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THE STRUCTURE OF BOUNDARY FUNCTIONS IN THE CLASSES $\mathfrak{G}^M$ AND $\mathfrak{W}^M$

In this paper we consider the structure of boundary functions with respect to a functional. In the first part of this work this functional is determined in the class $\mathfrak{G}^M$ of quasi-starlike functions and in the second part - in the class $\mathfrak{W}^M$ of quasi-convex functions.

Consider the class $\mathfrak{G}^M$ of functions

$$q(z) = \frac{1}{M} z + a_2 z^2 + \dots \quad \text{for} \quad |z| < 1,$$

determined by the equation

$$(1) \qquad F(q(z)) = \frac{1}{M} F(z) \quad \text{for} \quad |z| < 1$$

where

$$F(z) = z + A_2 z^2 + \dots \quad \text{for} \quad |z| < 1$$

is a starlike functions, and $M > 1$.

Functions of the class $\mathfrak{G}^M$ are said to be quasi-starlike.

In formula (1), let us replace the starlike function by a convex function; we get the class $\mathfrak{W}^M$. Functions of this class are said to be quasi-convex.

The class $\mathfrak{G}^M$ and the class $\mathfrak{W}^M$ are compact.

Assume that in $\mathfrak{G}^M(\mathfrak{W}^M)$ there is given a functional defined for every function $q \in \mathfrak{G}^M(\mathfrak{W}^M)$ by the formula

$$(2) \qquad F(q) = F(u_0, u_1, \ldots, u_n, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_n)$$

where F denotes a holomorphic function, whose derivatives of the first order do not disappear simultaneously in a suficiently large domain.

In the first part of this paper we shall consider the class $\mathfrak{G}^M$.

Let

$$(2') \qquad u_i = q^{(i)}(\zeta), \quad i = 0,1,\ldots,n, \quad \text{for} \quad q \in \mathfrak{G}^M$$

where ζ is an arbitrary complex number of the unit disc.

Let us consider the set D of values of the functional (2) and $\mathring{F}$ be an arbitrary point of the boundary of D. If there exists a point $\hat{F}$ of the complement of D that

$$(3) \qquad |F - \hat{F}| \geqslant |\mathring{F} - \hat{F}|$$

for all F of the set D then $\mathring{F}$ will be called a regular point. In the contrary the point of the boundary of the set D is called a singular point.

In [4] it has been proved that the set of the regular points of the boundary of set D is dense in the boundary of set D, so in order to determine the boundary of the set D it is sufficient to examine the set of its regular points.

A function for which the value of the functional belongs to the boundary of the set D will be called a boundary function with respect to this functional.

A function for which the value of the functional gives a regular boundary point will be called a regular boundary function.

Now we shall apply the variation formula (4) from [3] to the extremal problem in the $\mathfrak{G}^M$

$$(4) \quad q_\varepsilon(z) = q(z) + \varepsilon \frac{G(z)q'(z)}{G'(z)}\Big[Q(z,a) - Q(q(z),a)\Big] + o(\varepsilon)$$

where $|a|<1$, $|z|<1$, $M>1$, A is an arbitrary complex number,

(5) $$G(q(z)) = \frac{1}{M} G(z) \quad \text{and} \quad G \text{ is a starlike function.}$$

(6) $$Q(z,a) = A\, K(z,a) - \bar{A}\, K(z,\tfrac{1}{\bar{a}}) - \frac{A}{H(a)} L(z,a) - \frac{\bar{A}}{\overline{H(a)}} L(z,\tfrac{1}{\bar{a}})$$

(7) $$K(z,a) = \frac{z+a}{z-a} + H(z)$$

(8) $$L(z,a) = \frac{z+a}{z-a} H(z) + 1$$

(9) $$H(z) = \frac{z\, G'(z)}{G(z)} .$$

Let us take an arbitrary boundary and regular function q^* of the family $\mathfrak{G}^M$. Putting $q = q^*$, $z = \zeta$ and $a = z$ in the formula (4), we obtain

$$q_\varepsilon(\zeta) = q^*(\zeta) + \varepsilon \psi(\zeta,z) + o(\varepsilon)$$

where

(10) $$\psi(\zeta,z) = \frac{G^*(\zeta)q^{*\prime}(\zeta)}{G^{*\prime}(\zeta)} \left[Q(\zeta,z) - Q(q^*(\zeta),z)\right].$$

