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# ON INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH FUNCTIONAL ARGUMENTS

In paper [6] some theorems were proved concerning the existence of solutions of the first Fourier problem in a bounded domain for a system of parabolic equations with a linear main part and with a non-linear operator depending on unknown functions.

In the present paper we obtain the same results under weaker assumptions concerning the non-linear parts of equations. These results involve a system of integro-differential equations with functional arguments as a particular case.

## 1. Differential equations containing operators

Let $G$ be a bounded open domain of the Euclidean space $E_{n+1}$ of the variables $(x,t)=(x_1,\dots,x_n,t)$ whose boundary consists of two open domains lying on the planes $t=T_0=\text{const.}<0$ and $t=T=\text{const.}>0$, and of a side surface $S$ situated in the strip $\{(x,t): T_0 \leqslant t \leqslant T\}$. We define

$$\dot{G}^{\tau} = G \cap \{(x,t): 0 < t < \tau\}, \quad S^{\tau} = S \cap \{(x,t): 0 \leqslant t < \tau\},$$

$$\Sigma^{\tau} = S^{\tau} \cup (\bar{G} \cap \{(x,t): t \leqslant 0\}) \quad (0 < \tau \leqslant T),$$

where $\bar{Q}$ denotes the closure of $Q$.

In this section we derive an a priori estimate for the norm $|u|_{1+\beta}^{G^T}$ [1], where $u = (u^1,\dots,u^N)$ is a solution of the problem:

$$(1.1)\qquad L^k u^k = \sum_{i,j=1}^{n} a_{ij}^k(x,t)u_{x_i x_j}^k - u_t^k = B^k u,\ (x,t)\in \overline{G^T}\setminus\Sigma^T,$$

$$(1.2)\qquad u^k(x,t) = \varphi^k(x,t),\ (x,t)\in\Sigma^T\ (k=1,\dots,N).$$

With the aid of the above-mentioned estimate we show the existence of solutions of the problem (1.1), (1.2). This result enables us to obtain a maximum solution and a minimum solution.

The following assumptions are introduced (see sec. 1 of [6]):

(1.I) For any $(x,t)\in\overline{G^T}$ and $\xi\in E_n$ we have

$$a_{ij}^k(x,t) = a_{ji}^k(x,t),\ \sum_{i,j=1}^{n} a_{ij}^k(x,t)\xi_i\,\xi_j \geq A_o|\xi|^2$$
$$(A_o = \text{const.} > 0).$$

(1.II) The coefficients $a_{ij}^k$ satisfy the uniform Hölder condition with the exponent $\alpha$ $(0<\alpha<1)$ in $G^T$ and the uniform Lipschitz condition on the surface $S^T$.

Then for some constant $A_1 > 0$

$$\sum_{i,j=1}^{n}\left(|a_{ij}^k|_\alpha^{G^T} + |a_{ij}^k|_{1-0}^{S^T}\right) \leq A_1 .$$

(1.III) The surface $S^T$ belongs both to $\bar{\bar{C}}_{2+\alpha}$ and to $C_{2-0}$.

(1.IV) $B^k$ are operators defined on the set $C_{1,0}^N(\bar{G})$ with values belonging to the set $C(G^T)$.

(1.V) There are constants $A_i > 0$ $(i=2,3,4,5)$ and $0<\Theta<1$ such that for any $\tau$ $(0<\tau\leq T)$ and any $u\in C_{1,0}^N(\bar{G})$ we have

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[1] In this paper we shall use the norms $|v|_o^Q$, $|v|_{1-o}^Q$, $|v|_{m+\alpha}^Q$ (for a scalar-function v) and $|u|_{m+\alpha}^Q$ (for a vector-function $u=(u^1,\dots,u^N)$) and symbols $C_{m+\alpha}^N(Q)$, $\bar{\bar{C}}_{2+\alpha}$ and $C_{2-o}$ which are defined in sec. 1 and 2 of [4]. Moreover, the symbols $C(Q)$, $C^N(Q)$ and $C_{p,p+1}^N(Q)$ of sec. 1 of [6] will be used.

$$(1.3)\qquad |B^k u|_0^{G^\tau} \le A_2 + A_3 |u|_{1,0}^{G^{T_0,\tau}} + A_4\left(|u|_{1,0}^{G}\right)^\theta + A_5 |u|_{1,0}^{G},$$

where

$$G^{\nu,\tau} = G \cap \left\{(x,t):\ \nu < t < \tau\right\} \qquad (T_0 \le \nu < \tau \le T),$$

$$|u|_{1,0}^{Q} = \sum_{k=1}^{N} |u^k|_0^Q + \sum_{i=1}^{n}\sum_{k=1}^{N} |u^k_{x_i}|_0^Q .$$

(1.VI) The vector-function $\varphi = (\varphi^1,\ldots,\varphi^N)$, defined on $\Sigma^T$, belongs to $C^N_{1,0}\left(\overline{G^{T_0,0}}\right)$ and possesses an extension $\phi \in C^N_{1+\beta}(\overline{G^T}) \cap C^N_{2,1}(\overline{G^T})$ $(0<\beta<1)$.

(1.VII) If a function $\phi \in C^N_{2,1}(\overline{G^T}) \cap C^N_{1,0}(\bar G)$ is an extension of $\varphi$, then

$$B^k \phi = L^k \phi^k, \ (x,0) \in R_0 = S \cap \{t=0\}.$$

T h e o r e m 1. If assumptions (1.I) – (1.IV), (1.VI), (1.VII) are satisfied and if $u \in C^N_{1,0}(\bar G) \cap C^N_{2,1}(\overline{G^T})$ is a solution of the problem (1.1), (1.2), then $u \in C^N_{1+\beta}(G^T)$. Moreover, if assumption (1.V) is satisfied with a sufficiently small constant $A_5$[2], then the norm $|u|_{1+\beta}^{G^T}$ is bounded by a constant $M$ depending only on $A_i$ $(i=0,1,\ldots,6)$, $\theta$, $\beta$ and $G^T$, where

$$A_6 > |\varphi|_{1+\beta}^{G^T}, \quad A_6 > |\varphi|_{2,1}^{G^T}\ ^{3)}, \quad A_6 \ge |\varphi|_{1,0}^{G^{T_0,0}} .$$

P r o o f. We apply a method similar to that used in the proof of Theorem 1 of [6]. Let the function $\phi$ be an extension of $\varphi$ such that

2) More precisely, the constant $A_5$ is bounded by a sufficiently small constant depending only on $A_0, A_1, A_3$, $\beta$ and $G^T$.

