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POLYA DISTRIBUTION CONNECTED WITH THE PROBLEM OF BAYES

0. Introduction. In this paper we attempt to transfer some considerations carried out in [3] and [7] to case when we start not with a binomial or a hypergeometrical distribution but with a Polya distribution:

$$P(X=m) = \binom{n}{m} \frac{p^{[m,-\alpha]} \quad q^{[n-m,-\alpha]}}{1^{[n,-\alpha]}}, \qquad m=0,1,2,\dots,n, \qquad (1.1)$$

where: p, q, α are arbitrary real numbers which satisfy the conditions $p>0$, $q>0$, $p+q=1$, $n(-\alpha) \leqslant \min(p,q)$ and the expression $x^{[r,h]}$ is a factorial polynomial of the r-th degree with respect to x

$$x^{[r,h]} \equiv x(x-h)(x-2h)\dots(x-(r-1)h),$$

(we shall write $x^{[r]}$ instead of $x^{[r,1]}$ if $h=1$).

A corresponding model of drawing lots called in the paper the continuous scheme of Polya has been assigned to this distribution. According to the conditions of the scheme we perform n_1 experiments, the parameter p of Polya distribution being treated as a realization of the random variable P with the density $f(p)$. X_1 stands for a random variable whose values are the numbers of successes in the result of the whole experiment which consists in obtaining the number p in a random way and performing n_1 experiments, according to the continuous Polya scheme.

The distribution of this variable has been deduced for an arbitrary admissible density f(p) and in the special cases when f(p) is the density (1.7) and (1.8): the uniform and the beta distribution respectively. It has been found that a special case of the obtained distribution of the variable X_1 (if f(p) is given by the formula (1.8)) is again a Polya distribution (with positive β).

The ordinary and factorial moments of this variable, in particular the mean value and the variance, have been found for the general case as well as for the special cases of the random variable X_1. This is the contents of § 1 and § 2. In § 3 and § 4 the random variable X_2 whose values are the numbers of successes in the further n_2 random experiments performed according to the continuous Polya scheme under the condition that the mentioned experiment results in n_1 successes, is being considered; the distribution of the random variable $X_2|X_1$ and its moments have been deduced.

The distribution and their moments obtained in this paper are rather complicated (but also general); this follows from the fact that the starting point is the relatively general and complicated distribution (1.1). The problems discussed in the paper may be extended to the case of a discrete distribution of the random variable P, by replacing the Riemann integrals appearing in the formulas by Stieltjes integrals with respect to the distribution function F(p) of the discrete random variable P with the values from the interval (0;1). In this case integration has been replaced by summation. Practical examples corresponding to the model discussed in the paper are similar.

§ 1. THE DISTRIBUTION OF THE RANDOM VARIABLE X_1

1. The "continuous" Polya scheme. Polya distribution (1.1) is more general then the distribution

$$P(X=m) = \binom{n}{m} \frac{M^{[m,-s]} \ (N-M)^{[n-m,-s]}}{N^{[n,-s]}} \quad , \quad m=0,1,2,\dots,n, \quad (1.2)$$

connceted with the urn Polya scheme [2], because the probability $p = M/N$ in the latter is a rational number of the interval (0,1) while in the former it is an arbitrary real number from this interval. But the Polya scheme corresponds to Polya distribution (1.1) with the following modification. We perform n random experiments in such a way that the probability of a success varies from sample to sample: let p be the probability of a success in the first sample, $0<p<1$; if l successes and $k-l$ failures have been realized in the first k experiments, the probability of a success in the $(k+1)$-st experiment is defined as follows

$$\frac{p + l\alpha}{1 + k\alpha} . \quad (1.3)$$

Hence it follows that the probability of m successes in n experiments with an arbitrary order is given by formula (1.1). Probabilities (1.3) and (1.1) are also well defined for $\alpha < 0$ under the condition that $n(-\alpha) \leqslant \min(p,q)$; since then $p + l\alpha \geqslant 0$, $l = 0,1,2,\dots,n$, $1 + k\alpha > 0$, $k=0,1,2,\dots,n$ and $(p + l\alpha):(1+k\alpha) < 1$; it is evident that the last condition holds if, taking into account the foregoing, we write it in the form $(k-l)(-\alpha) < q$. The above described scheme may be termed the continuous Polya scheme, to distinguish it from the ordinary (discrete) urn scheme of Polya.

