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RIGHT INVERTIBLE OPERATORS AND FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH INVOLUTIONS

In the papers [2] and [3] the author has developped the theory of right invertible operators acting in a linear space (without any topological assumptions). This theory has numerous applications to ordinary and partial differential equations and to difference equations, all with variable coefficients.

In the present paper we shall indicate applications of the results obtained in [2] to some functional-differential equations. A partial result concerning those equations has been proved in [1].

To begin with, we shall repeat some definitions and theorems given in [2] which will be necessary for our further considerations.

Let X be a linear space over the field of real or complex scalars. Let A be a linear (i.e. additive and homogeneous) operator defined on a linear subset $\mathcal{D}_A$, called the domain of A, and mapping $\mathcal{D}_A$ into X. The set of all such operators will be denoted by L(X). Denote by Z_A the kernel of an $A \in L(X)$, i.e. the set $Z_A = \{x \in \mathcal{D}_A : Ax = 0\}$.

Definition 1. An operator $D \in L(X)$ is said to be <u>right invertible</u>, if there exists an operator $R \in L(X)$ such that $\mathcal{D}_R = X$, $RX \subset \mathcal{D}_D$ and $DR = I$, where I denotes the identity operator.

The operator R is called a <u>right inverse</u> of D. The set of all right invertible operators belonging to L(X) will be denoted by R(X).

D e f i n i t i o n 2. An operator $F \in L(X)$ is said to be an initial operator for $D \in R(X)$ corresponding to a right inverse R of D if

$$FX = Z_D, \quad F^2 = F \quad \text{and} \quad FR = 0 \quad \text{on} \quad X.$$

The kernel Z_D of the operator $D \in R(X)$ is said to be the space of constants for D.

Let $D \in R(X)$. Then every right inverse R of D induces in a unique way an initial operator F for D corresponding to R. Namely we have (Theorem 2.1 of [2]):

$$F = I - RD \quad \text{on} \quad \mathcal{D}_D. \tag{1}$$

Moreover, every projection $F \in L(X)$ onto Z_D is an initial operator for D corresponding to a right inverse uniquely determined (Theorem 2.4 of [2]). By a simple induction we obtain from (1) a Taylor Formula:

$$I = \sum_{k=0}^{N-1} R^k F D^k + R^N D^N \quad \text{on} \quad \mathcal{D}_{D^N} \quad (N=1,2,\ldots) \tag{2}$$

Write

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \text{ where } D \in R(X),\ Q_0,\ldots,Q_{N-1} \in L(X) \text{ and } Q_N = I. \tag{3}$$

An initial value problem for the operator Q(D) defined by (3) is to find all solutions of the equation

$$Q(D)x = y, \qquad y \in X \tag{4}$$

satysfying the initial conditions

(5) $FD^k x = y_k$, where $y_k \in Z_D$ $(k=0,1,\ldots,N-1)$, F is an initial operator for D.

The initial value problem is said to be well-posed, if this problem has a unique solution for every $y \in X, y_0, \ldots, y_{N-1} \in Z_D$. It means that a well-posed homogeneous initial value problem has only zero as a solution.

T h e o r e m 1. (c.f. Corollary 3.1 and Theorem 3.2 of [2]). Let $D \in R(X)$ and let F be an initial operator for D corresponding to a right inverse R of D. Then

1) If -1 is not an eigenvalue of the operator $\hat{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}$, then the initial value problem (4)-(5) is well-posed and its unique solution is

$$(6) \qquad x = R^N (I+\hat{Q})^{-1}\Big[y - \sum_{j=0}^{N-1} Q_j \sum_{k=j}^{N-1} R^{k-j} y_k\Big] + \sum_{k=0}^{N-1} R^k y_k \, .$$

2) If -1 is an eigenvalue of the operator $\hat{Q}$, then the problem (4)-(5) is ill-posed. However this problem has solutions if and only if

$$(7) \qquad \hat{y} = y - \sum_{j=0}^{N-1} Q_j \sum_{k=j}^{N-1} R^{k-j} y_k \in (I+\hat{Q})X.$$