3) For the definition of these norms see the footnote 3 of paper [6].

$$|\phi|_{1+\beta}^{G^T} \leq A_6, \qquad |\phi|_{2,1}^{G^T} \leq A_6.$$

Then the function $v(x,t) = u(x,t) - \phi(x,t)$ is a solution of the problem

$$(1.4) \qquad L^k v^k = B^k u - L^k \phi^k, \; (x,t) \in \overline{G^T} \setminus \Sigma^T ,$$

$$(1.5) \qquad v^k(x,t) = 0, \; (x,t) \in \Sigma^T \; (k=1,\dots,N) .$$

Since the functions $B^k u - L^k \phi^k$ are continuous in $\overline{G^T}$ and vanish on $R_0$ (by assumptions (1.IV), (1.VII)), therefore according to Lemma 2 of [4], $v^k \in C_{1+\beta}(G^T)$ and consequently $u \in C^N_{1+\beta}(G^T)$. Moreover

$$|v^k|_{1+\beta}^{G^\tau} \leq K(\beta)\tau^\gamma |B^k(v+\phi) - L^k\phi^k|_0^{G^\tau},$$

where $\gamma = (1-\beta)/2$, $0 < \tau \leq T$ and $K(\beta)$ is a constant depending only on $\beta$, $A_0$, $A_1$ and $G^T$. Hence, by (1.3), we have

$$(1.6) \qquad |v|_{1+\beta}^{G^\tau} \leq NK(\beta)\tau^\gamma \left[A_3 |v|_{1+\beta}^{G^\tau} + A_4 \left(|v|_{1+\beta}^{G^\tau}\right)^\theta + A_5 |v|_{1+\beta}^{G^\tau}\right] + K_{11}.$$

If

$$N(A_3 + A_5)K(\beta)T^\gamma < 1,$$

then (1.6) implies the estimate

$$|v|_{1+\beta}^{G^T} \leq K_{12}\left(|v|_{1+\beta}^{G^T}\right)^\theta + K_{13},$$

whence

$$|v|_{1+\beta}^{G^T} \leq K_{14} \quad \text{i.e.} \quad |u|_{1+\beta}^{G^T} \leq K .$$

Now we shall consider the case

$$NA_3K(\beta)T^\gamma \geq 1 .$$

We put

(1.7) $$\tau = p^{-1}\left[4NA_3K(\beta)\right]^{-1/\gamma},$$

where $p \geq 1$ is a number such that T is an integer multiple of $\tau$ . Let $\xi(t)$ be a function (defined for real t) with continuous derivative $\xi'(t)$ such that $0 \leq \xi(t) \leq 1$, $\xi(t) = 0$ for $t \leq \tau/2$ and $\xi(t) = 1$ for $t \geq \tau$ . We assume that the constant $A_5$ satisfies the following condition

(1.8) $$A_5 \leq \left[NK(\beta)\tau^{\gamma}\lambda^{i_0+1}\right]^{-1},$$

where $i_0 = 2T\tau^{-1} - 2$ while $\lambda \geq \max(1, \sup|\xi'(t)|)$ is a sufficiently large constant which will be specified later.

It follows from (1.7) and (1.8) that

(1.9) $$NA_3K(\beta)\tau^{\gamma} \leq 1/4,$$

(1.10) $$NA_5K(\beta)\tau^{\gamma} \leq \lambda^{-i_0-1}$$

Assuming that $\lambda^{i_0+1} \geq 4$ and using (1.9), (1.10) we obtain from (1.6) the estimate

(1.11) $$|v|^{G^{\tau}}_{1+\beta} \leq 2\lambda^{-i_0-1}|v|^{G^{\tau,T}}_{1+\beta} + K_1\left(|v|^{G^{\tau}}_{1+\beta}\right)^{\theta} + K_1'.$$

Now we shall estimate the norm $|v|^{G^{3\nu}}_{1+\beta}$ , where $\nu = \tau/2$. It is easy to see that the function $w(x,t) = \xi(t)v(x,t)$ is a solution of the problem.

$$L^k w^k = \xi(t)\left[B^k(v+\phi) - L^k\phi^k\right] - \xi'(t)v^k \equiv g^k(x,t), (x,t) \in \overline{G^{\nu,3\nu}} \setminus \Sigma^{\nu,3\nu},$$

$$w^k(x,t) = 0, \quad (x,t) \in \Sigma^{\nu,3\nu} = S^{3\nu} \cup \overline{G^{T_0,\nu}} \quad (k=1,\dots,N).$$

Since $g^k(x,\nu) = 0$ for $(x,\nu) \in S \cap \{t = \nu\}$, therefore (as before) we have

$$|w^k|^{G^{\nu,3\nu}}_{1+\beta} \leq K(\beta)\tau^{\gamma}|g^k|^{G^{\nu,3\nu}}_0$$

Hence, recalling that

$$w^k(x,t) = v^k(x,t) \text{ for } (x,t) \in \overline{G^{2\nu,3\nu}},$$

we get

$$|v^k|^{G^{2\nu,3\nu}}_{1+\beta} \leq NK(\beta)\tau^{\gamma}|B^k(v+\phi)|^{G^{\nu,3\nu}}_0 + \lambda K(\beta)\tau^{\gamma}|v^k|^{G^{\nu,2\nu}}_{1+\beta} + K_{21},$$

which, by (1.3), implies

$$|v|^{G^{2\nu,3\nu}}_{1+\beta} \leq NK(\beta)\tau^{\gamma}\left[A_3|v|^{G^{3\nu}}_{1+\beta} + A_5|v|^{G^T}_{1+\beta} + A_4(|v|^{G^T}_{1+\beta})^{\theta}\right] +$$

$$+ \lambda K(\beta)\tau^{\gamma}|v|^{G^{\nu,2\nu}}_{1+\beta} + K_{22}.$$

The last inequality together with (1.11) yields

$$(1.12)\qquad |v|^{G^{2\nu,3\nu}}_{1+\beta} \leq NK(\beta)\tau^{\gamma}\left[(A_3+A_5)(1+2\lambda^{-i_0-1})+2N^{-1}\lambda^{-i_0}\right]|v|^{G^{2\nu,3\nu}}_{1+\beta} +$$

$$+NK(\beta)\tau^{\gamma}\left[2A_3\lambda^{-i_0-1}+A_5(1+2\lambda^{-i_0-1})+2N^{-1}\lambda^{-i_0}\right]|v|^{G^{3\nu,T}}_{1+\beta} + K_{23}(|v|^{G^T}_{1+\beta})^{\theta} + K_{24}$$