2. Introduction of the random variable X_1. We perform the following composed experiment: a number $p \in (0,1)$ is obtained by a random method, as a realization of a random variable P with the density $f(p)$ for $p \in (0,1)$ and 0 for $p \bar{\in} (0,1)$; next we carry out n_1 experiments according to the continuous scheme of Polya defined above i.e. with the (conditional) probability of success S_{k+1} in the $(k+1)$-st experiment given by the formula

$$P(S_{k+1}|P=p) = \frac{p + l\alpha}{1 + k\alpha}, \quad k=0,1,2,\dots,n_1-1; \quad l \leqslant k. \quad (1.3')$$

Let X_1 be a random variable whose values are the numbers of successes resulting from the whole experiment; thus X_1 can have the values $0,1,2,\dots,n_1$. The conditional probability of the random variable X_1 with the condition $P=p$ i.e. the distribution of the random variable $(X_1|P=p)$ is independent of the distribution $f(p)$ of the random variable P, and it is a Polya distribution:

$$P(X_1=m_1|P=p) = \binom{n_1}{m_1} \frac{p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]}}{1^{[n_1,-\alpha]}}, \quad m_1=0,1,2,\dots,n_1. \quad (1.4)$$

The inconditional distribution of the random variable X_1 is obtained by applying the theorem on the total probability

$$P(X_1 = m_1) = \int_0^1 P(X_1 = m_1|P = p)\, f(p)\, dp =$$

$$= \frac{\binom{n_1}{m_1}}{1^{[n_1,-\alpha]}} \int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp, \quad m_1=0,1,2,\dots,n_1. \quad (1.5)$$

The function under the last integral is inconvenient for integrating because it involves except $f(p)$ also the product of factorial polynomials. But if we take into account the following relationships

$$p^{[m_1,-\alpha]} = \sum_{i=0}^{m_1} s_i^{m_1}(-\alpha)^{m_1-i}\, p^i;$$

$$q^{[n_1-m_1,-\alpha]} = \sum_{j=0}^{n_1-m_1} s_j^{n_1-m_1}(-\alpha)^{n_1-m_1-j}\, q^j$$

in which s_i^n stand for the Stirling numbers of the first kind i.e. the coefficients (with respect to x) in the identity

$$x^{[n]} \equiv s_0^n + s_1^n x^1 + s_2^n x^2 + \dots + s_{n-1}^n x^{n-1} + s_n^n x^n,$$

then formula (1.5) takes the following, more expanded form:

$$P(X_1=m_1) = \frac{\binom{n_1}{m_1}}{1^{[n_1,-\alpha]}} \sum_{i,j=0}^{m_1,n_1-m_1} s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} \int_0^1 p^i q^j f(p)dp, \qquad (1.6)$$

$$m_1 = 0,1,\dots,n_1.$$

3. Particular cases with respect to the distribution of the random variable P. The inconditional distribution of the random variable X_1 depends essentially on the density $f(p)$ of the random variable P. Consider now two particular cases of the distribution of the random variable P.

(a) The random variable P is subject to a uniform distribution with the density

$$f(p) \begin{cases} 1 & \text{for } 0<p<1 \\ 0 & \text{for } p\leqslant 0 \text{ or } p\geqslant 1 \end{cases} \qquad (1.7)$$

(the postulate of Bayes).

(b) The random variable P is subject to a beta distribution with the density

$$f(p) = \begin{cases} \frac{\Gamma(1/\beta)}{\Gamma(\vartheta/\beta)\Gamma((1-\vartheta)/\beta)} p^{\vartheta/\beta-1} q^{(1-\vartheta)/\beta-1} & \text{for } 0<p<1 \\ 0 & \text{for } p\leqslant 0 \text{ or } p\geqslant 1, \end{cases} \qquad (1.8)$$

ϑ and β being arbitary numbers which satisfy the conditions $0<\vartheta<1$, $\beta>0$.