We put

$$\Delta q^{*(k)} = q_\varepsilon^{(k)} - q^{*(k)} \qquad k=0,1,\dots,n.$$

Taking (4) and (10) into consideration, we get

$$\Delta q^{*(k)} = \varepsilon \psi_\zeta^{(k)}(\zeta,z) + o(\varepsilon).$$

Denoting by

$$\Delta F = F(q_\varepsilon) - F(q^*)$$

(11) $$L(q^*) = \sum_{k=0}^{n} \left[p_k \psi_\zeta^{(k)}(\zeta,z) + q_k \overline{\psi_\zeta^{(k)}(\zeta,z)}\right],$$

where

$$p_k = F'_{u_k}(q^*), \quad q_k = F'_{\bar{u}_k}(q^*).$$

Next we expand (2) in Taylor's series in a neighbourhood of $(u_0^*, u_1^*, \dots, u_n^*, \bar{u}_0^*, \bar{u}_1^*, \dots, \bar{u}_n^*)$. In view of (2′) as well as of (11), we obtain

$$\Delta F = L(q^*) + o(\varepsilon).$$

It is not difficult to see that for $\mathring{F} = F(q^*)$ (3) can be written as

$$\text{re}\left[\beta L(q^*)\right] \geqslant 0 \tag{12}$$

where

$$\beta = e^{-i \arg(\mathring{F} - \hat{F})}.$$

Next we denote

$$M(\zeta, z) = \frac{G^*(\zeta) q^{*\prime}(\zeta)}{G^{*\prime}(\zeta)} \left[K(\zeta, z) - K(q^*(\zeta), z)\right], \tag{13}$$

$$N(\zeta, z) = \frac{G^*(\zeta) q^{*\prime}(\zeta)}{G^{*\prime}(\zeta)} \left[L(\zeta, z) - L(q^*(\zeta), z)\right], \tag{13′}$$

and

$$\frac{\partial^k}{\partial \zeta^k} M(\zeta, z) = M^k(\zeta, z), \tag{14}$$

$$\frac{\partial^k}{\partial \zeta^k} N(\zeta, z) = N^k(\zeta, z). \tag{14′}$$

If in (12) in the place of the expressions with $\bar{A}$ we put their conjugates, then (12) can be written, in view (6), (10), (11), (13) and (14), in form

$$\mathrm{re}\Big\{A\sum_{k=0}^{n}\Big[\beta p_k M^k(\zeta,z) - \beta p_k M^k(\zeta,\tfrac{1}{\bar z}) - \frac{1}{H(z)}\Big(\beta p_k N^k(\zeta,z) + \overline{\beta p_k N^k(\zeta,\tfrac{1}{\bar z})}\Big) + \overline{\beta q_k} M^k(\zeta,z) - \beta q_k \overline{M^k(\zeta,\tfrac{1}{\bar z})} + - \frac{1}{H(z)}\Big(\overline{\beta q_k} N^k(\zeta,z) - \beta q_k \overline{N^k(\zeta,\tfrac{1}{\bar z})}\Big)\Big]\Big\} \geq 0.$$

Hence, because A is arbitrary, we have

$$H(z)\Big(\overline{S(\zeta,\tfrac{1}{\bar z})} - S(\zeta,z)\Big) = P(\zeta,z) + \overline{P(\zeta,\tfrac{1}{\bar z})}, \tag{15}$$

where

$$P(\zeta,z) = \sum_{k=0}^{n}\alpha_k N^k(\zeta,z), \tag{16}$$

$$S(\zeta,z) = \sum_{k=0}^{n}\bar\alpha_k M^k(\zeta,z), \tag{17}$$

$$\alpha_k = \beta p_k + \overline{\beta q_k}.$$

From (15) we see that the function G^* defined by the formula (5) for $q = q^*$ satisfies equation (18) for $z \in D$, where