Now let us take the constant $\lambda$ so large that

$$2NK(\beta)\tau^{\gamma}(A_3\lambda^{-i_0-1} + N^{-1}\lambda^{-i_0}) + \lambda^{-i_0-1}(1+2\lambda^{-i_0-1}) \leq 1/4 .$$

Then, by (1.9) and (1.10), the coefficient at $|v|^{G^{2\nu,3\nu}}_{1+\beta}$ on the right hand side of inequality (1.12) is less than or equal to 1/2. Thus, in virtue of (1.10), we get from (1.12) the inequality

$$(1.13)\qquad |v|^{G^{2\nu,3\nu}}_{1+\beta} \leq a_1\lambda^{-i_0}|v|^{G^{3\nu,T}}_{1+\beta} + K_{25}(|v|^{G^T}_{1+\beta})^{\theta} + K_{26},$$

where $a_1 \leq 2+2K(\beta)\tau^{\gamma}$. It follows from (1.11), (1.13) that

$$|v|^{G^T}_{1+\beta} \leqslant 2(1+a_1)\lambda^{-i_0}|v|^{G^{3\nu,T}}_{1+\beta} + K_{27}\left(|v|^{G^T}_{1+\beta}\right)^{\theta} + K_{28},$$

which together with (1.13) implies the estimate

$$|v|^{G^{3\nu}}_{1+\beta} \leqslant (2+3a_1)\lambda^{-i_0}|v|^{G^{3\nu,T}}_{1+\beta} + K_2\left(|v|^{G^T}_{1+\beta}\right)^{\theta} + K_2'.$$

In the next step we estimate the norm $|v|^{G^{4\nu}}_{1+\beta}$. We apply the method used in the previous step with $G^{\nu,3\nu}$ and $\xi(t)$ replaced by $G^{2\nu,4\nu}$ and $\xi(t-\nu)$, respectively.

Proceeding further in the above manner, step by step, we finally obtain the estimate

$$|v|^{G^{T-\nu}}_{1+\beta} \leqslant a_{i_0-1}\lambda^{-2}|v|^{G^{T-\nu,T}}_{1+\beta} + K_{i_0}\left(|v|^{G^T}_{1+\beta}\right)^{\theta} + K_{i_0}',$$

which in the next step implies

$$|v|^{G^T}_{1+\beta} \leqslant K'\left(|v|^{G^T}_{1+\beta}\right)^{\theta} + K''.$$

Hence

$$|v|^{G^T}_{1+\beta} \leqslant \bar{K} \quad \text{i.e} \quad |u|^{G^T}_{1+\beta} \leqslant M,$$

which completes the proof.

Let us note the following consequence from the proof of Theorem 1.

R e m a r k 1. Theorem 1 holds true for every solution $v \in C^N_{1,0}(\bar{G}) \cap C^N_{2,1}(\overline{G^T})$ of the problem

$$L^k v^k = \omega B^k(v+\phi) - \omega L^k \phi^k, \quad (x,t) \in \overline{G^T} \setminus \Sigma^T,$$

$$v^k(x,t) = 0; \quad (x,t) \in \Sigma^T \quad (k=1,\dots,N),$$

where $\phi$ is the extension of $\varphi$ occurring in the proof of Theorem 1 and $\omega$ is an arbitrary constant. Moreover, for $0 \leqslant \omega \leqslant 1$, the constant $M$ which bounds the norm $|v|^{G^T}_{1+\beta}$ is independent of $\omega$.

Now we discuss the existence of solutions of the problem considered. The following additional assumptions will be needed:

(1.VIII) The vector-function $\varphi$, defined on $\Sigma^T$, belongs to $C^N_{1+\alpha}(\overline{G^{T_0,0}})$ and possesses an extension $\phi \in C^N_{1+\beta}(G^T) \cap C^N_{2+\alpha}(G^T)$, where $0<\alpha<\beta<1$.

(1.IX) The operators $B^k$ map the space $C^N_{1+\alpha}(G)$ into the set $\bigcup_{0<\varepsilon<1} C_\varepsilon(G^T)$ and are continuous in the following sense: if

$$u, u_m \in C^N_{1+\alpha}(\bar{G}),\ u_m(x,t) = u(x,t),\ (x,t) \in \overline{G^{T_0,0}} \quad (m=1,2,\dots)$$

and

$$\lim_{m\to\infty} |u_m - u|^G_{1+\alpha} = 0,$$

then

$$\lim_{m\to\infty} |B^k u_m - B^k u|^{G^T}_0 = 0.$$

T h e o r e m 2. If assumptions (1.I) - (1.V), (1.VII)-(1.IX) are satisfied and the constant $A_5$ is sufficiently small (see the footnote 2), then there exists a solution $u(x,t)$ of the problem (1.1), (1.2); moreover $u \in C^N_{1+\beta}(G^T) \cap C^N_{2+\varepsilon}(G^T)$ for some $\varepsilon$, $0<\varepsilon<1$.

We apply the method of Leray-Schauder. Let us denote by $\Omega$ the set of all functions $v \in C^N_{1+\alpha}(\bar{G})$ such that $v(x,t)=0$ on $\Sigma^T$. Obviously $\Omega$ is a Banach space (as a subspace of $C^N_{1+\alpha}(\bar{G})$). For $v \in \Omega$ and $\omega \in [0,1]$ let us consider the problem

$$L^k w^k = \omega B^k(v+\phi) - L^k \phi^k, \quad (x,t) \in \overline{G^T} \setminus \Sigma^T,$$

$$w^k(x,t) = 0, \quad (x,t) \in \Sigma^T \ (k=1,\dots,N),$$

where $\phi \in C^N_{1+\beta}(G^T) \cap C^N_{2,1}(G^T)$ is an extension of $\varphi$. In view of Lemmas 1 and 2 of [4] there exists a unique solution $w(x,t) = (w^1(x,t),\dots,w^N(x,t))$ of this problem and, moreover, $w \in C^N_{1+\beta}(G^T)$