If the random variable P has uniform distribution (1.7), then formulas (1.5) and (1.6) become respectively

$$P(X_1=m_1) = \frac{\binom{n_1}{m_1}}{1^{[n_1,-\alpha]}} \int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} dp, \qquad m_1 = 0,1,\dots,n_1, \tag{1.9}$$

$$P(X_1=m_1) = \frac{\binom{n_1}{m_1}}{1^{[n_1,-\alpha]}} \sum_{i,j=0}^{m_1,n_1-m_1} s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} B(i+1,j+1), \qquad m_1 = 0,1,\dots,n_1, \tag{1.10}$$

B(x,y) being the beta function.

For P with beta distribution (1.8) formula (1.6) is transformed to the following

$$P(X_1=m_1) = \frac{\binom{n_1}{m_1}}{1^{[n_1,-\alpha]}} \sum_{i,j=0}^{m_1,n_1-m_1} \frac{s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} \vartheta^{[i,-\beta]} (1-\vartheta)^{[j,-\beta]}}{1^{[i+j,-\beta]}}, \tag{1.11}$$

since

$$\frac{\Gamma(1/\beta)}{\Gamma(\vartheta/\beta)\Gamma((1-\vartheta)/\beta)} \int_0^1 p^{\vartheta/\beta+i-1} q^{(1-\vartheta)/\beta+j-1} dp =$$

$$= \frac{\Gamma(1/\beta)}{\Gamma(\vartheta/\beta)\Gamma((1-\vartheta)/\beta)} B(\vartheta/\beta+i,\ (1-\vartheta)/\beta+j) =$$

$$= \frac{\Gamma(1/\beta)}{\Gamma(\vartheta/\beta)\Gamma((1-\vartheta)/\beta)} \frac{\Gamma(\vartheta/\beta+i)\Gamma((1-\vartheta)/\beta+j)}{\Gamma(1/\beta+i+j)} =$$

$$= \frac{\Gamma(1/\beta)}{\Gamma(\vartheta/\beta)\Gamma((1-\vartheta)/\beta)} \frac{\Gamma(\vartheta/\beta)(\vartheta/\beta)^{[i,-1]}\ \Gamma((1-\vartheta)/\beta)((1-\vartheta)/\beta)^{[j,-1]}}{\Gamma(1/\beta)(1/\beta)^{[i+j,-1]}} =$$

$$= \frac{(\vartheta/\beta)^{[i,-1]}\ ((1-\vartheta)/\beta)^{[j,-1]}}{(1/\beta)^{[i+j,-1]}} = \frac{\vartheta^{[i,-\beta]}\ (1-\vartheta)^{[j,-\beta]}}{1^{[i+j,-\beta]}}.$$

4. Particular cases with respect to the parameter α.

Let in the formulas (1.3), (1.4), (1.5), (1.6), (1.9), (1.10), (1.11) the parameter $\alpha = 0$; then these formulas become simpler and take the form

$$P(S_{k+1} | P = p) = p, \qquad (1.3a)$$

$$P(X_1 = m_1 | P = p) = \binom{n_1}{m_1} p^{m_1} q^{n_1-m_1}, \qquad (1.4a)$$

$$P(X_1 = m_1) = \binom{n_1}{m_1} \int_0^1 p^{m_1} q^{n_1-m_1} f(p)dp, \qquad \begin{matrix}(1.5a)\\(1.6a)\end{matrix}$$

$$P(X_1 = m_1) = \binom{n_1}{m_1} \int_0^1 p^{m_1} q^{n_1-m_1} dp =$$

$$(1.9a), (1.10a)$$

$$= \binom{n_1}{m_1} B(m_1+1, n_1-m_1+1) = \frac{1}{n_1+1},$$

$$P(X_1 = m_1) = \binom{n_1}{m_1} \frac{\vartheta^{[m_1,-\beta]} (1-\vartheta)^{[n_1-m_1,-\beta]}}{1^{[n_1,-\beta]}} \qquad (1.11a)$$

respectively.

The last formula yields Polya distribution (1.4) with ϑ and β instead of p and α (if we are confined only to the positive values of the parameter α); this can be explained as follows. The obtained distribution (1.11) may be treated as the composition of Polya distribution (1.4) with beta distribution (1.8); on the other hand the composition of a binomial distribution with a beta distribution with suitably chosen parameters yields a Polya distribution. The problem which has been considered here is more general (because the Polya distribution is more general than the binomial distribution). The two problems become equivalent if we accept $\alpha = 0$, hence formula (1.11) for $\alpha = 0$ is a Polya distribution.

