DEMONSTRATIO MATHEMATICA
Vol. V No 1 1973

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ON ALMOST DECOMPOSABLE VECTOR FIELDS

In a previous paper [2], we introduced the concept of almost product pseudo-Riemannian space and studied particular cases of such a space, viz almost hyperbolic Kähler space, almost semi-hyperbolic Kähler space, almost nearly-hyperbolic Kähler space, and almost quasi-hyperbolic Kähler space.

The purpose of the present paper is to define almost decomposable vector fields and to investigate their properties in such special spaces.

1. Introduction

We consider an n-dimensional almost product pseudo-Riemannian space [2] with an almost product structure $F_i{}^h$ and pseudo-Riemannian metric $g_{ji}\, d\xi^j\, d\xi^i$ such that

$$F_j{}^h F_i{}^j = A_i^h; \qquad F_j{}^t F_i{}^s g_{ts} = -g_{ji};$$

$$F_{ji} = -F_{ij}; \qquad F_{ji} = F_j{}^t g_{ti},$$

where A_j^h denotes the so called unit tensor, and indices $h,i,j,\ldots$ run over the range $1,2,\ldots,n$. An almost product pseudo-Riemannian space is said to be an almost hyperbolic Kähler space [2] iff,

$$F_{jih} \equiv \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0;$$

an almost semi-hyperbolic Kähler space [2] iff,

$$F_i \equiv -\nabla_j F_i^{\ j} = 0;$$

an almost nearly-hyperbolic Kähler space [2] iff,

$$G_{ji}^{\ \ h} \equiv \nabla_j F_i^{\ h} + \nabla_i F_j^{\ h} = 0,$$

and an almost quasi-hyperbolic Kähler space [2] iff $\nabla_j F_i^{\ h}$ is pure in j and i, where ∇_j denotes the covariant differentiation with respect to g_{ji}.

A tensor T_{ji} is said to be pure in j and i if

$$^{*}O_{ji}^{ts}\, T_{ts} = 0,$$

where

$$^{*}O_{ji}^{ts} = \frac{1}{2}(A_j^t A_i^s - F_j^{\ t} F_i^{\ s}),$$

while T_{ji} is called hybrid in j and i if

$$O_{ji}^{ts}\, T_{ts} = 0,$$

where

$$O_{ji}^{ts} = \frac{1}{2}(A_j^t A_i^s + F_j^{\ t} F_i^{\ s}).$$

We recall a lemma from [2], which shall be subsequently used.

L e m m a (1.1). A space which is an almost hyperbolic Kähler space or an almost nearly-hyperbolic Kähler space or an almost quasi-hyperbolic Kähler space is necessarily an almost semi-hyperbolic Kähler space.

2. Contravariant almost decomposable vector fields

Consider a contravariant vector field $x^h = (x^a, x^{\varkappa})$ in a locally product space. The vector field x^h is called contravariant decomposable if [3]

$$\partial_\lambda x^a = 0 \quad \text{and} \quad \partial_b x^\varkappa = 0; \qquad \begin{array}{l} a,b,\ldots = 1,2,\ldots,p \\ \varkappa,\lambda,\ldots = p+1,\ldots,p+q = n \end{array}$$

or equivalently

$$ {}^*O_{ir}^{sh} \, \partial_s \, x^r = 0 ,$$

i.e.,

$$\text{(2.1)} \qquad \underset{x}{\mathcal{L}} F_i{}^h = x^t \partial_t F_i{}^h - F_i{}^t \partial_t x^h + F_t{}^h \partial_i x^t = 0,$$

where $\underset{x}{\mathcal{L}}$ denotes the Lie derivative with respect to the vector field x^h.

We define a contravariant almost decomposable vector field in an almost product pseudo-Riemannian space to be contravariant vector field x^h which satisfies (2.1).

In an almost product pseudo-Riemannian space, the equation (2.1) may be written as

$$\underset{x}{\mathcal{L}} F_i{}^h = x^t \nabla_t F_i{}^h - F_i{}^t \nabla_t x^h + F_t{}^h \nabla_i x^t = 0,$$

or

$$\text{(2.2)} \qquad x^t \nabla_t F_{ih} - F_i{}^t \nabla_t x_h + F_{th} \nabla_i x^t = 0.$$

Taking the symmetric part of the above equation with respect to i and h, we obtain

$$O_{ji}^{ts} (\nabla_t x_s + \nabla_s x_t) = 0 \quad \text{or} \quad O_{ji}^{ts} (\underset{x}{\mathcal{L}} g_{ts}) = 0$$

and as a consequence

$$O_{ts}^{ji} (\nabla^t x^s + \nabla^s x^t) = 0 \quad \text{or} \quad O_{ts}^{ji} (\underset{x}{\mathcal{L}} g^{ts}) = 0,$$

which provides the proof of the following:

T h e o r e m (2.1). For a contravariant almost decomposable vector field x^h, in an almost product pseudo-Riemannian space, the Lie derivatives $\underset{x}{\mathcal{L}} g^{ji}$ and $\underset{x}{\mathcal{L}} g_{ji}$ are both hybrid tensor fields.

