu_1 > 0$ and $\nu - \gamma_1 \varkappa m \geq \nu_1 > 0$, and take into account the estimate

$$\sum_{\Omega_h} u^2(x) \leqslant d_o^2 \sum_{\Omega_h} (l_i u)^2 \tag{28}$$

where d_o is the diamater of the set Ω_h, we obtain

$$h^m \sum_{\Omega_h} \sum_{i=0}^{m} (l_i u)^2 \leqslant \frac{1}{\nu_1} S_o + c_3 h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left\{ \sum_{i=0}^{m} (l_i u)^2 + t^2 \right\}_{s\Delta x_0} \tag{29}$$

where

$$S_o = h^m \sum_{\Omega_h} \left\{ (l^o u)^2 + a_{ij} l_i u^T l_j u + m\gamma \Delta x_o h^m \sum_{i=1}^{m} (l_i u)^2 \right\}_{x_0=0} .$$

Now let us introduce

$$y(p) = h^m \Delta x_o \sum_{s=1}^{p} \sum_{\Omega_h} \left\{ \sum_{i=0}^{m} (l_i u)^2 \right\}_{s\Delta x_0} \tag{30}$$

and

$$h(p) = \frac{1}{\nu_1} S_o + c_3 h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} F^2(s\Delta x_o) , \tag{31}$$

then inequality (29) will take the form

$$\frac{y(p) - y(p-1)}{\Delta x_o} \leqslant c_3 y(p) + h(p) . \tag{32}$$

After certain manipulations this yields

$$y(p) = \frac{1}{1 - c_3 \Delta x_o} y(p-1) + \frac{\Delta x_o}{1 - c_3 \Delta x_o} h(p) .$$

The definition of $y(p)$ and the conditions of finite-difference problem explain the equality

$$(33) \qquad y(1) = \Delta x_o h^m \sum_{\Omega_h} \left[\psi^2 + \sum_{i=1}^{m} (l_i \varphi + \Delta x_o l_i \psi) \right] .$$

Denoting $E = \frac{1}{1-c_3\Delta x_o}$ and applying the last inequality (p-1) -times we obtain

$$y(p) \leqslant E^{p-1} y(1) + E \Delta x_o \sum_{s=2}^{p} E^{p-s} h(s).$$

If Δx_o is taken so small, that $1-c_3\Delta x_o \geqslant \frac{1}{2}$, then

$$(34) \qquad \begin{aligned} y(p) &\leqslant E^{p-1} y(1) + E^{p-1} \Delta x_o \sum_{s=2}^{p} h(s) \leqslant \\ &\leqslant E^{p-1} \Delta x_o (p-1) h(p) + E^{p-1} y(1) = E^{p-1} \big(ah(p) + y(1) \big) . \end{aligned}$$

Making use of the inequality

$$E^p = \left(1 + \frac{c_3 \Delta x_o}{1-c_3\Delta x_o} \right)^p \exp \frac{pc_3 \Delta x_o}{1-c_3\Delta x_o} \leqslant \exp\left(2c_3 a\right)$$

we obtain

$$y(p) \leqslant e^{2c_3 a} \big(y(1) + ah(r) \big) .$$

Now, the estimates

$$\Delta x_o h^m \sum_{\Omega_h} \left\{ \psi^2 + \sum_{i=1}^{m} (l_i \varphi + \Delta x_o l_i \psi)^2 \right\} =$$

$$= \Delta x_o h^m \sum_{\Omega_h} \left\{ \psi^2 + \sum_{i=1}^{m} \left(l_i \varphi + \Delta x_o \frac{\psi(x_j) - \psi(x_i - h)}{h} \right)^2 \right\}$$

$$\leqslant c_4\Delta x_o h^m \sum_{\Omega_h}\left\{\psi^2 + \sum_{i=1}^{m} (l_i \varphi)^2\right\},$$

$$h^m \sum_{\Omega_h}\left\{(l^o u)^2 + a_{ij} l_i u^T l_j u + m\gamma\Delta x_o h^m \sum_{i=1}^{m} (l_i u)^2\right\}_{x_0=0} =$$

