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CONVERGENCE OF SOME TRIGONOMETRIC SUMS

1. Preliminaries

Let E [resp. E^*] be the class of all real functions $f(t)$ Lebesgue-integrable over any finite interval, such that

$$\frac{1}{T}\int_{T}^{T+c}|f(t)|\,dt = o(1)\,[O(1)] \quad \text{and} \quad \frac{1}{T}\int_{-T-c}^{-T}|f(t)|\,dt = o(1)\,[O(1)]$$

as $T \to \infty$, for every fixed $c > 0$.

Consider an arbitrary function $f \in E^*$ and a positive number l. Write, for any $x \in (-\infty, \infty)$ and $n = 1, 2, \ldots$,

$$S_n^l(x;f) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{l} + b_k\sin\frac{k\pi x}{l}\right),$$

$$\tilde{S}_n^l(x;f) = \sum_{k=1}^{n}\left(a_k\sin\frac{k\pi x}{l} - b_k\cos\frac{k\pi x}{l}\right),$$

$$\sigma_n^l(x;f) = \frac{1}{n}\sum_{m=0}^{n-1} S_m^l(x;f),$$

$$\tilde{\sigma}_n^l(x;f) = \frac{1}{n}\sum_{m=1}^{n-1} \tilde{S}_m^l(x;f),$$

with

$$a_k = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi t}{l}\,dt, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi t}{l}\,dt.$$

An easy calculation shows (see [2], I, pp. 49, 88, 91) that

$$S_n^l(x;f) = \frac{1}{l}\int_{-l}^{l} f(u)D_n^l(u-x)du ,$$

$$\tilde{S}_n^l(x;f) = -\frac{1}{l}\int_{-l}^{l} f(u)\tilde{D}_n^l(u-x)du ,$$

$$\sigma_n^l(x;f) = \frac{1}{l}\int_{-l}^{l} f(u)K_n^l(u-x)du ,$$

$$\tilde{\sigma}_n^l(x;f) = -\frac{1}{l}\int_{-l}^{l} f(u)\tilde{K}_n^l(u-x)du,$$

where

$$D_n^l(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\frac{k\pi t}{l} = \frac{\sin(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}} ,$$

$$\tilde{D}_n^l(t) = \sum_{k=1}^{n} \sin\frac{k\pi t}{l} = \frac{1}{2}\cot\frac{\pi t}{2l} - \frac{\cos(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}} ,$$

$$K_n^l(t) = \frac{1}{n}\sum_{m=0}^{n-1} D_m^l(t) = \frac{1}{2n}\left(\frac{\sin\frac{n\pi t}{2l}}{\sin\frac{\pi t}{2l}}\right)^2 ,$$

$$\tilde{K}_n^l(t) = \frac{1}{n}\sum_{m=0}^{n-1} \tilde{D}_m^l(t) = \frac{1}{2}\cot\frac{\pi t}{2l} - \frac{\sin\frac{n\pi t}{l}}{n(2\sin\frac{\pi t}{2l})^2} .$$

In the paper [1], we gave some criteria of the Dini and Young type concerning the convergence of $S_n^l(x;f)$ as $l \to \infty$, $n \to \infty$ and $l/n \to 0$. Here, the remaining three sums will be examined.

In Sections 3 and 4 the following notation will be used:

$$\psi_x(t) = f(x+t) - f(x-t)\,,$$

$$\tilde{f}^l(x;h) = -\frac{1}{2l}\int_h^l \psi_x(t)\cot\frac{\pi t}{2l}\,dt \qquad (0<h<l)\,,$$

and

$$\tilde{f}^l(x) = -\frac{1}{2l}\int_{\to 0}^l \psi_x(t)\cot\frac{\pi t}{2l}\,dt = \lim_{h\to 0+}\tilde{f}^l(x;h)\,,$$

provided the limit exists (cf. [2], I, p. 51).