§ 2. THE MOMENTS OF THE RANDOM VARIABLE X_1

1. Factorial moments. An ordinary factorial moment of the random variable X_1 with distribution (1.5) (or in the transformed form (1.6)) is given by the formula

$$\alpha_{[r]} = \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \sum_{i=0}^{r} s_i^r(-\alpha)^{r-i} E(P^i), \tag{2.1}$$

where $E(P^i)$ is the average value of the random variable P^i (i.e. the i-th ordinary moment).

In fact, applying in succession: the definition of the r-th factorial moment, the distribution of the random variable X_1 given by formula (1.5), the following formula for the r-th moment of the Polya distribution [1]

$$\alpha_{[r]} = \frac{n_1^{[r]}\ p^{[r,-\alpha]}}{1^{[r,-\alpha]}},$$

the sum form of a factorial polynomial and the definition of the mean value of a function of a random variable we find

$$\alpha_{[r]} = \sum_{m_1=0}^{n_1} m_1^{[r]}\ P(X = m_1) =$$

$$= \sum_{m_1=0}^{n_1} m_1^{[r]} \int_0^1 \frac{\binom{n_1}{m_1} p^{[m_1,-\alpha]}\ q^{[n_1-m_1,-\alpha]}}{1^{[n_1,-\alpha]}} f(p)\,dp =$$

$$= \int_0^1 \sum_{m_1=0}^{n_1} m_1^{[r]} \frac{\binom{n_1}{m_1} p^{[m_1,-\alpha]}\ q^{[n_1-m_1,-\alpha]}}{1^{[n_1,-\alpha]}} =$$

$$= \int_0^1 \frac{n_1^{[r]} \; p^{[r,-\alpha]}}{1^{[r,-\alpha]}} \; f(p) \; dp =$$

$$= \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \int_0^1 \sum_{i=0}^{r} s_i^r \; (-\alpha)^{r-i} \; p^i \; f(p) \; dp =$$

$$= \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \sum_{i=0}^{r} s_i^r \; (-\alpha)^{r-i} \int_0^1 p^i \; f(p) \; dp =$$

$$= \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \sum_{i=0}^{r} s_i^r \; (-\alpha)^{r-i} \; E(P^i).$$

If now P has uniform distribution (1.7), then $E(P^i) = 1/(1+i)$ and formula (2.1) becomes:

$$\alpha_{[r]} = \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \sum_{i=0}^{r} \frac{s_i^r \; (-\alpha)^{r-i}}{1 + i} \; . \qquad (2.2)$$

For P with distribution beta (1.8) $E(P^i) = \vartheta^{[i,-\beta]}/1^{[i,-\beta]}$ and formula (2.1) is transformed to

$$\alpha_{[r]} = \frac{n_1^{[r]}}{1^{[r,-\alpha]}} \sum_{i=0}^{r} \frac{s_i^r \; (-\alpha)^{r-i} \; \vartheta^{[i,-\beta]}}{1^{[i,-\beta]}} \; . \qquad (2.3)$$

Let in formulas (2.1), (2.2), (2.3) the parameter $\alpha = 0$, then they become simpler being reduced to the form

$$\alpha_{[r]} = n^{[r]} \; E(P), \qquad (2.1a)$$

$$\alpha_{[r]} = \frac{n^{[r]}}{1 + r} , \qquad (2.2a)$$

$$\alpha_{[r]} = \frac{n^{[r]} \; \vartheta^{[r,-\beta]}}{1^{[r,-\beta]}} \; . \qquad (2.3a)$$

The last formula is the r-th factorial moment of the Polya distribution, what obviously could have been expected.