If this condition is satisfied, then solutions of the problem (4)-(5) exist and are of the form

$$(8) \qquad x = R^N (I+\hat{Q})_{-1}\, \hat{y} + \sum_{k=0}^{N-1} R^k y_k + \hat{x},$$

where $(I+Q)_{-1}\, \hat{y}$ denotes an element of the inverse image of $\hat{y}$ by the operator $I+\hat{Q}$ and $\hat{x}$ is an element of the eigenspace X_{-1} of the operator $\hat{Q}$ corresponding to the eigenvalue -1.

An immediate consequence is

C o r o l l a r y 1. Suppose that all assumptions of Theorem 1 are satisfied. If -1 is not an eigenvalue of the operator $\hat{Q}$ then all solutions of the equation (4) are of the form

$$x = R^N(I+\hat{Q})^{-1}\Big[y - \sum_{j=0}^{N-1} Q_j \sum_{k=j}^{N-1} R^{k-j}z_k\Big] + \sum_{k=0}^{N-1} R^k z_k. \tag{9}$$

where $z_0,\dots,z_{N-1} \in Z_D$ are arbitrary. If -1 is an eigenvalue of the operator $\hat{Q}$, then the equation (4) has solutions if and only if there exist $z_0,\dots,z_{N-1} \in Z_D$ such that

$$\hat{y} = y - \sum_{j=0}^{N-1}\sum_{k=j}^{N-1} Q_j R^{k-j} z_k \in (I+\hat{Q})X. \tag{10}$$

If this condition is satisfied, then solutions of (4) exist and are of the form

$$x = R^N(I+\hat{Q})_{-1}\hat{y} + \sum_{k=0}^{N-1} R^k z_k + \hat{x} , \tag{11}$$

where $z_0,\dots,z_{N-1}$ are determined by the condition (10), $(I+\hat{Q})_{-1}\hat{y}$ and $\hat{x}$ are described in Theorem 1.

C o r o l l a r y 2. (Corollary 3.2 of [2]).Suppose that all assumptions of Theorem 1 are satisfied. Moreover suppose that the operator $I-aR$ is invertible for every scalar a and $Q_k = q_k I$, where q_k $(k=0,1,\dots,N-1)$ are scalars. Then -1 is not an eigenvalue of the operator Q, so that the problem (4)-(5) is well-posed and has the unique solution $x = (I+\hat{Q})^{-1}\Big[R^N y + \sum_{j=0}^{N-1}\sum_{k=0}^{j} q_{N+k-j}R^j y_k\Big]$.

T h e o r e m 2. Let $D \in R(X)$ and let F be an initial operator for D corresponding to a right inverse R of D. Suppose we are given an involution $S \in L(X)$ such that

$$SF = FS, \tag{12}$$

$$DS = ASD, \text{ where } A \in L(X). \tag{13}$$

Then every solution of the equation

$$Q(D)x = Sx + y, \quad y \in X, \tag{14}$$

satisfies the equation

$$Hx = P(D_1)y + Sy, \text{ where } H = P(D_1)Q(D)-I, \ D_1 = A^{-1}D \in R(X) \tag{15}$$

$$P(D_1) = \sum_{k=0}^{N-1} P_k D_1^k, \quad P_k = SQ_kS \quad (k=0,1,\dots,N-1),$$

with the conditions

$$FD^kSQ(D)x = FD^kx + FD^kSy \quad (k=0,1,\dots,N-1). \tag{16}$$

P r o o f. Observe that the operator A is invertible and $A^{-1} = SAS$. Indeed, $A(SAS) = ASAS = ASAS(DR) = AS(ASD)R = ASDSR = (ASD)SR = (DS)SR = DS^2R = DR = I$ and $(SAS)A = S(ASAS)S = S^2 = I$. By a simple induction we obtain the following formulae