$$D = \Big\{z;\, |z| = 1 \text{ and } |G^*(z)| < \infty\Big\}$$

$$\mathrm{re}\,\frac{z\,G^{*\prime}(z)}{G^*(z)} = 0. \tag{18}$$

Condition (18) may be rewritten in the form

$$\arg G^*(z) = \text{const. for } z \in D. \tag{19}$$

Let $T(z)$ denote the right-hand side of formula (15)

$$T(z) = P(\zeta,z) + \overline{P(\zeta,\tfrac{1}{\bar z})}. \tag{20}$$

We see that, by (8), (13'), (14') and (16), the function $T(z)$ is a real and continuous function for z of the unit circle. Further, we see that $T(z)$ has no poles, and $T(z)$ has, at least, 4n+4 roots on the unit circle. From the above it follows that G^* has, at most, 4n+4 singular points on the unit circle. Next, it is well known that, by (18), the function G^* may be written in the form

$$(21) \qquad G(z) = \frac{z}{\prod_{k=1}^{N}(1-\sigma_k z)^{\beta_k}} \qquad \text{for } |z|<1$$

where

$$\sigma_k = e^{i\varphi_k},\ \operatorname{im}\varphi_k=0,\ \varphi_k \neq \varphi_{k'} \ \text{ for } k\neq k' \text{ and } k,k'=1,2,\dots,N,$$

$$\sum_{k=1}^{N}\beta_k = 2,\ \beta_k > 0 \ \text{ for } k=1,2,\dots,N.$$

Summarizing, we obtain the following theorem.

Theorem 1. Every quasi-starlike function q^*, which is a boundary and regular function with respect to the functional (2) satisfies the equation

$$G^*(q^*(z)) = \frac{1}{M}G^*(z) \qquad \text{for } |z|<1$$

where G^* is a starlike function of the form (21).

We shall now show that $N \leqslant 2n+2$.

It is obvious that the image of the unit circle under transformation of the form (21) is the plane without N half-lines of the form $w=w_0t$ where $t \geqslant 1$.

Let $e^{i\nu_j}$ denote the points whose images under transformation G^* are w_j where $j=1,2,\dots,N$. We may assume that

$$0 \leqslant \varphi_1 < \nu_1 < \dots < \nu_n < \varphi_1 + 2\pi .$$

Because

$$H(e^{i\varphi_j}) = 0 \quad \text{for} \quad j=1,2,\dots,N,$$

we have

$$(22) \qquad T(e^{i\varphi_j}) = 0 \quad \text{for} \quad j=1,2,\dots,N.$$

We shall now show that

$$(23) \qquad T(e^{i\varphi_j}) \geqslant 0 \quad \text{for} \quad j=1,2,\dots,N.$$

We shall use the variation formula (24) from [2].

$$(24) \qquad G_\varepsilon(z) = G(z) + \varepsilon G(z)\left[1 + H(z)\frac{z + e^{i\varphi_j}}{z - e^{i\varphi_j}}\right] + o(\varepsilon)$$

where $|z| < 1$.

We put

$$G(z) = w, \quad G_\varepsilon(z) = w_\varepsilon .$$

Hence, we see that

$$G^{-1}(w) = z = G_\varepsilon^{-1}(w)$$

$$G_\varepsilon^{-1}(G_\varepsilon(z)) = G_\varepsilon^{-1}\left(G(z) + \varepsilon G(z)\left[1 + H(z)\frac{z + e^{i\varphi_j}}{z - e^{i\varphi_j}}\right] + o(\varepsilon)\right).$$

We expand this function in Taylor's series in a neighbourhood of $\varepsilon = 0$. We then obtain

$$G^{-1}(w) = G_\varepsilon^{-1}(w) + \varepsilon w\left[1 + H(G^{-1}(w))\frac{G^{-1}(w) + e^{i\varphi_j}}{G^{-1}(w) - e^{i\varphi_j}}\right] G^{-1\prime}(w) + o(\varepsilon).$$