2. Convergence of $\sigma_n^l(x;f)$

Given any $f \in E^*$, let us start with the identities

$$\sigma_n^l(x;f) = \frac{1}{2l}\left\{\int_{-l-x}^{l-x} f(x+t)K_n^l(t)dt + \int_{-l+x}^{l+x} f(x-t)K_n^l(t)dt\right\}\,,$$

$$1 = \frac{1}{2l}\left\{\int_{-l-x}^{l-x} K_n^l(t)dt + \int_{-l+x}^{l+x} K_n^l(t)dt\right\}$$

which are valid for $x \in (-\infty,\infty)$, $l>0$, $n = 1, 2, \dots$. Denote by $g(x)$ an arbitrary real number. Then,

$$\sigma_n^l(x;f) - g(x) = \frac{1}{2l}\int_{-l+x}^{l-x}\left\{f(x+t) + f(x-t) - 2g(x)\right\}K_n^l(t)dt +$$

$$+ \frac{1}{2l}\int_{-l-x}^{-l+x}\left\{f(x+t) - g(x)\right\}K_n^l(t)dt +$$

$$+ \frac{1}{2l}\int_{l-x}^{l+x}\left\{f(x-t) - g(x)\right\}K_n^l(t)dt =$$

$$= I_n^l(x) + U_n^l(x) + V_n^l(x), \text{ whenever } l \geqslant x\,.$$

Clearly,

$$I_n^l(x) = \frac{1}{l}\int_0^{l-x}\left\{f(x+t) + f(x-t) - 2g(x)\right\} K_n^l(t)dt =$$

$$= \frac{1}{l}\left(\int_0^l - \int_{l-x}^l\right)\left\{f(x+t) + f(x-t) - 2g(x)\right\} K_n^l(t)dt = J_n^l(x) - R_n^l(x).$$

If $-l < a \leqslant x \leqslant b < l$ and $|g(x)| \leqslant K$ (a, b, K mean constants),

$$\left|R_n^l(x)\right| \leqslant \frac{1}{l}\left|\int_{l-x}^l |f(x+t)|\, K_n^l(t)dt\right| + \frac{1}{l}\left|\int_{l-x}^l |f(x-t)|\, K_n^l(t)dt\right| +$$

$$+ \frac{2}{l}\,|g(x)|\left|\int_{l-x}^l K_n^l(t)dt\right| \leqslant \frac{1}{2n\sin^2\frac{\pi(l-x)}{2l}}\left\{\frac{1}{l}\left|\int_{l-x}^l |f(x+t)|\,dt\right| +\right.$$

$$\left. + \frac{1}{l}\left|\int_{l-x}^l |f(x-t)|\,dt\right| + \frac{2}{l}\,|g(x)|\right\} \leqslant$$

$$\leqslant \frac{1}{2n\sin^2\frac{\pi(l-x)}{2l}}\left\{\frac{1}{l}\left|\int_l^{l+x} |f(u)|\,du\right| + \frac{1}{l}\left|\int_{x-l}^{2x-l} |f(u)|\,du\right| + \frac{2}{l}K\right\}.$$

Hence

$$\lim_{l,n\to\infty} R_n^l(x) = 0, \text{ uniformly in } x \in \langle a, b\rangle .$$

Analogously,

$$\lim_{l,n\to\infty} U_n^l(x) = 0 = \lim_{l,n\to\infty} V_n^l(x) .$$

Collecting the results, we obtain

$$(1) \qquad \sigma_n^l(x;f) - g(x) = J_n^l(x) + o(1) \quad \text{as } l, n \to \infty,$$

uniformly in $x \in \langle a, b\rangle$.

The behaviour of $J_n^l(x)$ will be investigated below.

L e m m a 1. If $f(t)$ is Lebesgue - integrable over every finite interval and if

$$\int_{|t|\geqslant 1} \frac{|f(t)|}{t^2}\, dt < \infty , \tag{2}$$

then, for each real a, b ($a \leqslant b$) and $\delta > 0$,

$$\lim_{1/n \to 0} \frac{1}{l}\int_{\delta}^{l} f(x \pm t) K_n^l(t)\, dt = 0 \qquad (l > \delta),$$

uniformly in $x \in \langle a, b\rangle$.

