DEMONSTRATIO MATHEMATICA
Vol. V No 2 1973
In memory of Professor
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ON THE NON-LINEAR RIEMANN PROBLEM FOR AN INFINITE SYSTEM OF FUNCTIONS WITH DEVIATED ARGUMENT

Let S^+ and L respectively denote the interior and the boundary of the unit circle with centre at the origin.

W. Leksiński suggested the following problem to the author.

Problem. Investigate whether there exists an infinite sequence $\{\phi_n(z)\}$ of functions, holomorphic in the domain S^+, whose boundary values $\phi_n^+(t) = u_n(t)+iv_n(t)$ are of Hölder class such that their real parts $u_n(t)$ and imaginary parts $v_n(t)$ satisfy at each point $t \in L$ the infinite system of boundary conditions

$$(1)\qquad a_n(t)u_n(t) - b_n(t)v_n(t) = c_n(t) + F_n\Big\{t, u_1(t), v_1(t), u_1[\alpha(t)], v_1[\alpha(t)], u_2(t), v_2(t), u_2[\alpha(t)], v_2[\alpha(t)], \dots\Big\}$$

$$n=1,2,3,\dots$$

We make the following assumptions.

1° The real functions $a_n(t)$, $b_n(t)$ and $c_n(t)$ of the complex variable t are defined on L and satisfy Hölder's condition

$$(2)\qquad \begin{aligned} |a_n(t)-a_n(t')| &\le K_n|t-t'|^h \\ |b_n(t)-b_n(t')| &\le K_n|t-t'|^h \\ |c_n(t)-c_n(t')| &\le K_n|t-t'|^h \end{aligned}$$

where $t, t' \in L$; $n=1,2,3,\ldots$; $0 < h < 1$ and $K_n > 0$ are some constants. Furthermore, $\inf_{t \in L} \left[a_n^2(t) + b_n^2(t) \right] > 0$ for $n = 1,2,3,\ldots$

2^0 The real functions $F_n(t, x_1 x_2, \ldots)$, $n = 1,2,\ldots$, of the complex variable t and real variables $x_1, x_2, \ldots$ are defined in the domain

$$(3) \qquad t \in L, \quad |x_i| < R \quad (R > 0, i=1,2,\ldots)$$

and satisfy the conditions

$$(4) \quad \left|F_n(t, x_1, x_2, \ldots)\right| \leqslant M_n\left(1 + \sum_{i=1}^{\infty} m_{in} |x_i|\right), \quad \left(Z_n = \sum_{i=1}^{\infty} m_{in} < \infty\right) \quad n=1,2,3,\ldots$$

and

$$(5) \quad \left| F_n(t, x_1, x_2, \ldots) - F_n(t', x_1', x_2', \ldots) \right| \leqslant$$

$$\leqslant B_n |t - t'|^h + \sum_{i=1}^{\infty} D_{ni} |x_i - x_i'|, \quad \left(D_n = \sum_{i=1}^{\infty} D_{ni} < \infty\right) \quad n=1,2,3,\ldots$$

where M_n, m_{in}, B_n and D_{ni} denote some positive constants, $0 < h < 1$.

3^0. The complex function $\mathcal{L}(t)$ defined for $t \in L$ is continously differentiable and maps L onto itself in a one-to-one way such that the orientation is preserved.

Solution. Consider the functional space Λ of all infinite sequences $U = \{\varphi_n(t)\}$, whose terms are complex functions continuous on L. The distance in this space is defined by the formula

$$(6) \quad \varrho(U,V) \underset{df}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|U-V\|_n}{1+\|U-V\|_n}, \text{ where } \quad \|U\|_n = \sup_{t \in L} |\varphi_n(t)|.$$

It is well-known that Λ is a space of the type B_0. Let $Z(R,\varkappa)$ be the set in Λ whose points $U = \{\varphi_n(t)\}$ are all infinite sequences of functions satisfying the inequalities

$$(7) \qquad |\varphi_n(t)| \leq R, \qquad |\varphi_n(t)-\varphi_n(t')| \leq \varkappa |t-t'|^h \quad n=1,2,3,\dots$$

where R and h are the constants appearing in assumptions 1° and 2°, and $\varkappa$ is some positive constant.

The set $Z(R,\varkappa)$ is closed, convex and compact.