Now by a strightforward calculation, we can prove

$$\frac{1}{2}(F_j{}^r F_{ri}{}^h + F_i{}^r F_{rj}{}^h) - G_{ji}{}^r F_r{}^h -$$
$$- F_j{}^t F_i{}^s \left[\frac{1}{2}(F_t{}^r F_{rs}{}^h + F_s{}^r F_{rt}{}^h) - G_{ts}{}^r F_r{}^h\right] = 0.$$

Consequently the symmetric tensor $\frac{1}{2}(F_j{}^r F_{ri}{}^h + F_i{}^r F_{rj}{}^h) - G_{ji}{}^r F_r{}^h$ is pure in j and i. Thus $\mathop{\mathcal{L}}_X g^{ji}$ being hybrid in j and i, we have

$$\frac{1}{2}(F_j{}^r F_{ri}{}^h + F_i{}^r F_{rj}{}^h)(\mathop{\mathcal{L}}_X g^{ji}) - G_{ji}{}^r F_r{}^h (\mathop{\mathcal{L}}_X g^{ji}) = 0.$$

Since X^h is a contravariant almost decomposable vector field, $\mathop{\mathcal{L}}_X F_i{}^h = 0$, and the above result reduces to

$$(2.3) \qquad \frac{1}{2} F_{ji}{}^h (\mathop{\mathcal{L}}_X F^{ji}) = G_{ji}{}^t F_t{}^h (\nabla^j X^i).$$

Now applying ∇^i to (2.1), we obtain

$$F_t{}^h \left[g^{ji} \nabla_j \nabla_i X^t + K_i{}^t X^i + F_i{}^t \mathop{\mathcal{L}}_X F^i + G_{ji}{}^s F_s{}^t (\nabla^j X^i)\right] = 0,$$

which on contraction with $F_h{}^l$ gives

$$(2.4) \quad g^{ji} \nabla_j \nabla_i X^h + K_i{}^h X^i + F_i{}^h \mathop{\mathcal{L}}_X F^i + G_{ji}{}^t F_t{}^h (\nabla^j X^i) = 0,$$

or

$$(2.5) \quad g^{ji} \nabla_j \nabla_i X^h + K_i{}^h X^i + F_i{}^h \mathop{\mathcal{L}}_X F^i + \frac{1}{2} F_{ji}{}^h \mathop{\mathcal{L}}_X F^{ji} = 0.$$

The above discussion is the proof of the following:

T h e o r e m (2.2). Equations (2.4) and (2.5) give necessary conditions for a vector field X^h in an almost product pseudo-Riemannian space to be contravariant almost decomposable.

Particular Cases:

Case 1: The case when the almost product pseudo-Riemannian space is an almost quasi-hyperbolic Kähler space.

In this case $\nabla_j F_i^{\ h}$ is pure in j and i and consequently $G_{ji}^{\ \ h}$ is also pure in j and i. On the other hand, for a contravariant almost decomposable vector field x^h, $\underset{x}{\mathcal{L}} g^{ji}$ is hybrid in j and i and as a consequence

$$G_{ji}^{\ \ h}(\nabla^j x^i) = -\frac{1}{2} G_{ji}^{\ \ h}(\underset{x}{\mathcal{L}} g^{ji}) = 0.$$

Making use of this in the equation (2.3), we get

$$F_{jih}(\underset{x}{\mathcal{L}} F^{ji}) = 0,$$

which together with the lemma (1.1), reduces (2.5) to

$$g^{ji}\nabla_j \nabla_i x^h + K_i^{\ h} x^i = 0.$$

These last two equations complete the proof of the following:

Theorem (2.3). In order that a vector field x^h in an almost quasi-hyperbolic Kähler space be contravariant almost decomposable it is necessary that

$$F_{jih}(\underset{x}{\mathcal{L}} F^{ji}) = 0$$

and

$$g^{ji}\nabla_j \nabla_i x^h + K_i^{\ h} x^i = 0.$$

Case 2: The case when the almost product pseudo-Riemannian space is an almost hyperbolic Kähler space.