$$SA^k = A^{-k}S, \quad SA^{-k} = A^kS \tag{17}$$

$$(k=1,2,\dots).$$

$$SD_1^k = D^kS, \quad SD^k = D_1^kS \tag{18}$$

By definition and Formulae (18)

$$P(D_1)S = SQ(D), \quad \text{hence} \quad P(D_1) = SQ(D)S. \tag{19}$$

Observe that the operator $D_1 = A^{-1}D \in R(X)$. Indeed, writing $R_1 = RA$, we find $D_1R_1 = A^{-1}DRA = A^{-1}A = I$. An initial operator for D corresponding to R_1 is $F_1 = I - R_1D_1 = I - RAA^{-1}D = I - RD = F$.

Since x is a solution of the equation (14), we have $Sx = Q(D)x-y$ and by Formula (19) we have

$$x = S^2x = D\left[Q(D)x - y\right] = SQ(D)x - Sy = P(D_1)Sx - Sy =$$

$$= P(D_1)\left[Q(D)x-y\right] - Sy = P(D_1)Q(D)x - P(D_1)y - Sy,$$

which implies

$$Hx = P(D_1)Q(D)x - x = P(D_1)y + Sy.$$

Hence x satisfies the equation (15). Moreover

$$FD^kSQ(D)x = FD^kS^2x + FD^kSy = FD^kx + FD^kSy \quad (k=0,1,\dots,N-1),$$

i.e. x satisfies also the conditions (16).

The converse statement is, in general, not true, as the following example shows:

E x a m p l e 1. Consider the following differential equation with reflection:

(20) $\quad x'(t) + a(t)x(t) = x(-t)$, where $a(t) = (1-e^{2t})/1+e^{2t})$.

In our case we have $D = d/dt$, $(Fx)(t) = x(0)$, $(Sx)(t) = x(-t)$, $DS = -SD$, hence $A = -I$, $y = 0$, $Q(t) = t + a$. The conditions (16) are of the form:

(21) $$x'(0) = x(0),$$

because $a(0) = 0$. Moreover $H = (-D+a)(D+a)-I = -\left[D^2-aD + Da + (1-a^2)I\right]$. Hence the equation (15) is of the form $x'' - ax' + (ax)' + (1-a^2)x = 0$, i.e. $x'' + (1+a'-a^2)x = 0$. It is easy to check that $1 + a' - a^2 = 0$. Thus we obtain the equation $x''=0$. Every solution of this equation satysfying the condition (21) is of the form $x(t) = C(t+1)$, where C is an arbitrary constant. But

$$x'(t) + a(t)x(t) - x(-t) = C + a(t)C(t+1) - C(-t+1) =$$

$$= C(1+t-e^{2t})/(1+e^{2t}) \equiv 0$$

if and only if $C = 0$. This means that only the function $x(t) \equiv 0$ satisfies the equation (20).

R e m a r k 1. If the polynomial $Q(D)$ has commutative coefficients, i.e. if $SQ_k = Q_kS$ for $k=0,1,\dots,N-1$, then $P_k = SQ_kS = Q_kS^2 = Q_k$, hence $P(D_1) = Q(D_1)$. In particular the last equality holds if $Q_k = q_kI$, where q_k are scalars $(k=0,1,\dots,N-1)$.

T h e o r e m 3. Suppose that all the assumptions of Theorem 2 are satisfied. Moreover, suppose that

$$SQ_k = Q_kS \quad (k=0,1,\dots,N-1) \tag{22}$$

and that the operators $I+\hat{Q}$, $I+\hat{Q}_1$, $I-\hat{H}$ are invertible, where we write

$$\hat{Q} = \sum_{k=0}^{N-1} Q_kR^{N-k}, \quad \hat{Q}_1 = \sum_{k=0}^{N-1} Q_kR_1^{N-k} = \sum_{k=0}^{N-1} Q_k(RA)^{N-k}, \tag{23}$$

$$\hat{H} = R^N(I+\hat{Q})^{-1}(RA)^N(I+Q_1)^{-1}.$$

Then x is a solution of the equation (14) if and only if x satisfies the equation (15) and the conditions (16), where $P(D_1) = Q(A^{-1}D)$.