Following the method presented in [2] and assuming the notation

$$F_n^*\{t,\varphi(t),\varphi[\alpha(t)]\} =$$

$$= F_n\left\{t, \frac{\varphi_1(t)+\overline{\varphi_1(t)}}{2}, \frac{\varphi_1(t)-\overline{\varphi_1(t)}}{2i}, \frac{\varphi_1[\alpha(t)]+\overline{\varphi_1[\alpha(t)]}}{2},\right.$$

$$\left.\frac{\varphi_1[\alpha(t)]-\overline{\varphi_1[\alpha(t)]}}{2i}\dots\right\}$$

we can transform the set defined above as follows

$$\psi_n(t) = [a_n(t)+ib_n(t)]^{-1}\left[c_n(t)+F_n^*\{t, \varphi(t), \varphi[\alpha(t)]\}\right. +$$

$$(8) \qquad + \frac{X_n^+(t)}{2\pi i}\int_L \frac{c_n(\tau) + F_n^*\{\tau,\varphi(\tau),\varphi[\alpha(\tau)]\}}{[a_n(\tau)+ib_n(\tau)]X_n^+(\tau)(\tau-t)}\,d\tau +$$

$$+ \frac{X_n^+(\tau)t^{\varkappa_n+1}}{2\pi i}\int_L \frac{c_n(\tau)+F_n^*\{\tau,\varphi(\tau),\varphi[\alpha(\tau)]\}}{\tau^{\varkappa_n+1}[a_n(\tau)+ib_n(\tau)]X_n^+(\tau)(\tau-t)}\,d\tau + X_n^+(t)P_n(t)$$

$$n=1,2,3,\dots$$

where $X_n^+(t)$ is the boundary value of the canonical solution $X_n(z)$ of the Hilbert problem

$$\Phi_n^+(t) = -\frac{a_n(t)-ib_n(t)}{a_n(t)+ib_n(t)}\;\Phi_n^-(t)$$

with index $\varkappa_n = \left[\arg \frac{a_n-ib_n}{a_n+ib_n}\right]_L$, where the orientation of L is consistent with the positive rotation on the complex variable plane, and $P_n(z)$ is a polynomial of a degree greater than $\varkappa_n$ ($P_n = 0$, if $\varkappa_n < 0$) with arbitrary positive coefficients β_k satisfying the conditions

$$\beta_{\varkappa_n - k} = \beta_k, \quad k=0,1,2,\dots,\varkappa_n$$

The above form of transformation (8) follows from consideration on the non-linear Riemann problem given in [1].

In view of (2)-(6) and (8) we obtain

$$\begin{aligned} &\left|F_n^*\left\{t,\varphi(t)\,\varphi\left[\alpha(t)\right]\right\}\right| \leq M_n\left\{1+m_{1n}\left|\frac{\varphi_1(t)+\overline{\varphi_1(t)}}{2}\right|+\right.\\ (9)\qquad &+m_{2n}\left|\frac{\varphi_1(t)-\overline{\varphi_1(t)}}{2i}\right|+m_{3n}\left|\frac{\varphi_1[\alpha(t)]+\overline{\varphi_1[\alpha(t)]}}{2}\right|+\\ &\left.+m_{4n}\left|\frac{\varphi_1[\alpha(t)]-\overline{\varphi_1[\alpha(t)]}}{2i}\right|+\dots\right\} \leq\\ &\leq M_n\,(1+R\sum_{i=1}^{\infty} m_{in}) = M_n(1+R\cdot Z_n) \end{aligned}$$

and

$$\begin{aligned} &\left|F_n^*\left\{t,\varphi(t),\varphi\left[\alpha(t)\right]\right\} - F_n^*\left\{t',\varphi(t'),\varphi\left[\alpha(t')\right]\right\}\right| \leq\\ (10)\qquad &\leq B_n\left|t-t'\right|^h + D_{n1}\left|\frac{\varphi_1(t)+\overline{\varphi_1(t)}}{2} - \frac{\varphi_1(t')+\overline{\varphi_1(t')}}{2}\right|+ \end{aligned}$$

$$+ D_{n2}\left|\frac{\varphi_1(t)-\overline{\varphi_1(t)}}{2i}-\frac{\varphi_1(t')-\overline{\varphi_1(t')}}{2i}\right|+$$

(10)

$$+ D_{n3}\left|\frac{\varphi_1[\alpha(t)]+\overline{\varphi_1[\alpha(t)]}}{2}-\frac{\varphi_1[\alpha(t')]+\overline{\varphi_1[\alpha(t')]}}{2}\right|+$$

$$+ D_{n4}\left|\frac{\varphi_1[\alpha(t)]-\overline{\varphi_1[\alpha(t)]}}{2i}-\frac{\varphi_1[\alpha(t')]-\overline{\varphi_1[\alpha(t')]}}{2i}\right|+\dots$$

$$\leq B_n|t-t'|^h+(D_{n1}+D_{n2})\varkappa|t-t'|^h+(D_{n3}+D_{n4})K_\alpha^h\varkappa|t-t'|^h+\dots$$

$$\leq B_n|t-t'|^h+K_\alpha D_n\varkappa|t-t'|^h .$$

Assumption 3^o assures that there exists a positive constant K_α appearing in estimation (10).
Taking into account estimations (9), (10) and considering properties of the canonical solution $X_n^+(t)$ we conclude that we can apply Plemelja-Privalov's theorem to the functions $\psi_n(t)$ defined by (8). This gives the following formulas