$$= h^m \sum_{\Omega_h}\left\{\psi^2 + a_{ij} l_i \varphi^T l_j \varphi + m\gamma\Delta x_o h^m \sum_{i=1}^{m} (l_i \varphi)^2\right\} \leqslant$$

$$\leqslant c_4 h^m \sum_{\Omega_h}\left\{\psi^2 + \sum_{i=1}^{m} (l_i \varphi)^2\right\}$$

with proper choice of constant c_5 yield

(35)
$$h^m \Delta x_o \sum_{s=1}^{p} \sum_{\Omega_h} \sum_{i=0}^{m}\left\{(l_i u)^2\right\}_{s\Delta x_0} \leqslant$$
$$\leqslant c_5 h^m \sum_{\Omega_h}\left\{\psi^2 + \sum_{i=1}^{m} (l_i \varphi)^2 + \Delta x_o \sum_{s=1}^{p} F^2(s\Delta x_o)\right\}.$$

The application of inequality $h^m \sum_{\Omega_h} u^2 \leqslant h^m \sum_{\Omega_h} \sum_{i=1}^{m} (l_i u)^2$ to the left hand side of the inequality (35) with a proper choice of constant c_6 leads to

(36)
$$h^m \Delta x_o \sum_{s=1}^{p} \sum_{\Omega_h}\left\{\sum_{i=0}^{m} (l_i u)^2 + u^2\right\}_{x_0=s\Delta x_0} \leqslant$$
$$\leqslant c_6\, h^m \sum_{\Omega_h}\left\{\psi^2 + \sum_{i=1}^{m} (l_i \varphi)^2 + \Delta x_o \sum_{s=1}^{p} F^2(s\Delta x_o)\right\},\quad p=1,2,\dots r.$$

The substitution of the right hand side of (29) into the same side of (35) and the application of (28), with a proper choice of the corresponding constant leads to inequality

(37)
$$h^m \sum_{\Omega_h}\left[u^2 + \sum_{i=0}^{m} (l_i u)^2\right] \leqslant$$

$$\leqslant c_7 h^m \sum_{\Omega_h} \left[\psi^2 + \sum_{i=1}^{m} (l_i \varphi)^2 + \Delta x_0 \sum_{s=1}^{p} F^2 \right\} , \qquad p=1,2,\dots,r.$$

The right hand sides of inequalities (36), (37) are commonly bounded with respect to h smaller than a positive constant.

Let h and Δx_0 take, respectively, the values $h_1, h_2, \dots, h_s \dots$ and $\varkappa h_1, \varkappa h_2, \dots, \varkappa h_s, \dots$ which tend to zero as $s \to \infty$.

Under these assumptions the following theorems hold.

T h e o r e m 1. Consider sequence $\{u_h\}$ of functions defined at the nodal points of Ω_h, with the property $h^m \sum_{\Omega_h} \left[u_h^2 + \sum_{i=1} (l_i u_h)^2 \right] \leqslant c$ [1] and u=0 on the boundary of Ω_h. Then, there exists the subsequence $\{u_{h_s}\}$ of such that it converges in the space $L_2(\Omega)$ and weakly converges in $W_2^{(1)}(\Omega)$ to a function $u(x) \in \mathring{D}(\Omega)$ for $\bar{x} \in \Omega$.

T h e o r e m 2. If $h^m \sum_{\Omega_h} u_h^2 \leqslant c$ and either one of the sequences $\{u_h'\}$, $\{u_{(m)}\}$ and $\{\tilde{u}_h\}$ weakly converges in $L_2(\Omega)$ to a function u(x) as $h \to 0$, then also the remaining two of these sequences converge weakly to u(x).

T h e o r e m 3. If the inequality $h^m \sum_{\Omega_h} \sum_{i=1}^{m} (l_i u)^2 \leqslant c$ holds true and if one of three sequences $\{u_h'\}$, $\{u_{(m)}\}$ and $\{\tilde{u}_h\}$ converges in $L_2(\Omega)$ to a function u(x), then also the remaining two sequence converge in $L_2(\Omega)$ to u(x) for $\bar{x} \in \Omega'$, where $\Omega' \subset \Omega$.