In this case, the equation (2.2) reduces to

$$x_t(-\nabla_i F_h^{\ t} + \nabla_h F_i^{\ t}) - F_i^{\ t}\nabla_t x_h + F_{th}\nabla_i x^t = 0.$$

Putting $\bar{x}_i \equiv -F_i^{\ t} x_t$ in the above equation, we get

$$\nabla_j \bar{x}_i - \nabla_i \bar{x}_j = F_j^{\ t}(\nabla_t x_i + \nabla_i x_t),$$

which yields:

T h e o r e m (2.4). A necessary and sufficient condition that a contravariant almost decomposable vector field x^h in an almost hyperbolic Kähler space be a Killing vector field is that $\bar{x}_i$ be closed.

Again in this case, the equation (2.5) reduces to

$$g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i = 0, \quad \text{(in view of Lemma (1.1)).}$$

The same result may be obtained in the following way. Since

$$\underset{x}{\mathcal{L}} (\nabla_j F_i{}^h) - \nabla_j \underset{x}{\mathcal{L}} F_i{}^h = (\underset{x}{\mathcal{L}} \left\{ {h \atop j\ t} \right\}) F_i{}^t - (\underset{x}{\mathcal{L}} \left\{ {t \atop j\ i} \right\}) F_t{}^h$$

and for a contravariant almost decomposable vector field x^h, $\underset{x}{\mathcal{L}} F_i{}^h = 0$, therefore

$$(2.6) \qquad \underset{x}{\mathcal{L}} (\nabla_j F_i{}^h) = (\underset{x}{\mathcal{L}} \left\{ {h \atop j\ t} \right\}) F_i{}^t - (\underset{x}{\mathcal{L}} \left\{ {t \atop j\ i} \right\}) F_t{}^h .$$

On the other hand, in an almost hyperbolic Kähler space [2]

$$N_{jih} = 2 F_j{}^t \nabla_h F_{it},$$

or

$$N_{ji}{}^h = 2 F_j{}^t (\nabla_t F_i{}^h - \nabla_i F_t{}^h),$$

which upon taking Lie derivative with respect to the vector field x^h gives

$$\underset{x}{\mathcal{L}} N_{ji}{}^h = 2 F_j{}^t \left\{ \underset{x}{\mathcal{L}} (\nabla_t F_i{}^h) - \underset{x}{\mathcal{L}} (\nabla_i F_t{}^h) \right\},$$

or

$$(2.7) \qquad \underset{x}{\mathcal{L}} N_{ji}{}^h = - 4 \, {}^*O_{ji}^{ts} \underset{x}{\mathcal{L}} \left\{ {h \atop t\ s} \right\}$$

by virtue of the equation (2.6).

Since the left hand side of the equation (2.7) is skew-symmetric in j and i while the right hand side is symmetric in j and i, therefore each side must be separately zero, that is,

$$\underset{x}{\mathcal{L}} N_{ji}{}^{h} = 0 \tag{2.8}$$

and

$$*O_{ji}^{ts} \underset{x}{\mathcal{L}} \left\{ {h \atop t\ s} \right\} = 0. \tag{2.9}$$

Transvecting (2.9) with g^{ji} and remembering that $g^{ji} *O_{ji}^{ts} = g^{ts}$, we have

$$g^{ji} \underset{x}{\mathcal{L}} \left\{ {h \atop j\ i} \right\} = g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i = 0 \tag{2.10}$$

for a contravariant almost decomposable vector field x^h. From the equations (2.8) and (2.10) we deduce the following:

Theorem (2.5). The Lie derivative of the Nijenhuis tensor with respect to a contravariant almost decomposable vector field in an almost hyperbolic Kähler space vanishes.

Theorem (2.6). A necessary condition for a vector field x^h in an almost hyperbolic Kähler space to be contravariant almost decomposable is that

$$g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i = 0 .$$

Case 3: The case when the almost product pseudo-Riemannian space is an almost nearly-hyperbolic Kähler space.