2. Ordinary moments. The ordinary moments of the random variable X_1 (and of its particular cases) are obtained by making use of the deduced formulas for the factorial moments and of the following formula which expresses the ordinary moments in terms of factorial moments

$$\alpha_r = \sum_{i=0}^{r} S_i^r \alpha_{[i]},$$

where S_i^r are Stirling numbers of the second kind i.e. the coefficients in the identity

$$x^r \equiv S_0^r + S_1^r x^{[1]} + S_2^r x^{[2]} + \dots + S_{r-1}^r x^{[r-1]} + S_r^r x^{[r]}.$$

Thus the ordinary moment α_r of the random variable X_1 with distribution (1.6) is given by the formula

$$\alpha_r = \sum_{j=0}^{r} S_j^r \frac{n_1^{[j]}}{1^{[j,-\alpha]}} \sum_{i=0}^{j} s_i^j (-\alpha)^{j-i} E(P^i). \qquad (2.4)$$

In particular if P has uniform distribution (1.7) or beta distribution (1.8), then the ordinary moment α_r of the random variable X_1 is expressed by appropriately simpler formulas:

$$\alpha_r = \sum_{j=0}^{r} S_j^r \frac{n_1^{[j]}}{1^{[j,-\alpha]}} \sum_{i=0}^{j} \frac{s_i^j (-\alpha)^{j-i}}{i+1}, \qquad (2.5)$$

$$\alpha_r = \sum_{j=0}^{r} S_j^r \frac{n_1^{[j]}}{1^{[j,-\alpha]}} \sum_{i=0}^{j} \frac{s_i^j (-\alpha)^{j-i} \vartheta^{[i,-\beta]}}{1^{[i,-\beta]}}. \qquad (2.6)$$

The formulas become simpler if we accept $\alpha = 0$:

$$\alpha_r = \sum_{j=0}^{r} S_j^r n_1^{[j]} E(P^j), \qquad (2.4a)$$

$$\alpha_r = \sum_{j=0}^{r} S_j^r \frac{n_1^{[j]}}{1+j} , \qquad (2.5a)$$

$$\alpha_r = \sum_{j=0}^{r} S_j^r \frac{n_1^{[j]} \vartheta^{[j,-\beta]}}{1^{[j,-\beta]}} . \qquad (2.6a)$$

The last three formulas are just the ordinary moments of the composition of a binomial distribution and of a distribution with the density $f(p)$. Formulas (2.5a) and (2.6a) are particular cases of formula (2.4a) which are obtained by accepting for $f(p)$ the densities (1.7) and (1.8) respectively.

3. Mean value and variance. Formulas (2.1) - (2.6) obtained above involve Stirling numbers of the first and the second kinds. To make use of them for finding moments (especially those of higher orders) one must have the tables of Stirling numbers or find them from the following reccurence formulas ([4], [6]):

$$S_i^{r+1} = i\, S_i^r + S_{i-1}^r ,$$

$$s_i^{r+1} = s_{i-1}^r - r s_i^r .$$

One may also proceed in another way, namely taking into account that the r-th (ordinary, factorial) moment of distribution (1.6) is the mean value of the r-th (ordinary factorial) moment of the Polya distribution, if the parameter p appearing in the formulas for the moments of the Polya distribution be treated as a realization of the random variable P with the density $f(p)$. i.e.

$$E(g(X_1)) = E(E(g(X_1)|P)) \text{ where } g(X_1) \equiv X_1^{[r]} \text{ or } g(X_1) \equiv X_1^r .$$

Now this method shall be applied to finding the mean value and the variance of the random variable X_1 (together with the particular cases). Making use of the first and the second moments of the Polya distribution we find

$$E(X_1) = E(E(X_1|P)) = E(n_1P) = n_1\ E(P), \qquad (2.7)$$

$$E(X_1^2) = E(E(X_1^2|P)) = E(n_1P(n_1P + 1 - P + n_1\alpha)/(1+\alpha)) =$$

$$= \frac{n_1}{1+\alpha}\ ((1 + n_1\alpha)\ E(P) + (n_1 - 1)\ E(P^2))$$

by which

$$D^2(X_1) = E(X_1^2) - E^2(X_1) =$$

$$= \frac{n_1}{1+\alpha}\ ((1+n_1\alpha)\ E(P) + (n_1-1)\ E(P^2) + n_1(1+\alpha)\ E^2(P)) = \qquad (2.8)$$

$$= \frac{n_1}{1+\alpha}\ (n_1\alpha\ (1 - E(P))\ E(P) + n_1D^2(P) + E(P) - E(P^2)).$$

