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NOTE ON STURM'S THEOREM FOR A STRONGLY ELLIPTIC SYSTEMS

In this paper Sturm's Theorem (see [1], [2], [3]) for a linear strongly elliptic systems of second order is given. The proof is qûite analogous to that for a single system given in [3].

Let G denote an n-dimensional bounded domain in the Euclidean space E_n and $\dot{G}$ the boundary of the domain G. In $\bar{G} = G \cup \dot{G}$, we consider the following problem

$$(1)\quad L[u]=\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + 2\sum_{i=1}^{n} C_i(x)\frac{\partial u}{\partial x_i} + F(x)u = 0 \qquad \text{for} \quad x \in G$$

$$(2)\qquad \sum_{ij=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_j}\cos(\bar{n},x_i) + S(x)u = 0 \qquad \text{for} \quad x \in \Gamma_1$$

$$u = 0 \qquad \text{for} \quad x \in \Gamma_2$$

$$(3)\quad K[V]=\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(B_{ij}(x)\frac{\partial V}{\partial x_j}\right) + 2\sum_{i=1}^{n} D_i(x)\frac{\partial V}{\partial x_i} + H(x)\cdot V = 0 \qquad \text{for} \quad x \in G$$

$$(4)\qquad \sum_{ij=1}^{n} B_{ij}(x)\frac{\partial V}{\partial x_j}\cos(\bar{n},x_i) + T(x)V = 0 \qquad \text{for} \quad x \in \Gamma_1$$

where $u(x) = \left(u_s(x)\right)_{s=1}^{m}$ is a column vector, $V(x)$ is a matrix of degree m, $x = (x_1,\dots,x_n) \in \bar{G}$, $A_{ij}(x)$, $B_{ij}(x)$, $C_i(x)$, $D_i(x)$, $F(x)$, $H(x)$, $S(x)$, $T(x)$ are real matrices of degree m, Γ_1 and Γ_2 are surfaces such that $\Gamma_1 \cup \Gamma_2 = \dot{G}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\bar{n}$ is a interior orthogonal direction.

We introduce the following notation

$$A^*_{ij} = \frac{1}{2}\left(A_{ij} + A^T_{ji}\right), \quad B^*_{ij} = \frac{1}{2}\left(B^*_{ij} + B^T_{ji}\right), \quad A = \left(A^*_{ij}\right)_{ij=1}^{n}$$

$$M_{ij} = B^*_{ij} - \sum_{kl=1}^{n} B^T_{ki}\, \bar{A}^*_{kl}\, B_{ij}, \quad N_i = D_i - \sum_{kl=1}^{n} C_k\, \bar{A}^*_{kl}\, B_{li},$$

$$M = \left(M_{ij}\right)_{ij=1}^{n}, \quad (\xi,\eta) = \sum_{j=1}^{m} \xi_j \cdot \eta_j, \quad \|\eta\| = \sqrt{(\eta,\eta)}$$

where $\bar{A}^*_{ij}$ and $\bar{M}_{ij}$ $(i,j=1,\dots,n)$ are matrices of degree m such that $A^{-1} = \left(\bar{A}^*_{ij}\right)_{ij=1}^{n}$, $M^{-1} = \left(\bar{M}_{ij}\right)_{ij=1}^{n}$, ξ_j and η_j are the components of m-dimensional vectors ξ and η.

S t u r m's T h e o r e m. If there exists a non-tunnel solution of (1), (2) and if

1° $A>0$, $M>0$, $T-S\geqslant 0$

2° $H-F-\sum_{ij=1}^{n} C_i\, \bar{A}^*_{ij}\, C^T_j - \sum_{ij=1}^{n} N_i\, \bar{M}_{ij}\, N^T_j > 0$

3° the matrix $V(x)$ is a solution of the problem (3), (4) and the matrices $V^T\left(\sum_{j=1}^{n} B_{ij} \frac{\partial V}{\partial x_j}\right)$ $(i=1,\dots,n)$ are symmetric,

then $\det V$ has zeroes for $x \in \bar{G}$.

P r o o f. From the formula of integration by parts we obtain

$$(5) \qquad \iint_G \Phi(x)dV + \int_{\Gamma_1} \Phi_1(x)d\sigma = -\iint_G (L[u],\, u)dV = 0$$

where

$$\Phi(x) = \sum_{ij=1}^{n} (A_{ij} \cdot \eta_j, \eta_i) + 2\sum_{j=1}^{n}\Big((P_j - C_j)\cdot \eta_j, \eta_o\Big) + \left(\left(\sum_{j=1}^{n} \frac{\partial P_j}{\partial x_j} - F\right)\cdot \eta_o, \eta_o\right), \quad \Phi_1(x) =$$

$$= \left(\left(\sum_{i=1}^{n} P_i \cos(\bar{n}, x_i) - S\right)\eta_o, \eta_o\right),$$

P_i are arbitrary symmetric matrices, $\eta_o = u$, $\eta_i = \frac{\partial u}{\partial x_i}$.