P r o o f. Observe, that, by our assumptions and Remark 1, we have $P(D_1) = Q(D_1)$, which implies $H = Q(D_1)Q(D) - I$. The necessity has been proved by Theorem 2.

Sufficiency. Suppose that x is a solution of (15)-(16). Put $u = Q(D)x - Sx - y$. Then, by Formula (19), since $Q(D_1)Q(D)x = x + Q(D_1)y + Sy$, we find

$$Q(D_1)u = Q(D_1)Q(D)x - Q(D_1)Sx - Q(D_1)y =$$

$$= Q(D_1)Q(D)x - SQ(D)x - Q(D_1)y = x - SQ(D)x + Sy =$$

$$= -S\left[Q(D)x - Sx - y\right] = - Su.$$

Write $v = Su$. Then $Q(D)v = Q(D)Su = SQ(D_1)u = -S^2u = -u = -Sv$. Hence

$$Hv = Q(D_1)Q(D)v - v = Q(D_1)(-Sv) - v = -SQ(D)v - v =$$

$$= -S(-Sv) - v = S^2v - v = 0.$$

Thus v satisfies the equation

(24) $$Hv = 0, \quad \text{where} \quad H = Q(D_1)Q(D) - I.$$

Since $FS = SF$, the conditions (16) and Formula (19) together imply

(25) $$FD^kv = 0 \quad \text{for} \quad k=0,1,\dots,N-1$$

Indeed,

$$FD^kv = FD^kSu = FD^kS\left[Q(D)x-Sx-y\right] = FD^kSQ(D)x-FD^kS^2x-FD^kSy =$$

$$= FD^kSQ(D)x - FD^kx - FD^kSy = 0.$$

Write $w = Q(D)v$. Formulae (25) and the equality $Q(D)v = - Sv$ together imply

(26) $$FD_1^kw = 0 \quad \text{for} \quad k=0,1,\dots,N-1$$

Indeed, for $k=0,1,\dots,N-1$ we have $FD_1^kw = FD_1^kQ(D)v = FD_1^k(-Sv) = -FD_1^kSv = -FSD^kv = - SFD^kv = 0$. From the equality (24) we conclude that

(27) $$Q(D_1)w = - Dw.$$

Indeed, $Q(D_1)w + Sw = Q(D_1)Q(D)v + SQ(D)v = Q(D_1)Q(D)v + S(-Sv) = Q(D_1)Q(D)v - S^2v = Q(D_1)Q(D)v - v = Hv = 0$. Since by our assumption the operator $I+\hat{Q}$ is invertible. Formulae (26), (27), (5), (6) together imply

(28) $$w = R_1^N(I+\hat{Q}_1)^{-1}v$$

Indeed

$$w = R_1^N(I+\hat{Q}_1)^{-1}(-Sw - \sum_{j=0}^{N-1}\sum_{k=j}^{N-1} R_1^{k-j}FD_1^k w) + \sum_{k=0}^{N-1} R_1^k F D_1^k w =$$

$$= -R_1^N(I+\hat{Q}_1)^{-1}Sw = - R_1^N(I+\hat{Q}_1)^{-1}SQ(D)v = - R_1^N(I+\hat{Q}_1)^{-1}S(-Sv) =$$

$$= R_1^N(I+\hat{Q}_1)^{-1}S^2v = R_1^N(I+\hat{Q}_1)^{-1}v,$$

because $F_1 = F$.