(11)

$$\left.\begin{aligned}|\psi_n(t)| &\leq W_o^n + H_o^n D_n\varkappa + C_o^n M_n R\\ |\psi_n(t)-\psi_n(t')| &\leq (W^n+H^nD_n\varkappa+C^nM_nR)|t-t'|^h\end{aligned}\right\}$$

where the positive constants W_o^n, W^n, C_o^n, H_o^n and H^n, $n=1,2,3,\dots$, are independent of the point $\{\varphi_n(t)\}$ in the set Z and of the constants $\varkappa, R, M_n, B_n$. Hence the operation (8) transforms the set $Z(R,\varkappa)$ into itself provided that the following inequalities are satisfied:

$$(12) \qquad \left.\begin{aligned} W_o^n + H_o^n D_n \varkappa + C_o^n M_n R &\leqslant R \\ W^n + H^n D_n \varkappa + C^n M_n R &\leqslant \varkappa \end{aligned}\right\}$$

$n=1,2,3,\ldots$

It can be shown that if the constants D_n and M_n are sufficiently small, namely if

$$(13) \qquad D_n < \frac{1}{H_o^n + H^n} \quad \text{and} \quad M_n < \frac{1}{C_o^n + C^n} \qquad n=1,2,3,\ldots$$

then the constants R and $\varkappa$ can be selected in such a way that system (12) is satisfied. Since R is the constant in assumption 2^o, we have to assume that

$$(14) \qquad R \geqslant \frac{W^n H_o^n D_n + (1 - H^n D_n) W_o^n}{(1 - C_o^n M_n)(1 - H^n D_n) - H_o^n D_n C^n M_n} \qquad n=1,2,3,\ldots$$

Hence if the constants M_n, D_n, R of our problem satisfy conditions (14) and (15), the operation (8) transforms the set $Z(R,\varkappa)$ into itself.

Repeating the reasoning given in [2] one can prove that transformation (8) is continuous. This allows to apply Schauder-Tichonov's theorem [3], [4], which results in the following corollary:

The infinite system of integral equations which is obtained from (8) upon replacing $\psi_n(t)$ by $\varphi_n(t)$ possesses at least one solution. If among them there is a solution $\{\varphi_n^*(t)\}$ which satisfies the condition

$$(15) \qquad \int_L q_n(\tau)\, H_n(\tau)\, d\tau = 0,$$

where $q_n(t)$ is a polynomial of a degree not greater than $-\varkappa_n - 2$ ($q_n \equiv 0$, in case $-\varkappa_n - 2 < 0$) and where

$$H_n(t)=\left[X_n^+(t)\right]^{-1}\left[a_n(t)+ib_n(t)\right]^{-1}\left\{c_n(t)+F_n^*(t,\varphi^*(t),\varphi^*[\alpha(t)])\right\}$$

then according to the theory of linear Riemann's problem, the problem in question has at least one solution of the form

$$\phi_n(z)=\frac{X_n(z)}{2\pi i}\int_L\frac{c_n(\tau)+F_n^*(\tau,\varphi^*(\tau),\varphi^*[\alpha(\tau)])}{\left[a_n(\tau)+ib_n(\tau)\right]X_n^+(\tau)(\tau-z)}d\tau+ \tag{16}$$

$$+\frac{X_n(z)z^{\varkappa_n+1}}{2\pi i}\int_L\frac{c_n(\tau)+F_n^*(\tau,\varphi^*(\tau),\varphi^*[\alpha(\tau)])}{\tau^{\varkappa_n+1}\left[a_n(\tau)+ib_n(\tau)\right]X_n^+(\tau)(\tau-z)}d\tau+$$

$$+X_n(z)P_n(z)\,,\qquad n=1,2,3,\dots$$

Condition (15) is always satisfied provided that all the indices $\varkappa_n$ are non-negative. Consequently, we can state the theorem:

If the constants R, M_n, D_n satisfy conditions (13), (14) and if condition (15) holds, then the problem in question (1) has at least one solution of the form (16).

REFERENCES

[1] W. Pogorzelski: Równania całkowe i ich zastosowania. T. III. (Integral equations and their applications. Vol. III), Warszawa 1960.

[2] D. Przeworska-Rolewicz: Sur un système d'équations intégrales non-linéaires de seconde espèce à une inifinité de fonctions inconnues à singularité forte. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 5 (1957) 467-470.

[3] J. Schauder: Der Fixpunktsatz in Funktionalräumen. Studia Math. 2 (1930). 171-180.

[4] A.N. Tichonov: Ein Fixpunktsatz. Math. Ann. 111 (1935) 767-776.

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Received January 18th, 1972.