To find the variance $D^2(X_1)$ we could have also employed the formula

$$D^2(X) = \alpha_{[2]} - \alpha_1^2 + \alpha_1 .$$

If P has distribution (1.7), then the mean value and the variance of the random variable X_1 are represented as follows

$$E(X_1) = \frac{n_1}{2}, \qquad (2.9)$$

$$D^2(X_1) = \frac{n_1}{12\ (1+\alpha)}\ (3\ n_1\alpha + n_1 + 2). \qquad (2.10)$$

For P with distribution beta (1.8)

$$E(X_1) = n_1\vartheta, \qquad (2.11)$$

$$D^2(X_1) = \frac{n_1\vartheta}{1+\alpha}\left(n_1\alpha(1-\vartheta) + n_1\beta\,\frac{1-\vartheta}{1+\beta} - \frac{\vartheta+\beta}{1+\beta} + 1\right), \quad (2.12)$$

since then:

$$E(P) = \vartheta, \quad E(P^2) = \frac{\vartheta(\vartheta+\beta)}{1+\beta}, \quad D^2(P) = \frac{\beta\vartheta(1-\vartheta)}{1+\beta}.$$

When $\alpha = 0$, then the formulas for mean values (2.7), (2.9), (2.11), do not involve the parameter α and remain unchanged, while variances (2.8), (2.10), (2.12) are transformed as follows

$$D^2(X_1) = n_1\,(n_1 D^2(P) + E(P) - E(P^2)), \quad (2.8a)$$

$$D^2(X_1) = \frac{n_1\,(n_1+2)}{12}, \quad (2.10a)$$

$$D^2(X_1) = \frac{n_1\vartheta\,(1-\vartheta)\,(1+n_1\beta)}{1+\beta}. \quad (2.12a)$$

The last formula yields the variance of the Polya distribution.

§ 3. THE DISTRIBUTION OF THE RANDOM VARIABLE $X_2|X_1$

1. Introduction of the random variable $X_2|X_1$. On performing a composed experiment in which m_1 successes have been realized, we carry out further n_2 experiments according to the conditions of the continuous Polya scheme. Let X_2 be a random variable whose values are the numbers of successes in those n_2 experiments. The probability of success in the first experiment of the second series is thus

$$\frac{p + m_1\alpha}{1 + n_1\alpha}. \quad (3.1)$$

The conditional probability of the random variable X_2 with the conditions $X_1 = m_1$, $P = p$ is again a Polya distribution:

$$P(X_2 = m_2 | X_1 = m_1, P = p) =$$

$$= \binom{n_2}{m_2} \frac{(p + m_1\alpha)^{[m_2,-\alpha]} (q+(n_1-m_1)\alpha)^{[n_2-m_2,-\alpha]}}{(1+n_1\alpha)^{[n_2,-\alpha]}}, \quad m_2=0,1,\ldots,n_2. \quad (3.2)$$

If $\alpha < 0$ it is assumed that the condition: $(n_1+n_2)(-\alpha) \leq \min(p,q)$ holds.

Now we shall be concerned with the random variable $X_2|X_1$, its unconditional probability (with respect to P) is obtained by applying the theorem on the tatal probability

$$P(X_2=m_2|X_1=m_1) = \int_0^1 P(P=p|X_1=m_1)\, P(X_2=m_2|X_1=m_1, P=p)\, dp. \quad (3.3)$$

The second factor under the integral is given by formula (3.2), the first is obtained by applying the appropriate case of the Bayes theorem and taking into consideration formula (1.4):

$$P(P = p|X_1 = m_1) = \frac{P(X_1 = m_1|P = p)\, f(p)}{\int_0^1 P(X_1 = m_1|P = p)\, f(p)\, dp} =$$

$$= \frac{p^{[m_1,-\alpha]}\, q^{[n_1-m_1,-\alpha]}\, f(p)}{\int_0^1 p^{[m_1,-\alpha]}\, q^{[n_1-m_1,-\alpha]}\, f(p)\, dp}.$$