$\cap C^N_{2+\varepsilon}(G^T)$ for some $0<\varepsilon<1$. This enables us to define a transformation Z by formula $Z(v,\omega) = w$. Proceeding further like in the proof of Theorem 4 of [6] and using Remark 1 one can show the existence of a solution $v(x,t)$ (which belongs to $C^N_{1+\beta}(G^T) \cap C^N_{2+\varepsilon}(G^T)$ for some $\varepsilon$, $0<\varepsilon<1$) of the problem

$$L^k v^k = B^k(v+\phi) - L^k \phi^k, \quad (x,t)\in \overline{G^T}\setminus\Sigma^T,$$

$$v^k(x,t) = 0, \quad (x,t)\in \Sigma^T \ (k=1,\dots,N).$$

Hence it immediately follows that the function $u(x,t) = v(x,t) + \phi(x,t)$ is a solution of the problem (1.1), (1.2) and $u \in C^N_{1+\beta}(G^T) \cap C^N_{2+\varepsilon}(G^T)$, which was to be proved.

Adding to assumptions of Theorem 2 the Lipschitz condition for operators $B^k$ we obtain the existence and uniqueness of solutions of the problem in question.

(1.X) For any $u,v \in C^N_{1,0}(\bar G) \cap C^N_{1+\alpha}(G^T)$ such that u=v in $\overline{G^{T_0,0}}$ and $|u|^{G^T}_{1+\alpha}$, $|v|^{G^T}_{1+\alpha} \le M$ (M being the constant occurring in Theorem 1 in the case $\beta=\alpha$) we have for any $\tau$ $(0<\tau\le T)$

$$|B^k u - B^k v|^{G^\tau}_0 \le A_7 |u-v|^{G^\tau}_{1+\alpha} + A_8 |u-v|^{G^T}_{1+\alpha},$$

where $A_7, A_8$ are certain positive constants independent of $\tau$.

T h e o r e m 3. If assumptions (1.I)-(1.V), (1.VII)-(1.X) are satisfied and the constants $A_5$, $A_8$ [4] are sufficiently small, then the problem (1.1), (1.2) has a unique solution $u = \{u^k\}$ in the space $C^N_{1,0}(\bar G) \cap C^N_{2,1}(\overline{G^T})$. Moreover, $u \in C^N_{1+\beta}(G^T) \cap C^N_{2+\varepsilon}(G^T)$ for some $\varepsilon$, $0<\varepsilon<1$.

P r o o f. We have to show only the uniqueness of solutions. So let $u,v \in C^N_{1,0}(\bar G) \cap C^N_{2,1}(\overline{G^T})$ be two solutions of the problem (1.1), (1.2). It means that

[4] Concerning the constant $A_5$ see the footnote 2 whereas $A_8$ is bounded by a small constant depending only on $A_0$, $A_1$, $\alpha$, $A_7$ and $G^T$.

$$L^k(u^k - v^k) = B^k u - B^k v, \ (x,t)\in \overline{G^T}\setminus \Sigma^T,$$

$$u^k - v^k = 0 \ \text{ on } \ \Sigma^T \ \ (k=1,\dots,N).$$

Since, by Theorem 1,

$$u,v\in C^N_{1+\alpha}(G^T) \quad \text{and} \quad |u|^{G^T}_{1+\alpha}, \ |v|^{G^T}_{1+\alpha} \leqslant M,$$

therefore taking into considerations the last relation and using Lemma 2 of [4] and assumption (1.X) we obtain,for any $\tau(0<\tau\leqslant T)$ the following inequality

$$(1.14) \qquad |u-v|^{G^\tau}_{1+\alpha} \leqslant NA_7K(\alpha)\tau^{\gamma}|u-v|^{G^\tau}_{1+\alpha} + NA_8K(\alpha)\tau^{\gamma}|u-v|^{G^T}_{1+\alpha}$$

$$(\gamma=(1-\alpha)/2).$$

If

$$(1.15) \qquad N(A_7 + A_8)K(\alpha)T^{\gamma} < 1,$$

then (1.14) implies the identity $u\equiv v$ in $\overline{G^T}$ i.e. $u\equiv v$ in $\bar{G}$.

In the case when (1.15) does not hold we proceed like in the proof of Theorem 1. Therefore we only outline the further argumentation.

Let us put

$$\tau = p^{-1}\left[4\ NA_7K(\alpha)\right]^{-1/\gamma} \quad (\gamma=(1-\alpha)/2)$$

and assume that

$$A_8 \leqslant \left[NK(\alpha)\tau^{\gamma}\lambda^{i_0+1}\right]^{-1},$$

where the symbols $p$, $\lambda$, $i_0$, $\xi(t)$ retain their previous meaning. Then it follows from (1.14) that

$$(1.16) \qquad |u-v|^{G^\tau}_{1+\alpha} \leqslant 2\lambda^{-i_0-1}|u-v|^{G^{\tau,T}}_{1+\alpha},$$

where $\lambda^{i_0+1} \geqslant 4$.

In order to estimate the norm $|u-v|^{G^{3\nu}}_{1+\alpha}$ $(\nu=\tau/2)$ observe that the functions $\bar{u}(x,t)=\xi(t)u(x,t)$, $\bar{v}(x,t)=\xi(t)\,v(x,t)$ fulfil the following relations

$$L^k(\bar{u}^k-\bar{v}^k) = \xi(t)(B^k u-B^k v) - \xi'(t)(u^k-v^k),\ (x,t)\in \overline{G^{\nu,3\nu}}\setminus\Sigma^{\nu,3\nu};$$

$$u^k-v^k=0 \quad \text{on} \quad \Sigma^{\nu,3\nu} \quad (k=1,\dots,N).$$

Hence by (1.16) we get (taking a suitable enlarged $\lambda$ )

$$(1.17) \qquad |u-v|^{G^{3\nu}}_{1+\alpha} \leqslant b_1\,\lambda^{-i_0}\ |u-v|^{G^{3\nu,T}}_{1+\alpha}.$$

Replacing in the above reasoning $\xi(t)$, $G^{\nu,3\nu}$ and $\Sigma^{\nu,3\nu}$ by $\xi(t-\nu)$, $G^{2\nu,4\nu}$ and $\Sigma^{2\nu,4\nu}$, respectively (and using (1.17) instead of (1.16)), one can show that

$$|u-v|^{G^{4\nu}}_{1+\alpha} \leqslant b_2\lambda^{-i_0+1}\,|u-v|^{G^{4\nu,T}}_{1+\alpha}.$$

Finally, for sufficiently large $\lambda$, we obtain the estimate

$$|u-v|^{G^{T-\nu}}_{1+\alpha} \leqslant b_{i_0-1}\lambda^{-2}\,|u-v|^{G^{T-\nu,T}}_{1+\alpha},$$

which in the next step implies

$$|u-v|^{G^T}_{1+\alpha} \leqslant 0 .$$

Thus $u\equiv v$ in $\overline{G^T}$, and the proof is completed.