P r o o f. Given any $\varepsilon > 0$, we choose a $\Delta > \max(\delta, |a| + 1, |b| + 1)$ such that

$$\left(\int_{-\infty}^{b-\Delta} + \int_{a+\Delta}^{\infty}\right) \frac{|f(u)|}{u^2}\, du < \frac{\varepsilon}{4} .$$

Then, in case $1/n \leqslant 1$ and $x \in \langle a, b\rangle$,

$$\frac{1}{l}\left|\int_{\Delta}^{l} f(x \pm t) K_n^l(t)\, dt\right| \leqslant \frac{1}{2n} \int_{\Delta}^{\infty} \frac{|f(x \pm t)|}{t^2}\, dt < \frac{\varepsilon}{2} .$$

By the mean-value theorem,

$$\frac{1}{l}\int_{\delta}^{\Delta} f(x \pm t) K_n^l(t)\, dt =$$

$$= \frac{1}{2ln} \sin^{-2} \frac{\pi\delta}{2l} \int_{\delta}^{\xi} f(x \pm t) \sin^2 \frac{n\pi t}{2l}\, dt \qquad (\delta \leqslant \xi \leqslant \Delta).$$

Hence

$$\frac{1}{l}\left|\int_{\delta}^{\Delta} f(x \pm t) K_n^l(t)\, dt\right| \leqslant \frac{1}{2n\delta^2} \int_{\delta}^{\Delta} |f(x \pm t)|\, dt < \frac{\varepsilon}{2},$$

uniformly in $x \in \langle a, b\rangle$, whenever $1/n$ is small enough.

Thus, the proof is completed.

L e m m a 2. Suppose that $f(t)$ is Lebesgue-integrable over any finite interval, and that $f(t)/t^2$ is of bounded variation over some intervals $(-\infty, -H\rangle$, $\langle H, \infty)$ $(H>0)$. Then, for each real a, $b (a \leqslant b)$ and $\delta > 0$,

$$\lim_{l^2/n \to 0} \frac{1}{l} \int_{\delta}^{l} f(x \pm t)\, K_n^l(t)dt = 0 \qquad (l > \delta),$$

uniformly in $x \in \langle a, b\rangle$.

P r o o f. By the well-known Jordan theorem,

$$\frac{f(t)}{t^2} = f_1(t) - f_2(t) \quad \text{for} \quad t \in \langle H, \infty),$$

where $f_j(t)$ $(j = 1, 2)$ are non-negative and non-increasing in this interval. Therefore

$$\frac{1}{l}\int_{\Delta}^{l} f(x+t)K_n^l(t)dt = \frac{1}{l}\int_{\Delta}^{l} \frac{f(x+t)}{(x+t)^2}(x+t)^2K_n^l(t)dt =$$

$$= \frac{1}{l}\int_{\Delta}^{l}\Big\{f_1(x+t) - f_2(x+t)\Big\}(x+t)^2K_n^l(t)dt, \quad \text{if } \Delta \geqslant \delta\ \ a+\Delta \geqslant H.$$

Applying the mean-value theorem, we have

$$\frac{1}{l}\int_{\Delta}^{l} f_j(x+t)(x+t)^2K_n^l(t)dt = f_j(x+\Delta)\frac{1}{l}\int_{\Delta}^{\xi_j}(x+t)^2K_n^l(t)dt \quad (\Delta \leqslant \xi_j \leqslant l).$$

Hence

$$\frac{1}{l}\left|\int_{\Delta}^{l} f_j(x+t)(x+t)^2K_n^l(t)dt\right| \leqslant f_j(a+\Delta)\frac{1}{l}\int_{\Delta}^{l}\left|x^2+2xt+t^2\right|K_n^l(t)dt.$$