T h e o r e m 4. If a function u_h depends on parameter x_0, where $x_0 = s\Delta x_0$, $s=0,1,\dots,\left[\frac{a}{\Delta x_0}\right]+1$, and vanishes on the boundary of Ω_h and outside of Ω_h and if $h^m \Delta x_0 \sum_{\Omega_h} \sum_{s=0} \left[u_h^2 + \sum_{i=0}^{m} (l_i u_h)^2 \right] \leqslant c$ then there exists a subsequence $\{u_{h_s}\}$ which converges in the norm of $L_2(\Omega)$ to a function of the class $\mathring{D}_1(Q)$ uniformly with respect to $x_0 \in [0,a]$ and which converges

[1] c is a positive constant.

weakly in the norm $W_2^{(1)}(Q)$. Uniform convergence in $L_2(\Omega)$ means that:

$$\bigwedge_{\varepsilon>0} \bigvee_{N(\varepsilon)} \bigwedge_{s>N(\varepsilon)} \left\| u_{h_s} - u \right\|_{L_2(\Omega)} < \varepsilon .$$

The functions entering in formulation of these theorems are defined as follows

$$u_h'(x) = u(x_1,\dots,x_m) \prod_{s=1}^{m} (x_s - k_s h) +$$

$$+ \sum_{r=1}^{m} u(x_1,\dots,x_{r-1},x_{r+1},\dots,x_m) \prod_{s=1}^{m} (x_s-k_sh)+\dots+u_h$$

$$u_{(r)}(x) = u(x_1,\dots,x_{r-1},x_{r+1},\dots,x_m) \prod_{\substack{s=1\\ s\neq r}}^{m} (x_s-k_sh)+\dots+u_h$$

$$\tilde{u}_h(x) = u(k_1h,k_2h,\dots,k_mh), \quad k_ih \leqslant x_i < (k_i+1)h .$$

The sequence $\left\{u_{h_s}\right\}$ satisfies the assumptions of Theorem 1, therefore the sequence $\left\{u_{h_s}'\right\}$ converges weakly in the norm of $W_2^{(1)}(\Omega)$ to a function of the class $\hat{D}_1(Q)$. Moreover, this sequence $\left\{u_{h_s}'\right\}$ converges in $L_2(\Omega)$ on each hyperplane $x_0 = \text{const}$. The convergence is uniform with respect to $x_0 \in [0,a]$. Consequently, we infer that $u(x)$ belongs to $L_2(\Omega)$ and depends continuously on x_0, and hence $u(0,\bar{x}) = \varphi(\bar{x})$.

Let us prove that $u(x)$ is a weak solution of the problem (1),(2). We have proved above, that the boundary and initial conditions are satisfied. Thus, we have only to prove that equations (1′) and (2′) are satisfied.

Let $\Phi \in \mathring{D}_2^1(Q)$ and assume h so small that $\Phi = 0$ on S_h; Φ is extended over all semispace $x_0 > 0$.

Let us notice that u_h satisfies the equation

$$h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \Phi^T \left[l^i a_{ij} l_j u + \sum_{i=0}^{m} A_i(x) l_i u + Au - l^o l_o u \right] = \Phi^T F.$$

After transformation we obtain

$$h^m \sum_{\Omega_h} \Delta x_o \sum_{s=1}^{p} \left[a_{ij} l^i \Phi^T l_j u + \Phi^T \sum_{i=0}^{m} A_i l_i u + \right.$$
$$\left. + \Phi^T Au + l^o \Phi^T l^o u + \Phi^T F \right] + h^m \left\{ \sum_{\Omega_h} \Phi^T l^o u \right\}_{x_0=0} = 0 .$$