At present we shall consider the existence of a maximum solution and a minimum solution of the problem (1.1),(1.2)[5]. For this purpose we state the following theorem.

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[5] The solution $u(x,t)$ of (1.1), (1.2) is called a maximum (minimum) solution if for every solution $w(x,t)$ of (1.1), (1.2) the inequalities $w^k(x,t)\leqslant u^k(x,t)$ $(w^k(x,t)\geqslant u^k(x,t))$ hold in $\overline{G^T}$ $(k=1,\dots,N)$.

T h e o r e m 4. Let assumptions (1.I)-(1.V) (with $A_5$ sufficiently small) and (1.IX) be satisfied and suppose that

(1.XI) the vector-function $\varphi$, defined on $\Sigma^T$, belongs to $C^N_{1+\alpha}(\overline{G^{T_0,0}})$ and possesses an extension $\phi \in C^N_{1+\beta}(\overline{G^T}) \cap C^N_{2,1}(\overline{G^T})$.

Under these assumptions there exists a solution $u(x,t) \in C^N_{1+\beta}(G^T) \cap W^N_{2+\varepsilon}(G^T)$ [6] of the problem (1.1), (1.2), where $\varepsilon\,(0 < \varepsilon < 1)$ is a certain constant.

P r o o f. Procceding in the same manner as in the proof of Theorem 1 and using, instead of Lemma 2 of [4], Lemma 2 of [5], one can derive an a priori estimate of the norm $|u|^{G^T}_{1+\beta}$ for a solution u of problem (1.1), (1.2). The further argumentation is similar to that used in the proof of Theorem 2; namely we apply the method of Leray-Schauder, making use of the above estimate and of Lemmas 1 and 2 of [5].

Theorem 4 enables us to prove, by the same considerations as those for Theorem 2 of [5], the following theorem.

T h e o r e m 5. Let the assumptions of Theorem 4 and the following one be satisfied:

(1.XII) If the functions $u=(u^1,\dots,u^N)$ and $v=(v^1,\dots,v^N)$ belong to $C^N_{1,0}(\bar{G}) \cap C^N_{2,1}(\overline{G^T})$ and fulfil the inequalities

$$L^k u^k - B^k u > L^k v^k - B^k v, \quad (x,t) \in \overline{G^T} \setminus \Sigma^T \quad (k=1,\dots,N),$$

$$u(x,t) < v(x,t), \quad (x,t) \in \Sigma^T,$$

then $u(x,t) < v(x,t)$ in $\overline{G^T}$.

These being assumed the problem (1.1), (1.2) has a maximum solution $v=\{v^k\}$ and a minimum solution $u = \{u^k\}$; moreover, $v,u \in C^N_{1+\beta}(G^T) \cap W^N_{2+\varepsilon}(G^T)$ for some $\varepsilon$, $0 < \varepsilon < 1$.

As in paper [5], one can obtain a theorem on weak inequalities which is a counterpart of Theorem 3 of [5].

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6) For the definition of $W^N_{2+\alpha}(Q)$ see sec. 1 of [5] and the footnote 4 of [5].

## 2. Differential equations with functional arguments

In this section we give examples of operators $B^k$ for which theorems of the previous section hold true. For these examples we formulate the corollaries only from Theorems 2, 3 and 5. At first we will discuss the general case

$$(2.1) \qquad B^k u = f^k(x,t,\psi_1^k(u),\ \psi_2^k(u),\ \psi_3^k(u)) \qquad (k=1,\dots,N) .$$

Next we shall consider the special case of operators $\psi_i^k$ given by formulas

$$(2.2) \qquad \psi_i^k(u) = (u(V_{3i-2}^k(x,t)),\quad u_x(W_i^k(x,t)),$$

$$\int_{G_t} u(V_{3i-1}^k(y,t))\mu_i^k(x,t;dy), \int_{G^t} u(V_{3i}^k(y,\tau)\nu_i^k(x,t;dy,d\tau)),$$

where

$$u(V_l^k(x,t)) = \left\{u^j(V_l^{kj}(x,t))\right\} \qquad (j=1,\dots,N),$$

$$\int_{G_t} u(V_{3i-1}^k(y,t))\mu_i^k(x,t;dy) =$$

$$= \left\{\int_{G_t} u^j(V_{3i-1}^{kj}(y,t))\mu_i^{kj}(x,t;dy)\right\} \qquad (j=1,\dots,N),$$

$$\int_{G^t} u(V_{3i}^k(y,\tau))\nu_i^k(x,t;dy,d\tau) = \left\{\int_{G^t} u^j(V_{3i}^{kj}(y,\tau)\nu_i^{kj}(x,t;dy,d\tau)\right\}$$

$$(j = 1,\dots,N),$$

$$u_x(W_i^k(x,t)) = \left\{u_{x_l}^j(W_{il}^{kj}(x,t))\right\} \qquad (j=1,\dots N;\quad l=1,\dots,n),$$

$$G_t = \left\{ x: (x,t) \in \overline{G^T} \setminus S^T \right\}.$$

The following assumptions will be introduced:

(2.I) The functions $f^k(x,t,p_1^1,\dots,p_1^{N_0},p_2^1,\dots,p_2^{N_0},p_3^1,\dots p_3^{N_0})$ $(k=1,\dots,N)$, defined on $\overline{G^T} \times E_{3N_0}$, satisfy a uniform Hölder condition in every bounded domain $\overline{G^T} \times H$ $(H \subset E_{3N_0})$. Moreover, there are constants $M_i > 0$ $(i=1,2,3,4)$, $0 < \theta_1 < 1$ such that

$$\left| f^k(x,t,p_1,p_2,p_3) \right| \le M_1 + M_2 |p_1| + M_3 |p_2|^{\theta_1} + M_4 |p_3|,$$

where

$$|p_i| = \sum_{j=1}^{N_0} |p_i^j|.$$

(2.II) $\psi_i^k$ $(k=1,\dots,N;\ i=1,2,3)$ are operators defined on $C_{1,0}^N(\bar{G})$ with values belonging to $C^{N_0}(\overline{G^T})$. There exist constants $M_i > 0$ $(i=5,\dots,10)$ and $0 < \theta_2 \le 1$ such that for any $\tau (0 < \tau \le T)$ and any $u \in C_{1,0}^N(\bar{G})$ we have

$$\left| \psi_1^k(u) \right|_0^{G^\tau} \le M_5 + M_6 |u|_{1,0}^{G^{T_0,\tau}} \text{ 7)},$$

$$\left| \psi_2^k(u) \right|_0^{G^\tau} \le M_7 + M_8 \left( |u|_{1,0}^{G} \right)^{\theta_2},$$

$$\left| \psi_3^k(u) \right|_0^{G^\tau} \le M_9 + M_{10} |u|_{1,0}^{G}.$$

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7) For a vector-function $v=(v^1,\dots,v^{N_0})$ the norm $|v|_0^Q$ is defined by the formula

$$|v|_0^Q = \sum_{k=1}^{N_0} |v^k|_0^Q$$

Moreover, operators $\psi_i^k$ map the space $C^N_{1+\alpha}(G)$ into the set $\bigcup_{0<\varepsilon<1} C^{N}_{\varepsilon}{}^{o}(G^T)$ and are continuous in the same sense as operators $B^k$ (see assumption (1.IX)).

It is easy to see that assumptions (2.I), (2.II) imply (1.IV), (1.V), (1.IX). Therefore we obtain, as a corollary from Theorem 2, the following theorem.

T h e o r e m 6. Let assumptions (1.I) - (1.III), (2.I), (2.II), (1.VIII) and (1.VII) (in the case (2.1)) be satisfied and let the product $M_4M_{10}$ be sufficiently small. Then Theorem 2 is true in the case (2.1).

In order to formulate a corollary from Theorem 3 for the case (2.1) we make the following assumptions:

(2.III) For any $(x,t) \in G^T$ and $p_i$, $\bar{p}_i$ (i=1,2,3) such that

$$|p_1|,\ |\bar{p}_1| \leq M_6M,\ |p_2|,\ |\bar{p}_2| \leq M_8M,\ |p_3|,\ |\bar{p}_3| \leq M_{10}M ,$$

where M is the bound of the norm $|u|^{G^T}_{1+\alpha}$ of the solution u of the problem (1.1), (1.2) in the case (2.1), we have

$$\left|f^k(x,t,p_1,p_2,p_3)-f^k(x,t,\bar{p}_1,\bar{p}_2,\bar{p}_3)\right| \leq \tag{2.3}$$

$$\leq M_{11}|p_1-\bar{p}_1| + M_{12}(|p_2-\bar{p}_2|+|p_3-\bar{p}_3|) ,$$

$M_{11}$, $M_{12}$ being positive constants.

(2.IV) For any $u,v \in C^N_{1,0}(\bar{G}) \in C^N_{1+\alpha}(G^T)$ such that

$$u=v \ \text{ in } \ G^{T_0,0} \quad \text{and} \quad |u|^{G^T}_{1+\alpha},\ |v|^{G^T}_{1+\alpha} \leq M$$

(M being the same as in (2.III)) and for any $\tau$ $(0<\tau\leq T)$ there hold the inequalities:

$$\left|\psi^k_1(u) - \psi^k_1(v)\right|^{G^\tau}_0 \leq M_{13}\ |u-v|^{G^\tau}_{1+\alpha} ,$$

$$\left|\psi^k_i(u) - \psi^k_i(v)\right|^{G^\tau}_0 \leq M_{14}|u-v|^{G^T}_{1+\alpha} \qquad (i=2,3).$$

Since assumptions (2.III), (2.IV) imply (1.X), therefore from Theorem 3 we immediately obtain the following theorem.

Theorem 7. Let the assumptions of Theorem 6 and (2.III), (2.IV) with a sufficiently small product $M_9M_{10}$ be satisfied. Then the conclusion of Theorem 3 holds true in the case (2.1).

Now we shall consider the existence of a maximum solution and a minimum solution of the problem (1.1), (1.2) for operators $B^k$ given by formulas:

$$B^k u = f^k(x,t,u(x,t),u_x^k(x,t),\ \psi^k(u)), \tag{2.4}$$

where $u_x^k = \left(u_{x_1}^k,\dots,u_{x_n}^k\right)$.

The following assumptions will be used:

(2.V) The functions $f^k(x,t,p_1,\dots,p_N,q_1,\dots,q_n,r_1,\dots,r_{N_1})$ $(k=1,\dots,N)$, defined on $\overline{G}^T \times E_{N+n+N_1}$, are non-increasing with respect to the variables $p_1,\dots,p_{k-1},p_{k+1},\dots,p_N,r_1,\dots,r_{N_1}$.

(2.VI) The functions $f^k(x,t,p,q,r)$ satisfy a uniform Hölder condition in every bounded domain $\overline{G}^T \times H$ $(H \subset E_{N+n+N_1})$. Moreover, there are constants $M_{15},M_{16} > 0$ such that for any $(x,t,p,q,r) \in \overline{G}^T \times E_{N+n+N_1}$ we have

$$\left|f^k(x,t,p,q,r)\right| \leqslant M_{15}+M_{16}\left(\sum_{i=1}^{N}|p_i| + \sum_{i=1}^{n}|q_i| + \sum_{i=1}^{N_1}|r_i|\right).$$

(2.VII) $\psi^k$ $(k=1,\dots,N)$ are operators defined on $C_{1,0}^N(\overline{G})$ with values belonging to $C^{N_1}(\overline{G}^T)$ which are non-decreasing in the following sense: if $u \leqslant v$ in $G^{T_0,\tau}$ $(0<\tau\leqslant T)$, then $\psi^k(u) \leqslant \psi^k(v)$ in $G^\tau$.