Denoting

$$M = \frac{1}{\lambda}, \quad G(q(z)) = \lambda G(z) \quad \text{and} \quad w = \lambda G(z)$$

we see that

$$q(z) = G_{\varepsilon}^{-1}(\lambda G(z)) + \varepsilon\lambda \frac{G(z)q'(z)}{G'(z)}\left[1+H(q(z))\frac{q(z)+e^{i\varphi_j}}{q(z)-e^{i\varphi_j}}\right] + o(\varepsilon).$$

On the other hand we have

$$G_{\varepsilon}^{-1}(\lambda G(z)) = G_{\varepsilon}^{-1}\left(\lambda G(z) + \varepsilon\lambda G(z)\left[1 + H(z)\frac{z+e^{i\varphi_j}}{z-e^{i\varphi_j}}\right] + o(\varepsilon)\right),$$

$$q_{\varepsilon}(z) = G_{\varepsilon}^{-1}(\lambda G(z)) + \varepsilon \frac{G(z)q'(z)}{G'(z)}\left[1+H(z)\frac{z+e^{i\varphi_j}}{z-e^{i\varphi_j}}\right] + o(\varepsilon).$$

Finaly we obtain

$$q_{\varepsilon}(z)-q(z) = \varepsilon \frac{G(z)q'(z)}{G'(z)}\left[H(z)\frac{z+e^{i\varphi_j}}{z-e^{i\varphi_j}} - H(q(z))\frac{q(z)+e^{i\varphi_j}}{q(z)-e^{i\varphi_j}}\right] + o(\varepsilon).$$

If in the above formula we put ζ in place of z, then (11) can be written

$$L(q) = \sum_{k=0}^{n} p_k \frac{\partial^k}{\partial\zeta^k}\left\{\frac{G(\zeta)q'(\zeta)}{G'(\zeta)}\left[H(\zeta)\frac{\zeta+e^{i\varphi_j}}{\zeta-e^{i\varphi_j}} - H(q(\zeta))\frac{q(\zeta)+e^{i\varphi_j}}{q(\zeta)-e^{i\varphi_j}}\right]\right\} +$$

$$+ \sum_{k=0}^{n} q_k \frac{\partial^k}{\partial\zeta^k}\left\{\frac{G(\zeta)q'(\zeta)}{G'(\zeta)}\left[H(\zeta)\frac{\zeta+e^{i\varphi_j}}{\zeta-e^{i\varphi_j}} - H(q(\zeta))\frac{q(\zeta)+e^{i\varphi_j}}{q(\zeta)-e^{i\varphi_j}}\right]\right\}.$$

Further by (12), we have

$$\mathrm{re}\left\{\sum_{k=0}^{n} p_k\beta \frac{\partial^k}{\partial\zeta^k}\left(\frac{G(\zeta)q'(\zeta)}{G'(\zeta)}\left(L(\zeta,e^{i\varphi_j}) - L(q(\zeta),e^{i\varphi_j})\right)\right) + \sum_{k=0}^{n} q_k\beta \frac{\partial^k}{\partial\zeta^k}\left(\frac{G(\zeta)q'(\zeta)}{G'(\zeta)}\left(L(\zeta,e^{i\varphi_j}) - L(q(\zeta),e^{i\varphi_j})\right)\right)\right\} \geqslant 0.$$

Consequently we have (23).

Let us consider the only two possible cases.

1^0 $T(e^{i\varphi_j}) = 0$ for every $j=1,2,\dots,N$.
Then by (22) we have that $T(z)$ has, at least, $2N$ roots on the unit circle. Therefore we obtain that $N \leqslant 2n+2$.

2^0 $T(e^{i\varphi_j}) > 0$ for some j.
In this case the unit circle may be divided into disjoint arces which inital and terminal points are $e^{i\varphi_j}$ for which $T(e^{i\varphi_j}) > 0$. For every arc $T(z)$ has, at least, two times more roots then the number of points $e^{i\nu_j}$ belonging to the arc. Then $T(z)$ has, at least, $2N$ roots, therefore we have that $N \leqslant 2n+2$.

Hence, we have proved the following theorem.