In this case, the equation (2.3) reduces to

$$F_{jik} (\underset{x}{\mathcal{L}} F^{ji}) = 0, \tag{2.11}$$

or equivalently

$$(\nabla_k F_{ji}) (\underset{x}{\mathcal{L}} F^{ji}) = 0 ,$$

i.e.,

$$(2.12)\quad (\nabla_k F_{ji})\,(x^h \nabla_h F^{ji} - F^{jt} \nabla_t x^i - F^{ti} \nabla_t x^j) = 0 .$$

In an almost nearly-hyperbolic Kähler space, the following relations are satisfied [2]

$$(2.13)\quad (\nabla_k F_{ji})(\nabla_h F^{ji})\, x^h = (K^*_{kh} - K_{kh})\, x^h$$

and

$$(2.14)\quad \frac{1}{4} N_{jih} = F_{ht} (\nabla_j F_i{}^t),$$

where K_{kh} is the Ricci tensor, $K^*_{kh} = H_{kt} F_h{}^t$, and $H_{kj} \equiv \equiv \frac{1}{2} K_{kjih} F^{ih}$. Solving the equation (2.12) with the help of (2.13) and (2.14), we get

$$(2.15)\quad (K^*_{kh} - K_{kh})\, x^h + \frac{1}{2} N_{kji} (\nabla^j x^i) = 0.$$

On the other hand, the equation (2.5) in this case reduces to

$$(2.16)\quad g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i = 0.$$

Equations (2.11), (2.15), and (2.16) provide the proof of the following:

T h e o r e m (2.7). In order that x^h be contravariant almost decomposable vector field in an almost nearly-hyperbolic Kähler space, it is necessary that

$$F_{ji}{}^h \underset{x}{\mathcal{L}} F^{ji} = 0,$$

or equivalently,

$$(K^*_{kh} - K_{kh})\, x^h + \frac{1}{2} N_{kji} (\nabla^j x^i) = 0,$$

and

$$g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i = 0.$$

Case 4: The case when the almost product pseudo-Riemannian space is an almost semi-hyperbolic Kähler space.

In this case $F_i = 0$ and consequently the equation (2.5) gives:

Theorem (2.8). A necessary condition for a contravariant vector field x^h in an almost semi-hyperbolic Kähler space to be contravariant almost decomposable is

$$g^{ji} \nabla_j \nabla_i x^h + K_i{}^h x^i + \frac{1}{2} F_{ji}{}^h \mathcal{L}_x F^{ji} = 0.$$

3. Covariant almost decomposable vector fields

Let us consider a covariant vector field $x_i = (x_b, x_\lambda)$ in a locally product space. The vector field x_i is said to be covariant decomposable if [3]

$$\partial_\mu x_b = 0 \quad \text{and} \quad \partial_c x_\lambda = 0 ,$$

or equivalently

$${}^*O_{ji}^{ts} \partial_t x_s = 0 ,$$

i.e.,

$$(3.1) \qquad x_h(\partial_j F_i{}^h - \partial_i F_j{}^h) - F_j{}^t \partial_t x_i + F_i{}^t \partial_j x_t = 0.$$

In an almost product pseudo-Riemannian space, we define a covariant almost decomposable vector field as a covariant vector field which satisfies (3.1).

In an almost product pseudo-Riemannian space, the equation (3.1) may be written as

$$(3.2) \qquad x_h(\nabla_j F_i{}^h - \nabla_i F_j{}^h) - F_j{}^t \nabla_t x_i + F_i{}^t \nabla_j x_t = 0,$$

which on taking the symmetric part with respect to j and i gives

$${}^*O_{ji}^{ts} (\nabla_t x_s - \nabla_s x_t) = 0.$$

Hence we can state:

Theorem (3.1). If x_i is a covariant almost decomposable vector field in an almost product pseudo-Riemannian space, then $(\nabla_j x_i - \nabla_i x_j)$ is pure in j and i.

Since in this space F^{ji} is hybrid in j and i [2], therefore

$$F^{ji}(\nabla_j x_i - \nabla_i x_j) = 0,$$

or

$$F^{ji} \nabla_j x_i = 0,$$

which implies the following:

Theorem (3.2). If x_i is a covariant almost decomposable vector field in an almost product pseudo-Riemannian space, then $F^{ji} \nabla_j x_i$ vanishes.

Putting $$\bar{x}_i \equiv -F_i^{\ t} x_t,$$

the equation (3.2) can be written as

$$(3.3) \qquad \nabla_j \bar{x}_i - \nabla_i \bar{x}_j + F_j^{\ t}(\nabla_t x_i - \nabla_i x_t) = 0,$$

which completes the proof of the following:

Theorem (3.3). If x_i and $\bar{x}_i$ are both closed in an almost product pseudo-Riemannian space, then x_i is covariant almost decomposable.

Transvection by the equation (3.3) with $F_l^{\ j}$, we get

$$(3.4) \qquad \nabla_j x_i - \nabla_i x_j + F_j^{\ t}(\nabla_t x_i - \nabla_i x_t) = 0.$$

From equations (3.3) and (3.4) we deduce:

Theorem (3.4). In an almost product pseudo-Riemannian space, x_i is covariant almost decomposable iff $\bar{x}_i$ is covariant almost decomposable.