(2.VIII) There exist constants $M_{17},M_{18} > 0$ such that for any $\tau$ $(0<\tau\leqslant T)$ and any $u \in C_{1,0}^N(\overline{G})$ we have

$$|\psi^k(u)|_0^{G^\tau} \leq M_{17} + M_{18}|u|_{1,0}^{G^{\tau_0,\tau}}.$$

Moreover, the operators $\psi^k$ map $C^N_{1+\alpha}(G)$ into the set $\bigcup_{0<\varepsilon<1} C_\varepsilon^{N_1}(G^T)$ and are continuous in the same sense as the operators $B^k$ (assumption (1.IX)).

T h e o r e m 8. If assumptions (1.I) - (1.III), (2.V)-(2.VIII) are satisfied, then the assertion of Theorem 5 is true in the case (2.4).

This theorem is a consequence of Theorem 5 and of the following lemma.

L e m m a 1. Let assumptions (1.I), (1.III), (2.V) and (2.VII) be satisfied. Suppose that functions $u,v \in C^N_{1,0}(\bar{G}) \cap C^N_{2,1}(G^T)$ fulfil the inequalities

$$L^k u^k - f^k(x,t,u,u^k_x, \psi^k(u)) > L^k v^k - f^k(x,t,v,v^k_x,\psi^k(v))$$

$$(x,t) \in \overline{G^T} \setminus \Sigma^T \ (k=1,\dots,N), \ u(x,t) < v(x,t), (x,t) \in \Sigma^T.$$

Under these assumptions we have $u(x,t) < v(x,t)$ in $\bar{G}$.

The method of proving this lemma is the same as that used to prove the theorem on strong differential inequalities (see [3], p.191).

At present we shall treat the case (2.1) with operators $\psi^k_i$ given by (2.2) (shortly the case (2.1),(2.2)). The following assumptions will be needed:

(2.IX) Operators $V^{kj}_i$, $W^{kj}_{lm}$ $(i=1,\dots,9;\ k,j=1,\dots,N;\ l=1,2,3;\ m=1,\dots,n)$ map $\overline{G^T}$ into $\bar{G}$ and satisfy the uniform Hölder condition:

$$d(V^{kj}_i(P),V^{kj}_i(P')) \leq M_{19}[d(P,P')$$

$$(0<\alpha_0 \leq 1)$$

$$d(W^{kj}_{lm}(P), W^{kj}_{lm}(P')) \leq M_{19}[d(P,P')]^{\alpha_0},$$

where

$$P=(x,t),\quad P'=(x',t'),\ d(P,P')=\left(|x-x'|^2+|t-t'|\right)^{1/2},$$

$$|x-x'|=\left[\sum_{i=1}^{n}(x_i-x_i')^2\right]^{1/2}.$$

Moreover, for every $\tau\,(0<\tau\leq T)$ the operators $V_i^{kj}$, $W_{1m}^{kj}$ ($i = 1,2,3;\ k,j=1,\dots,N;\ m=1,\dots,n$) map the domain $\overline{G^{\tau}}$ into $\overline{G^{\tau_0,\tau}}$.

(2.X) Let $\mathfrak{m}$ (resp. $\mathfrak{n}$) denote the $\sigma$-field of all Lebesgue measurable subsets of $\overline{G^T}$ (resp. $D_0=\overline{\bigcup_{0\leq t\leq T}G_t}$). By $\nu_i^{kj}(x,t;D)$ and $\mu_i^{kj}(x,t;D)$ ($i=1,2,3;\ k,j=1,\dots,N$) we will denote finite non-negatives measures (depending on $(x,t)\in\overline{G^T}$) defined on $\mathfrak{m}$ and $\mathfrak{n}$, respectively. The following conditions are imposed:

$1^0$ There is a constant $M_{20}>0$ such that for any $(x,t)\in\overline{G^T}$

$$\nu_i^{kj}(x,t;\overline{G^T}),\qquad \mu_i^{kj}(x,t;D_0)\leq M_{20}.$$

$2^0$ There exists a finite non-negative measure $\bar\nu$ (resp. $\bar\mu$) defined on $\mathfrak{m}$ (resp. $\mathfrak{n}$) such that for any $D\in\mathfrak{m}$ (resp. $D\in\mathfrak{n}$) and for any points $P(x,t)$, $P'(x',t')$ of the domain $\overline{G^T}$ we have

$$\left|\nu_i^{kj}(x,t;D)-\nu_i^{kj}(x',t';D)\right|\leq M_{21}\,\bar\nu(D)\left[d(P,P')\right]^{\alpha_1}$$

$$\left(\text{resp. } \left|\mu_i^{kj}(x,t;D)-\mu_i^{kj}(x',t';D)\right|\leq M_{21}\bar\mu(D)\left[d(P,P')\right]^{\alpha_1}\right),$$

where $M_{21}>0$ and $0<\alpha_1<1$ are certain constants.

$3^0$ There is a constant $M_{22}>0$ such that for any $D\in\mathfrak{m}$ (resp. $D\in\mathfrak{n}$) we have

$$\nu_i^{kj}(x,t;D)\leq M_{22}\,m_1(D)\quad(\text{resp. } \mu_i^{kj}(x,t;D)\leq M_{22}\,m_2(D)),$$

$m_1(D)$ ($m_2(D)$) being the (n+1)-dimensional (n-dimensional) Lebesgue measure of D.

L e m m a 2. Let $\nu(x,t;D)$ be a measure satisfying all the conditions imposed on the measures $\nu_i^{kj}(x,t;D)$ in assumption (2.X). Suppose that V is a continuous operator mapping the domain $\overline{G}^T$ into $\overline{G}$. Under these assumptions if $w(x,t) \in C(\overline{G})$, then the function

$$\overline{w}(x,t) = \int_{G^t} w(V(y,\tau))\,\nu(x,t;dy,d\tau)$$

belongs to $C_{\alpha_1}(G^T)$.

P r o o f. Let $P(x,t)$, $P'(x',t') \in G^T$. Without loss of generality we may assume that $t \geq t'$. Then

$$(2.5) \qquad \overline{w}(x,t)-\overline{w}(x',t')=I_1+I_2-I_3,$$

where

$$I_1 = \int_{G^t \setminus G^{t'}} w(V(y,\tau))\nu(x,t;dy,d\tau),$$

$$I_2 = \int_{G^{t'}} w(V(y,\tau))\nu(x,t;dy,d\tau),$$

$$I_3 = \int_{G^{t'}} w(V(y,\tau))\nu(x',t';dy,d\tau).$$

With the aid of conditions $1^o$ and $3^o$ of assumption (2.X) it follows that

$$(2.6) \qquad |I_1| \leq M_{20}M_{22}\,|w|_o^G\,|t-t'|\,.$$

A direct application of the definition of integral and condition $2^o$ of (2.X) yields the estimate

$$(2.7) \qquad |I_2 - I_3| \leq M_{21}\,|w|_o^G\,\overline{\nu}(G^{t'})\left[d(P,P')\right]^{\alpha_1}.$$

Relations (2.5) - (2.7) imply the inequality

$$\left|\overline{w}(x,t)-\overline{w}(x',t')\right| \leq \text{const.}\left[d(P,P')\right]^{\alpha_1},$$

which completes the proof.