Observe now that, by definition, $Q(D)v = w$, and that the operator $I+\hat{Q}$ is invertible by our assumption. Hence in the same way, as above, we conclude from Formulae (28), (25), (5), (6) that

$$v = R^N(I+\hat{Q})^{-1}\left[w - \sum_{j=0}^{N-1} Q_j \sum_{k=j}^{N-1} R^{k-j}FD^k v\right] = R^N(I+Q)^{-1}w =$$

$$= R^N(I+\hat{Q})^{-1}R_1^N(I+\hat{Q}_1)^{-1}v = \hat{H}v.$$

Thus $(I-\hat{H})w = 0$. Since the operator $I-\hat{H}$ is invertible by our assumption, we have $v = 0$ and $Q(D)x - Sx - y = u = Sv = 0$, which proves that x is a solution of the equation (14).

R e m a r k 2. Theorem 3 was proved in [1] only for equations with scalar coefficients and under an addditional assumption that $I-\lambda\hat{R}$ is invertible for all scalars λ. This was also supposed in the proof of Theorem 2.

E x a m p l e 2. Consider the following ordinary functional-differential equation of Carleman type

$$(29)\qquad x''(t) + Q_1(t)x'(t) + Q_0(t)x(t) = x(g(t)) + y(t),$$

where we assume that: 1) $g(t) \not\equiv t$ is a continuously differentiable function mapping the interval (a,b) (where we may have $a=-\infty$, $b=+\infty$) onto itself and satysfying so-called <u>Carleman condition</u>: $g(g(t)) \equiv t$ for all $t\in(a,b)$; 2) Q_0, Q_1 are real-valued continuously differentiable functions determined on (a,b) and such that $Q_k(g(t)) = Q_k(t)$ for $k=0,1$ and $t\in(a,b)$; 3) y is a real-valued n-times continuously differentiable function determined on (a,b).

It follows from our assumptions, that $g'(t) \neq 0$ for all $t\in(a,b)$ and that there exists a unique fix-point of g, i.e. such a point $c\in(a,b)$ that $g(c) = c$ (c.f. [4], Chapter VIII, Section 1). We put $D = d/dt$, $(Fx)(t) = x(c)$, $(Rx)(t)=\int_c^t x(s)ds$, $(Sx)(t)=x(g(t))$. We conclude that F is an initial operator for D corresponding to the right inverse R of D and that S is an involution. Moreover, by our assumptions,

$$(SFx)(t) = x(g(c)) = (FSx)(t);$$

$(Ax)(t) = g'(t)x(t)$, hence A is invertible;

$$(SQ_kx)(t) = Q_k(g(t))x(g(t)) = Q_k(t)\, x(g(t)) =$$
$$= (Q_kSx)(t) \qquad (k=0,1).$$

We have also

$$(Qx)(t) = (Q_0R^2x + Q_1Rx)(t) = \int_c^t M(t,s)x(s)ds,$$

where $M(t,s) = Q_0(t-s) + Q_1(t)$ is a continuous function for $a < t,s < b$,

$$(Q_1x)(t) = Q_0(RA)^2x + Q_1(RA)x(t) = \int_c^t M_1(t,s)x(s)ds$$

where $M_1(t,s) = \left\{Q_0(t)\left[g(t) - g(s)\right] + Q_1(t)\right\}g'(s)$ is a continuous function for $a < t,s < b$. Therefore the operators $I+\hat{Q}, I+\hat{Q}_1$ are invertible in the space $C[a_1,b_1]$, where $a < a_1 < c < b_1 < b$ are arbitrarily fixed. Moreover we conclude that the operator $I-\hat{H} = I-R^2(I+\hat{Q})^{-1}(RA)^2(I+\hat{Q}_1)^{-1}$ is also invertible in X. all conditions of Theorem 3 are satisfied. Then $x \in X$ is a solution of the equation (28) if and only if (15)-(16) hold. But in our case

$$H = \left[(g')^{-1}\frac{d}{dt}(g')^{-1}\frac{d}{dt} + (g')^{-1}Q_1\frac{d}{dt} + Q_0\right]\left[\frac{d^2}{dt^2} + Q_1\frac{d}{dt} + Q_0\right] =$$

$$= (g')^{-2}\left\{\frac{d^2}{dt^2} + \left[g'Q_1 - (g')^{-1}g''\right]\frac{d}{dt} + g'^2Q_0\right\}\left[\frac{d^2}{dt^2} + Q_1\frac{d}{dt} + Q_0\right].$$