But

$$\frac{1}{l}\int_{\Delta}^{l} K_n^l(t)dt \leqslant \frac{1}{2ln}\cdot\frac{1}{\sin^2\frac{\pi\Delta}{2l}} \leqslant \frac{l}{2n\Delta^2},$$

$$\frac{1}{l}\int_{\Delta}^{l} K_n^l(t)dt \leqslant \frac{l}{2n\Delta^2},$$

$$\frac{1}{l}\int_{\Delta}^{l} tK_n^l(t)dt \leqslant \frac{l^2}{2n\Delta},$$

$$\frac{1}{l}\int_{\Delta}^{l} t^2K_n^l(t)dt \leqslant \frac{1}{2nl}\int_{\Delta}^{l}\left(\frac{t}{\sin\frac{\pi t}{2l}}\right)^2 dt \leqslant \frac{l^2}{2n}.$$

Consequently,

$$\lim_{l^2/n\to 0}\frac{1}{l}\int_{\Delta}^{l} f(x+t)K_n^l(t)dt = 0, \text{ uniformly in } x\in\langle a, b\rangle .$$

It is easy to see that, in the last relation, the function $f(x + t)$ can be replaced by $f(x-t)$.

The integral of $f(x\pm t)K_n^l(t)$, extended over $\langle\delta,\Delta\rangle$ is small as we please (see above). Thus, the result follows.

Assuming that $f\in E^*$ and $l\to\infty$, $n\to\infty$, we have

T h e o r e m 1. If $f(t)$ satisfies the conditions of Lemma 1 [resp. Lemma 2], then, for all real x such that

$$\lim_{h\to 0+}\frac{1}{h}\int_0^h\left|f(x+t) + f(x-t) - 2g(x)\right|dt = 0, \tag{3}$$

the relation

$$\lim_{l/n\to 0}\sigma_n^l(x;f) = g(x) \qquad \left[\lim_{l^2/n\to 0}\sigma_n^l(x;f) = g(x)\right] \tag{4}$$

holds (cf. [2], I, p. 90; II, pp. 242-3).

P r o o f. In view of (1), it is enough to show that

$$J_n^l(x) = \frac{1}{l}\int_0^l\left\{f(x+t) + f(x-t) - 2g(x)\right\}K_n^l(t)dt$$

tends to zero as $l/n\to 0$ $\left[\text{resp. } l^2/n\to 0\right]$.

Let us put

$$\varphi_x(u) = f(x+u) + f(x-u) - 2\,g(x), \qquad \lambda(t) = \int_0^t\left|\varphi_x(u)\right|du$$

and choose, for a given $\varepsilon > 0$, an arbitrary $\eta > 0$ such that

$$\lambda(t) \leqslant \varepsilon t, \text{ whenever } 0 < t \leqslant \eta .$$

Write

$$J_n^l(x) = \frac{1}{l}\left(\int_0^{l/n} + \int_{l/n}^{\eta} + \int_{\eta}^{l}\right)\varphi_x(t)K_n^l(t)dt = A + B + C .$$

Clearly,

$$|A| \leqslant \frac{1}{l}\int_0^{l/n}\left|K_n^l(t)\right|d\lambda(t) \leqslant \frac{n}{2l}\lambda\left(\frac{l}{n}\right) \leqslant \frac{\varepsilon}{2}, \text{ if } \frac{l}{n} \leqslant \eta .$$

In this case

$$|B| \leqslant \frac{1}{l}\int_{l/n}^{\eta}\left|\varphi_x(t)\right|K_n^l(t)dt \leqslant \frac{l}{2n}\int_{l/n}^{\eta}\frac{1}{t^2}\,d\lambda(t) =$$

$$= \frac{l}{2n}\left\{\frac{\lambda(\eta)}{\eta^2} - \frac{\lambda(l/n)}{(l/n)^2} + 2\int_{l/n}^{\eta}\frac{\lambda(t)}{t^3}\,dt\right\} ;$$

whence

$$|B| \leqslant \frac{\varepsilon l}{2n}\left[\frac{1}{\eta} + \frac{n}{l} + 2\int_{l/n}^{\eta}\frac{1}{t^2}dt\right] \leqslant \frac{3\varepsilon}{2} .$$

Applying Lemma 1 [resp. Lemma 2], we get

$$|C| < \frac{\varepsilon}{2} ,$$

provided l/n [resp. l^2/n] is small enough, and the proof is completed.