We have $\Phi(x) = \sum_{ij=1}^{n} (A_{ij}^{*} \eta_j, \eta_i) + 2\sum_{j=1}^{n}\Big((P_j - C_j)\, \eta_j, \eta_o\Big) +$

$$+\left(\left(\sum_{j=1}^{n} \frac{\partial P_j}{\partial x_j} - F\right)\cdot \eta_o, \eta_o\right) = (A\eta, \eta) + (2\, C\eta, \eta_o) +$$

$$+\left(\left(\sum_{j=1}^{n} \frac{\partial P_j}{\partial x_j} - F\right)\eta_o, \eta_o\right) = \left\| A_1^{-1}\eta - A_1^{T}\, C^{T}\, \eta_o \right\|^2 + (Q\cdot\eta_o, \eta_o)$$

where $Q = \sum_{j=1}^{n} \frac{\partial P_j}{\partial x_j} - \frac{1}{2} F - \frac{1}{2} F^T - \sum_{ij=1}^{n} (P_i - C_i)\bar{A}_{ij}^{*}(P_j - C_j^T)$, $C =$ $= (P_1 - C_1, \ldots, P_n - C_n)$, A_1 is a solution of the matrix equation $A_1 \cdot A_1^T = A^{-1}$, $\eta = (\eta_i)_{i=1}^{n}$.

Suppose that $\det V \neq 0$ for $x \in \bar{G}$. We take

$$P_i = -\left(\sum_{j=1}^{n} B_{ij} \frac{\partial V}{\partial x_j}\right) V^{-1}.$$

We have

$$\Phi_1(x) = \Big((T - S)\, \eta_o, \eta_o\Big) \tag{6}$$

$$\frac{\partial P_i}{\partial x_i} = - \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(B_{ij}\frac{\partial V}{\partial x_j}\right)\cdot V^{-1} + \sum_{j=1}^{n}\left(B_{ij}\frac{\partial V}{\partial x_j}\right)V^{-1}\frac{\partial V}{\partial x_i} V^{-1}$$

$$\sum_{i=1}^{n} V^T \frac{\partial P_i}{\partial x_i} V = \sum_{i=1}^{n} V^T D_i \frac{\partial V}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial V^T}{\partial x_i} D_i^T V +$$

$$+ \frac{1}{2} V^T(H + H^T)V + \sum_{ij=1}^{n} \frac{\partial V^T}{\partial x_i} B_{ij}^{*} \frac{\partial V}{\partial x_i}$$

$$V^T Q V = \sum_{ij=1}^{n} \frac{\partial V^T}{\partial x_i} M_{ij} \frac{\partial V}{\partial x_j} + \sum_{i=1}^{n} V^T N_i \frac{\partial V}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial V^T}{\partial x_i} N_i^T V + V^T Z V =$$

$$(7) \qquad = V^T\tilde{V} + V^T N M_1 \tilde{V} + \tilde{V}^T M_1^T N^T V + V^T Z V =$$

$$= (\tilde{V}^T + V^T N M_1)(\tilde{V} + M_1^T N^T V) + V^T\left[Z - \sum_{ij=1}^{n} N_i \bar{M}_{ij} N_j^T\right]V$$

where

$$N = (N_1,\dots,N_n),\quad Z = \frac{1}{2} H + \frac{1}{2} H^T - \frac{1}{2} F - \frac{1}{2} F^T - \sum_{ij=1}^{n} C_i \bar{A}_{ij}^{*} C_j^T ,$$

and M_1 is a solution of the matrix equation $M_1 M_1^T = M^{-1}$, $\tilde{V} = M_1^{-1}\cdot V_1$, $V_1 = \left(\frac{\partial V}{\partial x_i}\right)_{i=1}^{n}$.

From (5), (6), (7) we obtain that the unique solution of the problem (1), (2) is $u(x) \equiv 0$, so our assumption was false.

If moreover, the matrices C_i $(i = 1,\dots,n)$ are symmetric, then condition 2^0 may be replaced by the condition:

$$M - F + \sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i} - \sum_{ij=1}^{n} N_i \bar{M}_{ij} N_j^T > 0 .$$

In this case we prove the theorem by a similar argument taking $P_i = -\left(\sum_{j=1}^{n} B_{ij}\frac{\partial V}{\partial x_j}\right)\cdot V^{-1} + C_i$.

REFERENCES

[1] Л. Кукс: Теорема Штурма и колеблечость решений сильно элиптических систем. Dokł. Akad. Nauk SSSR 142(1962)32-35.

[2] K. Kreith: A Sturm Theorem for strongly elliptic systems and applications. Bull.Amer.Math.Soc. 75 (1969) 1025-1027.

[3] H. Protter: A comparison theorem for elliptic equations. Proc.Amer.Math.Soc. 10 (1959) 296-299.

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