Again from equation (3.3) we readily verify:

Theorem (3.5). In an almost product pseudo-Riemannian space, a necessary and sufficient condition that a covariant almost decomposable vector field x_i be closed is that $\overline{x}_i$ be closed.

For a covariant almost decomposable vector field x_i, we have

$$N_{ji}{}^h x_h = \left[F_j{}^t (\nabla_t F_i{}^h - \nabla_i F_t{}^h) - F_i{}^t (\nabla_t F_j{}^h - \nabla_j F_t{}^h) \right] x_h,$$

$$= F_j{}^t \left\{ F_t{}^l \nabla_l x_i - (\nabla_t x_l) F_i{}^l \right\} - F_i{}^t (F_t{}^l \nabla_l x_j - F_j{}^l \nabla_t x_l),$$

(from the equation (3.2))

$$= {}^*O_{ji}^{ts} (\nabla_t x_s - \nabla_s x_t)$$

$$= 0, \qquad \text{(from Theorem (3.1))}$$

hence we can state:

T h e o r e m (3.6). For a covariant almost decomposable vector field x_i in an almost product pseudo-Riemannian space, we have

$$N_{ji}{}^h x_h = 0.$$

In particular, if the space is almost hyperbolic Kähler, then we have

$$\nabla_j F_i{}^r - \nabla_i F_j{}^r = - \nabla^r F_{ji}$$

and then the equation (3.2) reduces to

$$(\nabla^t F_{ji}) x_t + F_j{}^t (\nabla_t x_i) + F_{ti} \nabla_j x^t = 0 ,$$

or

$$\mathop{\mathcal{L}}_{x} F_{ji} + F_j{}^t (\nabla_t x_i - \nabla_i x_t) = 0,$$

which leads to the following:

T h e o r e m (3.7). If a covariant almost decomposable vector field x_i in an almost hyperbolic Kähler space is closed, then we have

$$\underset{x}{\mathcal{L}} F_{ji} = 0 .$$

Suppose that a vector field x^h is contravariant as well as covariant almost decomposable in an almost hyperbolic Kähler space, that is,

$$(3.5) \qquad x^t \nabla_t F_{ih} - F_i{}^t(\nabla_t x_h) - F_h{}^t(\nabla_i x_t) = 0$$

and

$$(3.6) \qquad x^t \nabla_t F_{ih} + F_i{}^t (\nabla_t x_h) - F_h{}^t (\nabla_i x_t) = 0 .$$

Subtracting the equation (3.5) from the equation (3.6), we get

$$\nabla_j x_h = 0 .$$

Thus we can state:

T h e o r e m (3.8). If a vector field x^h is contravariant as well as covariant almost decomposable in an almost hyperbolic Kähler space, then it is covariantly constant.

From Lemma (1.1), for almost hyperbolic Kähler space, almost semi-hyperbolic Kähler space, almost nearly-hyperbolic Kähler space, and almost quasi-hyperbolic Kähler space, we have $F_i = 0$.

Now for the divergence of $\bar{x}_j$, we have

$$\nabla^j \bar{x}_j = -\nabla^j (F_j{}^i x_i) = -F^{ji} \nabla_j x_i = 0$$

by virtue of the Theorem (3.2).

This combined with the equation (3.3), provides the proof of the following:

T h e o r e m (3.9). In each of the almost hyperbolic Kähler, almost semi-hyperbolic Kähler, almost nearly-hyperbolic Kähler, and almost quasi-hyperbolic Kähler spaces, if a covariant almost decomposable vector field x_i is closed, then $\bar{x}_i$ is harmonic.

On the other hand, under the assumptions of the Theorem (3.9) the divergence of x_i can be written as

$$\nabla^j x_j = \nabla^j (F_j{}^i F_i{}^t x_t) = -\nabla^j (F_j{}^i \bar{x}_i) = F^{ji}(\nabla_j \bar{x}_i)$$

$$= \frac{1}{2} F^{ji} (\nabla_i \bar{x}_j - \nabla_j \bar{x}_i) = 0 ,$$

which together the equation (3.4) provides the proof of the following:

T h e o r e m (3.10). In each of the almost hyperbolic Kähler, almost semi-hyperbolic Kähler, almost nearly-hyperbolic Kähler, and almost quasi-hyperbolic Kähler spaces, if a covariant almost decomposable vector field x_i is closed, then it is harmonic.

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Received December 13th, 1972.