R e m a r k 1. If there exist two real numbers $f_1(x)$, $f_2(x)$ for which

$$\lim_{h\to 0+}\frac{1}{h}\int_0^h |f(x+t) - f_1(x)|\,dt = 0 = \lim_{h\to 0+}\frac{1}{h}\int_0^h |f(x-t) - f_2(x)|\cdot dt,$$

then relations (3) – (4) hold with $g(x) = \{f_1(x) + f_2(x)\}/2$.

R e m a r k 2. For any $f \in E^*$ continuous at every point of a finite interval $\langle a, b\rangle$, the convergence (4) is uniform in $x \in \langle a, b\rangle$ and $g(x) = f(x)$ in this interval.

3. Behaviour of $\tilde{S}_n^l(x;f)$

Considering f's of class E, we have

$$\tilde{S}_n^l(x;f) = -\frac{1}{l}\int_{-l-x}^{l-x} f(x+t)\tilde{D}_n^l(t)dt = \frac{1}{l}\int_{-l+x}^{l+x} f(x-t)\tilde{D}_n^l(t)dt$$

for $x \in (-\infty, \infty)$, $l > 0$, $n = 1, 2, \ldots$. Consequently, if $l \geqslant x$,

$$\tilde{S}_n^l(x;f) = -\frac{1}{2l}\int_{-l+x}^{l-x} \psi_x(t)\tilde{D}_n^l(t)dt - \frac{1}{2l}\int_{-l-x}^{-l+x} f(x+t)\tilde{D}_n^l(t)dt +$$

$$+\frac{1}{2l}\int_{l-x}^{l+x} f(x-t)\tilde{D}_n^l(t)dt = G_n^l(x) - P_n^l(x) + Q_n^l(x).$$

It is easily seen,

$$G_n^l(x) = -\frac{1}{l}\int_0^{l-x} \psi_x(t)\tilde{D}_n^l(t)dt = -\frac{1}{l}\int_0^{l}\psi_x(t)\tilde{D}_n^l(t)dt +$$

$$+\frac{1}{l}\int_{l-x}^{l} \psi_x(t)\tilde{D}_n^l(t)dt = M_n^l(x) + W_n^l(x)$$

and

$$M_n^l(x) = -\frac{1}{l}\int_{\to 0}^{l}\frac{\psi_x(t)}{2\tan\frac{\pi t}{2l}}dt + \frac{1}{l}\int_{\to 0}^{l}\psi_x(t)\cdot\frac{\cos(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}}dt =$$

$$= \tilde{f}^l(x) + Y_n^l(x),$$

whenever $\tilde{f}^l(x)$ is finite. In this case

$$\tilde{S}_n^l(x;f) - \tilde{f}^l(x) = Y_n^l(x) + W_n^l(x) - P_n^l(x) + Q_n^l(x) .$$

If $-l<a\leqslant x\leqslant b<l$ (a, b = const), then

$$\left|W_n^l(x)\right|\leqslant \frac{1}{l}\left|\int_{l-x}^{l}|f(x+t)|\left|\tilde{D}_n^l(t)\right|dt\right| + \frac{1}{l}\left|\int_{l-x}^{l}|f(x-t)|\left|\tilde{D}_n^l(t)\right|dt\right| \leqslant$$

$$\leqslant \frac{1}{\sin\frac{\pi(l-x)}{2l}}\left\{\frac{1}{l}\left|\int_{l}^{l+x}|f(u)|du\right| + \frac{1}{l}\left|\int_{x-l}^{2x-l}|f(u)|du\right|\right\} .$$

Considering $x\geqslant 0$ and $x<0$, separately, we obtain

$$\lim_{l,n\to\infty} W_n^l(x) = 0, \text{ uniformly in } x\in\langle a, b\rangle .$$

The terms $P_n^l(x)$, $Q_n^l(x)$ behave similarly. Hence

$$\tilde{S}_n^l(x;f) - \tilde{f}^l(x) = Y_n^l(x) + o(1) \quad \text{as } l, n\to\infty, \tag{5}$$

uniformly in $x\in\langle a, b\rangle$, provided $\tilde{f}^l(x)$ is finite in this interval. Further ivestigations will be concentrated about $Y_n^l(x)$ and $\tilde{f}^l(x)$.