T h e o r e m 2. Every quasi-starlike function q^*, which is boundary and regular function with respect to the functional (2) satisfies the equation

$$G^*(q^*(z)) = \frac{1}{M} G^*(z) \qquad \text{for} \quad |z| < 1$$

where G^* is a starlike function of the form (21) for which $N \leqslant 2n+2$.

Next let us consider the clas $\mathfrak{W}^M$ of functions quasi convex. Now we shall use the variation formula (25) from [3] to extremal problem in the $\mathfrak{W}^M$.

$$(25) \quad h_\varepsilon(z) = h(z) + \varepsilon \frac{h'(z)}{f'(z)}\left[\int_0^z f'(z)\left[Q(z,a) - Q(h(z),a)\right] dz\right] + o(\varepsilon)$$

where $|z| < 1$, $|a| < 1$, A is an arbitrary complex number, $h \in \mathfrak{W}^M$, f is a convex function

$$f(h(z)) = \lambda f(z), \qquad 0 < \lambda < 1$$

$$Q(z,a) = AK(z,a) - \bar{A}\, K\left(z, \frac{1}{\bar{a}}\right) - \frac{A}{H(a)}\, L(z,a) - \frac{\bar{A}}{\overline{H(a)}}\, L\left(z, \frac{1}{\bar{a}}\right)$$

$$K(z,a) = \frac{z+a}{z-a} + H(z)$$

$$L(z,a) = \frac{z+a}{z-a} H(z) + 1$$

$$H(z) = \frac{z f''(z)}{f'(z)} + 1. \tag{27}$$

In exactly the same way as previously we show that every quasi-convex function h^* which is boundary and regular function with respect to the functional (2) satisfies the equation

$$H(z)\left(E(\zeta,z) - \overline{E\left(\zeta, \frac{1}{\bar z}\right)}\right) = R(\zeta,z) + \overline{R\left(\zeta, \frac{1}{\bar z}\right)} \tag{28}$$

where

$$E(\zeta,z) = \sum_{k=0}^{n} \alpha_k \frac{\partial^k}{\partial \zeta^k}\left(\frac{h^{*\prime}(\zeta)}{f^{*\prime}(\zeta)} \int_0^{\zeta} f^{*\prime}(\zeta)\left(K(\zeta,z) - K\left(h^*(\zeta),z\right)\right) d\zeta\right) \tag{29}$$

$$R(\zeta,z) = \sum_{k=0}^{n} \bar\alpha_k \frac{\partial^k}{\partial \zeta^k}\left(\frac{h^{*\prime}(\zeta)}{f^{*\prime}(\zeta)} \int_0^{\zeta} f^{*\prime}(\zeta)\left(L(\zeta,z) - L\left(h^*(\zeta),z\right)\right) d\zeta\right). \tag{30}$$

Hence, denoting by $\mathring{T}(z) = R(\zeta,z) + \overline{R\left(\zeta, \frac{1}{\bar z}\right)}$, by (26) and (30) we have that $\mathring{T}(z)$ is meromorphic in the annulus $R_1 < |z| < R_2$, for given $R_1 < 1$ and $R_2 > 1$. Consequently, $\mathring{T}(z)$ has, at most, a finite number of roots on the unit circle.

From (27) and (28) we see that $\operatorname{re}\left(\frac{z f^{*\prime\prime}(z)}{f^{*\prime}(z)} + 1\right) = 0$ for $z \in D$, where

$$D = \left\{z;\ |z| = 1 \ \text{and}\ |h^*(z)| < \infty\right\}.$$

This implies, that $\arg G^*(z) = \text{const.}$ for $z \in D$ where $G^*(z) = z f^{*\prime}(z)$.

Hence, we have proved the following theorem.

T h e o r e m 3. Every quasi-convex function h^* which is boundary and regular function with respect to the functional (2) satisfies the equation

$$f^*(h^*(z)) = \lambda f^*(z) \qquad \text{for } |z| < 1$$

where f^* is a convex function of the form

$$f^*(z) = \int_0^z \frac{G^*(z)}{z}\, dz$$

where G^* is a starlike function of the form (21).

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