T h e o r e m 9. Let assumptions (1.I)-(1.III),(2.I)with $N_0=3N+nN$, (2.IX),(2.X),(1.VIII) and (1.VII) (in the case (2.1), (2.2)) be satisfied and let the constant $M_4$ (in (2.I))be sufficiently small. Then the assertion of Theorem 2 is true in the case (2.1), (2.2).

For the proof it suffices to observe that assumptions (2.IX), (2.X) imply, by Lemma 2 and Lemma 4 of [4], the assumptions (2.VII), (2.VIII) and then to apply Theorem 6.

T h e o r e m 10. Let the assumptions of Theorem 9 be satisfied. Denote by M the bound of the norm $|u|^{G^T}_{1+\alpha}$ of solution u of the problem (1.1), (1.2) in the case (2.1),(2.2). We assume that for any $(x,t)\in\overline{G^T}$ and $p_i$, $\overline{p}_i$ (i=1,2,3) such that

$$|p_i|,|\overline{p}_i| \leq (1+2M_{20})M$$

there are fulfilled inequalities (2.3) with a sufficiently small constant $M_{12}$. Under these assumptions the conclusion of Theorem 3 is true in the case (2.1), (2.2).

This theorem is a consequence of Theorem 7.

Finally, in the case (2.4) with $\psi^k$ defined by the formula

$$(2.8)\quad \psi^k(u) = \left(\left\{u^i(V_1^{ki}(x,t))\right\},\ \left\{\int_{G_t} u^i(V_2^{ki}(y,t))\mu^{ki}(x,t;dy)\right\},\right.$$
$$\left.\left\{\int_{G^t} u^i(V_3^{ki}(y,\tau))\nu^{ki}(x,t;dy,d\tau)\right\}\right)\quad (i=1,\dots,N),$$

we easily obtain, as a corollary from Theorem 8,the following theorem.

T h e o r e m 11. If assumptions (1.I)-(1.III), (2.V), (2.VI) with $N_1=3N$, (2.IX) for $V_i^{kj}$ and (2.X) (with $\mu_i^{kj}$ and $\nu_i^{kj}$ replaced by $\mu^{kj}$ and $\nu^{kj}$, respectively) are satisfied, then the assertion of Theorem 5 remains valid in the case (2.4), (2.8).

R e m a r k 2. According to the Radon-Nikodym theorem (see for example [2], p.299) condition $3^0$ of assumption (2.X) implies the existence of non-negative functions $\varrho_i^{kj}(x,t,y)$, $\delta_i^{kj}(x,t,y,\tau)$ such that

$$\mu_i^{kj}(x,t;D) = \int_D \varrho_i^{kj}(x,t,y)dy,$$

$$\nu_i^{kj}(x,t;D) = \int_D \delta_i^{kj}(x,t,y,\tau)dy\, d\tau$$

(see also Remark 4 of [6]). However, if $G^T$ is a cylindrical domain, then the above-mentioned condition with regard to the measures $\mu_i^{kj}$ is superfluous in Theorems 9-11. Moreover, if in this case we replace the integrals over $G^t$ with respect to the measures $\nu_i^{kj}$ $(i=2,3;\ k,j=1,\ldots,N)$ (appearing in functions $f^k$) by integrals over $G^T$, then condition $3^0$ with respect to these measures may also be omitted.

We conclude this section by giving an example of operators $V_i^{kj}$, $W_{lm}^{kj}$ which fulfil assumption (2.IX) in the case when G is a cylindrical domain i.e. $G = D_0 \times (T_0,T)$. Namely, let $v_i^{kj}$, $w_{lm}^{kj}$ $(i=1,\ldots,9;\ k,j=1,\ldots,N;\ l=1,2,3;\ m=1,\ldots,n)$ be operators mapping the domain $D_0$ into itself and satisfying the uniform Hölder condition

$$|v_i^{kj}(x)-v_i^{kj}(x')| \leq M_{23}|x-x'|^{\alpha_0},$$

$$|w_{lm}^{kj}(x)-w_{lm}^{kj}(x')| \leq M_{23}|x-x'|^{\alpha_0}.$$

Moreover, take into considerations functions $g_i^{kj}(t)$, $h_{lm}^{kj}(t)$ $(i=1,\ldots,9;\ k,j=1,\ldots,N;\ l=1,2,3;\ m=1,\ldots,n)$ mapping the interval $[0,T]$ into $[T_0,T]$, satisfying the uniform Hőlder condition with exponent $\alpha_0/2$ and such that

$g_i^{kj}(t) \leq t$, $h_{1m}^{kj}(t) \leq t$ for $i=1,2,3;\ k,j=1,\ldots,N;\ m=1,\ldots,n.$

Obviously the operators $V_i^{kj}$, $W_{lm}^{kj}$ defined by the formulas

$$V_i^{kj}(x,t)=(v_i^{kj}(x),g_i^{kj}(t)),\qquad W_{lm}^{kj}(x,t) = (w_{lm}^{kj}(x),h_{lm}^{kj}(t))$$

fulfil assumption (2.IX).

REFERENCES

[1] A. F r i e d m a n: Partial differential equations of parabolic type. Englewood Cliffs 1964.

[2] R. S i k o r s k i: Funkcje rzeczywiste, t.I. Warszawa 1958.

[3] J. S z a r s k i: Differential inequalities. Warszawa 1965.

[4] H. U g o w s k i: On integro-differential equations of parabolic and elliptic type. Ann. Polon. Math, 22(1970), 255-275.

[5] H. U g o w s k i: On integro-differential equations of parabolic type. Ann.Polon.Math.25(1971),9-22.

[6] H. U g o w s k i: Some theorems on the estimate and existence of solutions of integro-differential equations of parabolic type. Ann. Polon.Math. 25(1972),311-323.

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