Assuming that $l\to\infty$, $n\to\infty$, we have

L e m m a 3. If $f(t)$ is Lebesgue-integrable over every finite interval and if

$$\int_{|t|\geqslant 1}\frac{|f(t)|}{t}dt<\infty ,$$

then, for each real a, b $(a \leqslant b)$ and $\delta > 0$,

$$\lim_{l/n \to 0} \frac{1}{l} \int_{\delta}^{l} f(x \pm t) \frac{\cos(2n+1)\frac{\pi t}{2l}}{2 \sin \frac{\pi t}{2l}} dt = 0 ,$$

uniformly in $x \in \langle a, b \rangle$.

The proof runs as in Section 3 of [1].

T h e o r e m 2. If $f(t)$ satisfies the conditions of Lemma 3, then, at every x for which

$$\int_0^1 \frac{|\psi_x(t)|}{t} dt < \infty, \tag{6}$$

the relation

$$\lim_{l/n \to 0} \tilde{S}_n^l(x;f) = -\frac{1}{\pi}\int_0^\infty \frac{\psi_x(t)}{t} dt$$

holds (cf. [2], I, p. 52; II, pp. 242-3).

P r o o f. The assumption of Lemma 3 imply $f \in E$ and

$$\int_1^\infty \frac{|\psi_x(t)|}{t} dt < \infty \quad \text{for every} \quad x \in (-\infty, \infty).$$

Consider a point x such that (6) is fulfilled. Then the Lebesgue-integrals

$$\int_0^\infty \frac{|\psi_x(t)|}{t} dt, \int_0^\infty \frac{\psi_x(t)}{t} dt , \int_0^l \psi_x(t) \cot \frac{\pi t}{2l} dt \quad (l > 0)$$

are finite. Moreover,

$$\tilde{f}^l(x) = -\frac{1}{2l}\int_0^l \psi_x(t) \cot \frac{\pi t}{2l} dt \quad (l > 0) .$$

We shall prove that

$$(7) \qquad \lim_{l\to\infty} \tilde{f}^l(x) = -\frac{1}{\pi}\int_0^\infty \frac{\psi_x(t)}{t}\,dt .$$

Write

$$Z_n^l(x) = \tilde{f}^l(x) + \frac{1}{\pi}\int_0^l \frac{\psi_x(t)}{t}\,dt = \int_0^l \psi_x(t)\left\{\frac{1}{\pi t} - \frac{1}{2l}\cot\frac{\pi t}{2l}\right\} dt .$$

Choose, for a given $\varepsilon > 0$, a positive Δ such that

$$\int_\Delta^\infty \frac{|\psi_x(t)|}{t}\,dt < \varepsilon .$$

Evidently, if $l > \Delta$,

$$Z_n^l(x) = \frac{1}{\pi}\left(\int_0^\Delta + \int_\Delta^l\right)\frac{\psi_x(t)}{t}\left[1 - \frac{\pi t}{2l}\cot\frac{\pi t}{2l}\right]dt = A + B$$

and

$$|B| \leqslant \frac{1}{\pi}\int_\Delta^l \frac{|\psi_x(t)|}{t}\,dt \leqslant \frac{1}{\pi}\int_\Delta^\infty \frac{|\psi_x(t)|}{t}\,dt < \frac{\varepsilon}{2} .$$

Further,

$$|A| \leqslant \frac{1}{\pi}\left(1 - \frac{\pi\Delta}{2l}\cot\frac{\pi\Delta}{2l}\right)\int_0^\Delta \frac{|\psi_x(t)|}{t}\,dt < \frac{\varepsilon}{2}$$

for l large enough, and relation (7) follows.