The conditions (16) are of the form

$$0 = x''(g(c)) + Q_1(g(c))x'(g(c)) + Q_0(g(c))x(g(c)) - x(c) - y(g(c)) =$$

$$= x''(c) + Q_1(c)x'(c) + \left[Q_0(c) - 1\right]x(c) - y(c),$$

$$0 = \left\{\frac{d}{dt}\left[x''(g(t)) + Q_1(g(t))x'(g(t)) + Q_0(g(t))x(g(t)) - \right.\right.$$

$$\left.\left. - x(t) - y(g(t))\right]\right\}_{t=c} = g'(c)\left\{x'''(c) + Q_1(c)x''(c) + \right.$$

$$\left. + \left[Q_1'(c) + Q_0(c) - (g'(c))^{-1}\right]x'(c) + (g'(c))^{-1}Q_0'(c)x(c) - y'(c)\right\}.$$

For instance, if we put $g(t) = -t$, $Q_1(t) \equiv 0$, $Q_0(t) \equiv \lambda$, we have $g'(t) \equiv -1$, $g''(t) \equiv 0$, $c = 0$ and we conclude that every solution of the equation $x'' + \lambda x = x(-t) + y(t)$ is a solution of the equation

$$x^{(IV)}(t) + 2\lambda x''(t) + \lambda^2 x(t) = y''(t) + y(t) + y(-t)$$

satysfying the conditions

$$x''(0) + (\lambda -1)x(0) = y(0); \quad x'''(0)+(\lambda+1)x'(0) = y'(0)$$

E x a m p l e 3. Consider the partial functional-differential equation

$$(30) \qquad \frac{\partial^2 x(t,s)}{\partial t\, \partial s}+P(t,s)x(t,s) = x(s,t)+y(t,s),$$

where we assume that $P, y \in C^1(\Omega)$, $\Omega = [a,b]\times[a,b]$ and moreover $P(s,t) = P(t,s)$ in Ω. We are looking for solutions of (30) belonging to $C^2(\Omega)$. We put $D = \partial^2/\partial t\,\partial s$, $(Fx)(t,s) = x(s,s) + \int_s^t x_t'(u,u)du$, $(Rx)(t,s) = \int_s^t \left[\int_u^s x(u,w)\,dw\right]du$ and $(Sx)(t,s) = x(s,t)$. We conclude that F is an initial operator for D corresponding to the right inverse R of D and that S is an involution. Moreover, since $(SDx)(t,s) = (DSx)(t,s)$, we have $A = I$. Also $SF = FS$ and $SP = PS$. Since $A = I$, in our case $I + Q_1 = I + Q = I + RP$ is obviously invertible and also $\hat{H} = I - [R(I+RP)^{-1}]^2$. Therefore all conditions of Theorem 3 are satisfied. Observe that $H = [Q(D)]^2 = (D+P)^2 = D^2 + PD + DP+P^2$. Hence x is a solution of the equation (30) if x satisfies the equation

$$(31) \qquad x_{ttss}^{(IV)} + 2Px_{ts}'' + P_s'x_t' + P_t'x_s' + (2P_{ts}'' + P^2)x=w,$$

$$\text{where} \quad w(t,s) = y_{ts}''(t,s) + P(t,s)y(t,s) + y(s,t),$$

with the condition

$$x_{ts}'' (s,s) + P(s,s)x(s,s) - y(s,s) + \int_s^t \Big[x_{tts}'''(u,u) +$$

$$+ P(u,u)x_t'(u,u) + P_t'(u,u)x(u,u) - y_t'(u,u)\Big]du = 0.$$

General solution of the equation (31) can be easily obtained using Corollary 1 (where N=2).

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