This equation may be written as an integral relation by application of step functions

$$(38) \quad \int_{\Omega} \int_{0}^{a} \left[(\tilde{l}^o \tilde{\Phi}^T)(\tilde{l}^o \tilde{u}) - \tilde{a}_{ij}(x)(\tilde{l}^i \tilde{\Phi}^T)(\tilde{l}_j \tilde{u}) + \tilde{\Phi}^T \sum_{i=0}^{m} \tilde{A}_i \tilde{l}_i \tilde{u} + \right.$$
$$\left. + \tilde{\Phi}^T \tilde{A} \tilde{u} - \tilde{\Phi}^T \tilde{F} \right] dx + \int_{\Omega} \tilde{\Phi}^T(0,\bar{x}) \tilde{\psi}(\bar{x}) d\bar{x} = 0 .$$

These step functions $\tilde{\Phi}^T, (\tilde{l}^i \tilde{\Phi}^T), \tilde{a}_{ij}, \tilde{A}_i, \tilde{F}, \tilde{\psi}$ are uniformly convergent as $h \to 0$. For $h = h_s$, under assumptions stated above on h, the functions $\tilde{u}$ and $(\tilde{l}^i \tilde{u})$ converge weakly to u and $\frac{\partial u}{\partial x_i}$ (see Theorem 1). After taking limit, the equation (38) leads to

$$(39) \quad \int_{Q} \left[\frac{\partial \Phi^T}{\partial x_o} \frac{\partial u}{\partial x_o} - \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial \Phi^T}{\partial x_i} \frac{\partial u}{\partial x_j} + \Phi^T \sum_{i=0}^{m} A_i \frac{\partial u}{\partial x_i} + \right.$$
$$\left. + \Phi^T Au - \Phi^T F \right] dx + \int_{\Omega} \Phi^T(0,\bar{x}) \psi(\bar{x}) d\bar{x} = 0 .$$

Let $\Phi \in \mathring{D}_2(Q)$, then there exists $\Phi_k \in \mathring{D}_2(Q)$ such that $\| \Phi - \Phi_k \|_{W_2^{(1)}(Q)} \to 0$ as $k \to \infty$ and the convergence is uniform with respect to $x_o \in (0,a)$. For all functions Φ equality (39) holds true, and since $u \in \mathring{D}_1(\Omega)$ we also have $u \in W_2^{(1)}(Q)$.

If the above conditions are satisfied, then the following estimates may be derived

$$
(40) \qquad \left| \int_Q \left[\left(\frac{\partial \Phi^T}{\partial x_0} - \frac{\partial \Phi_k^T}{\partial x_0} \right) \frac{\partial u}{\partial x_0} - a_{ij}(x) \left(\frac{\partial \Phi^T}{\partial x_i} - \frac{\partial \Phi_k^T}{\partial x_i} \right) \frac{\partial u}{\partial x_j} + \right. \right.
$$

$$
\left. + \left(\Phi^T - \Phi_k^T \right) \left(\sum_{i=0}^{m} A_i \frac{\partial u}{\partial x_i} + Au - F \right) \right] dx +
$$

$$
\left. + \int_\Omega \left(\Phi^T - \Phi_k^T \right)_{x_0=0} \psi(\bar{x}) d\bar{x} \right| \leqslant
$$

$$
\leqslant c_8 \left[\|u\|_{W_2^{(1)}(Q)} \|\Phi - \Phi_k\|_{W_2^{(1)}(Q)} + \|\psi\|_{L_2(\Omega)} \|\Phi - \Phi_k\|_{L_2(\Omega)} \right]
$$

where the right hand term converges uniformly to zero as $h \to 0$. From this estimate we conclude that for an arbitrary function $\Phi \in \overset{\circ}{D}_2(Q)$ identity (39) holds true. This fact proves the existence of a weak solution of the problem (1),(2), under assumption that the functions $F(x)$, $\varphi(\bar{x})$, $\frac{\partial \varphi}{\partial x_i}$, $\psi(\bar{x})$ are continuous in Q, Ω, resp., and that φ and ψ vanish in a close vicinity of the boundary of Ω.

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