Take into account the expression

$$Y_n^l(x) = \frac{1}{l}\int_0^l \psi_x(t)\,\frac{\cos(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}}\,dt$$

and a positive δ for which

$$\int_0^{\delta} \frac{|\psi_x(t)|}{t}\, dt < \varepsilon .$$

Then, by Lemma 3,

$$(8) \qquad |Y_n^l(x)| \leqslant \frac{\varepsilon}{2} + \frac{1}{l}\left|\int_{\delta}^{l} \psi_x(t)\, \frac{\cos(2n+1)\frac{\pi t}{2l}}{2\sin\frac{\pi t}{2l}}\, dt\right| < \varepsilon ,$$

if l/n is small.

Applying (5), (7), (8), we get at once the desired assertion.

R e m a r k. It can easily be observed that if $f \in E$ and

$$\int_0^{\infty} \frac{1}{t}|\psi_x(t)|\, dt < \infty ,$$

the thesis of Theorem 2 is also true.

4. Property of $\tilde{\sigma}_n^l(x;f)$

If $f \in E$,

$$\tilde{\sigma}_n^l(x;f) = -\frac{1}{l}\int_{-l-x}^{l-x} f(x+t)\tilde{K}_n^l(t)dt = \frac{1}{l}\int_{-l+x}^{l+x} f(x-t)\tilde{K}_n^l(t)dt$$

for $x \in (-\infty, \infty)$, $l>0$, $n = 1, 2, \dots$. Hence

$$\tilde{\sigma}_n^l(x;f) = -\frac{1}{l}\int_0^{l}\psi_x(t)\tilde{K}_n^l(t)dt + \frac{1}{2l}\int_{l-x}^{l}\psi_x(t)\tilde{K}_n^l(t)dt +$$

$$+ \frac{1}{2l}\int_{-l}^{-l+x}\psi_x(t)\tilde{K}_n^l(t)dt + \frac{1}{2l}\int_{l-x}^{l+x} f(x-t)\tilde{K}_n^l(t)dt - \frac{1}{2l}\int_{-l-x}^{-l+x} f(x+t)\tilde{K}_n^l(t)dt.$$

Putting

$$F_n^l(x) = -\frac{1}{l}\int_0^{l}\psi_x(t)\tilde{K}_n^l(t)dt ,$$

and reasoning as in preceding Section, we obtain

(9) $$\tilde{\sigma}_n^l(x;f) = F_n^l(x) + o(1) \qquad \text{as } l,\ n \to \infty,$$

uniformly in x over any finite interval $\langle a, b\rangle$.

T h e o r e m 3. Suppose that $f \in E$ satisfies condition (2). Consider an arbitrary $x \in (-\infty, \infty)$ for which

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |\psi_x(t)|\, dt = 0.$$

Then

$$\lim_{l/n \to 0} \tilde{\sigma}_n^l(x;f) = -\frac{1}{\pi} \int_{\to 0}^{\to\infty} \frac{\psi_x(t)}{t}\, dt \qquad (l,\ n \to \infty),$$

provided that the last integral is finite (cf. [2], I, p.92; II, p. 246).

P r o o f. Setting

$$\Phi_n^l(t) = \frac{1}{n}\left(2 \sin \frac{\pi t}{2l}\right)^{-2} \sin \frac{n\pi t}{l},$$

we have

$$\tilde{K}_n^l(t) = \frac{1}{n} \sum_{m=0}^{n-1} \tilde{D}_m^l(t) = \frac{1}{2} \cot \frac{\pi t}{2l} - \Phi_n^l(t) \qquad (l > 0,\ n \geqslant 1).$$

It is easily seen,

(10) $$\frac{1}{l}\left|\tilde{K}_n^l(t)\right| \leqslant \frac{1}{ln} \sum_{m=1}^{n} m \leqslant \frac{n}{l} \quad \text{always},$$

(11) $$\frac{1}{l}\left|\Phi_n^l(t)\right| \leqslant \frac{l}{4nt^2}, \quad \text{if } 0 < |t| \leqslant l.$$

Let us write

$$F_n^l(x) = -\frac{1}{l}\left(\int_0^{l/n} + \int_{l/n}^{l}\right)\psi_x(t)\tilde{K}_n^l(t)dt =$$

$$= \tilde{f}^l(x;1/n) + \frac{1}{l}\int_{1/n}^{l}\psi_x(t)\,\Phi_n^l(t)dt - \frac{1}{l}\int_0^{1/n}\psi_x(t)\tilde{K}_n^l(t)dt\,.$$

Choose, for a given $\varepsilon > 0$, a positive Δ such that

$$\mu(h) \equiv \int_0^h |\psi_x(u)|du < \varepsilon h, \text{ whenever } 0<h\leqslant\Delta\,.$$

Then, by (10),

$$\frac{1}{l}\left|\int_0^{1/n}\psi_x(t)\tilde{K}_n^l(t)dt\right| \leqslant \frac{n}{l}\int_0^{1/n}|\psi_x(t)|dt<\varepsilon, \quad \text{if } \frac{1}{n}\leqslant\Delta.$$

In view of (11),

$$\frac{1}{l}\left|\int_{1/n}^{\Delta}\psi_x(t)\,\Phi_n^l(t)dt\right| \leqslant \frac{1}{4n}\int_{1/n}^{\Delta}\frac{|\psi_x(t)|}{t^2}\,dt =$$

$$= \frac{1}{4n}\left\{\frac{\mu(\Delta)}{\Delta^2} - \frac{\mu(1/n)}{(1/n)^2} + 2\int_{1/n}^{\Delta}\frac{\mu(t)}{t^3}\,dt\right\} \leqslant$$

$$\leqslant \frac{1}{4n}\left\{\frac{\varepsilon}{\Delta} + \frac{\varepsilon}{1/n} + 2\varepsilon\left(\frac{1}{1/n} - \frac{1}{\Delta}\right)\right\} < \frac{3\varepsilon}{4}, \text{ when } \frac{1}{n}\leqslant\Delta.$$

The assumption (2) and inequality (11) imply

$$\lim_{1/n\to 0}\frac{1}{l}\int_{\Delta}^{l}\psi_x(t)\,\Phi_n^l(t)dt = 0 \qquad (l,n\to\infty).$$

Hence

(12) $\left|F_n^l(x) - \tilde{f}^l(x;\,1/n)\right| < 2\varepsilon$ as $1/n$ are small enough.

Further,

$$\tilde{f}^l(x;\,1/n) + \frac{1}{\pi}\int_{1/n}^{l}\frac{\psi_x(t)}{t}\,dt = \int_{1/n}^{l}\psi_x(t)\left\{\frac{1}{\pi t} - \frac{1}{2l}\cot\frac{\pi t}{2l}\right\}dt =$$

$$=\left(\int_{l/n}^{D}+\int_{D}^{l}\right)\frac{\psi_x(t)}{t}\left\{1-\frac{\pi t}{2l}\cot\frac{\pi t}{2l}\right\}dt = A + B ,$$

where D is a positive number such that

$$\left|\int_{T_1}^{T_2}\frac{\psi_x(t)}{t}\,dt\right| < \varepsilon , \qquad \text{if} \quad T_2 \geqslant T_1 \geqslant D .$$

By the mean-value theorem,

$$B = \int_{\xi}^{l}\frac{\psi_x(t)}{\pi t}\,dt, \text{ with a certain } \xi\in\langle D,\ l\rangle .$$

Hence

$$|B| < \varepsilon/2 \quad \text{for} \quad l \geqslant D .$$

Also, this theorem yields

$$A = \left(1-\frac{\pi D}{2l}\cot\frac{\pi D}{2l}\right)\int_{\tau}^{D}\frac{\psi_x(t)}{\pi t}\,dt \qquad \left(\frac{1}{n}\leqslant\tau\leqslant D\right) .$$

Consequently,

$$|A| < \varepsilon/2 \ \text{ for sufficiently large } l.$$

Thus, we have

$$\lim_{l/n\to 0}\ \tilde{f}^{l}(x;\ 1/n) = -\frac{1}{\pi}\int_{\to 0}^{\to\infty}\frac{\psi_x(t)}{t}\,dt \qquad (l,\ n\to\infty)$$

and by (9) and (12) the result follows.

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