On substituting (3.2) and (3.4) into (3.3) we get

$$P(X_2 = m_2|X_1 = m_1) =$$

$$= \frac{\binom{n_2}{m_2}}{(1+n_1\alpha)^{[n_2,-\alpha]}} \; \frac{\int_0^1 p^{[m,-\alpha]}\, q^{[n-m,-\alpha]}\, f(p)dp}{\int_0^1 p^{[m_1,-\alpha]}\, q^{[n_1-m_1,-\alpha]}\, f(p)dp}, \quad \begin{array}{l} m = m_1+m_2 \\ n = n_1+n_2. \end{array} \quad (3.5)$$

On replacing the product of the factorial polynomials appearing under the integral of the last formula by the sum, this formula is reduced to the form

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \frac{\binom{n_2}{m_2}}{(1+n_1\alpha)^{[n_2,-\alpha]}} \frac{\sum_{i,j=0}^{m,n-m} s_i^m \, s_j^{n-m} (-\alpha)^{n-i-j} \int_0^1 p^i \, q^j \, f(p) \, dp}{\sum_{i,j=0}^{n_1,n_1-m_1} s_i^{m_1} \, s_j^{n_1-m_1} \, (-\alpha)^{n_1-i-j} \int_0^1 p^i \, q^j \, f(p) \, dp}, \qquad (3.6)$$

where $m = m_1 + m_2$, $n = n_1 + n_2$,
with simpler integrals.

2. Particular cases with respect to the distribution of the random variable P. If uniform distribution (1.7) of the random variable P is assumed, then formulas (3.5) and (3.6) are reduced to the following simpler form

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \frac{\binom{n_2}{m_2}}{(1+n_1\alpha)^{[n_2,-\alpha]}} \frac{\int_0^1 p^{[m,-\alpha]} \, q^{[n-m,-\alpha]} \, dp}{\int_0^1 p^{[m_1,-\alpha]} \, q^{[n_1-m_1,-\alpha]} \, dp}, \qquad m = m_1+m_2, \quad n = n_1+n_2 \qquad (3.7)$$

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \frac{\binom{n_2}{m_2}}{(1+n_1\alpha)^{[n_2,-\alpha]}} \frac{\sum_{i,j=0}^{m,n-m} s_i^m \, s_j^{n-m} (-\alpha)^{n-i-j} \, B(i+1,j+1)}{\sum_{i,j=0}^{m_1,n_1-m_1} s_i^{m_1} \, s_j^{n_1-m_1} \, (-\alpha)^{n_1-i-j} \, B(i+1,j+1)}. \qquad (3.8)$$

For a random variable P with beta distribution (1.8), distribution (3.6) becomes

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \frac{\binom{n_2}{m_2}}{(1+n_1\alpha)^{[n_2,-\alpha]}} \frac{\sum_{i,j=0}^{m,n-m}\left(s_i^m s_j^{n-m}(-\alpha)^{n_1-i-j} \vartheta^{[i,-\beta]} (1-\vartheta)^{[j,-\beta]} / 1^{[i+j,-\beta]}\right)}{\sum_{i,j=0}^{m_1,n_1-m_1}\left(s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} \vartheta^{[i,-\beta]} (1-\vartheta)^{[j,-\beta]}/1^{[i+j,-\beta]}\right)}.$$

3. Particular cases with respect to the parameter. Let the parameter $\alpha = 0$ in formulas (3.1), (3.2), (3.4), (3.5), (3.7), (3.8), (3.9), this means the reduction of a continuous Polya scheme to a Bernoulli scheme. These formulas become simpler assuming respectively the forms: (3.1a) P (of the success in the first experiment of the second series) = p,

$$P(X_2 = m_2 | X_1 = m_1, P = p) = \binom{n_2}{m_2} p^{m_2} q^{n_2-m_2}, \quad (3.2a)$$

$$P(P = p | X_1 = m_1) = \frac{p^{m_1} q^{n_1-m_1} f(p)}{\int_0^1 p^{m_1} q^{n_1-m_1} f(p)\, dp}, \quad (3.4a)$$

$$P(X_2 = m_2 | X_1 = m_1) = \binom{n_2}{m_2} \frac{\int_0^1 p^m q^{n-m} f(p)dp}{\int_0^1 p^{m_1} q^{n_1-m_1} f(p)dp}, \quad \begin{matrix} m = m_1+n_1 \\ n = n_1+n_2 \end{matrix} \quad (3.5a)$$

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \binom{n_2}{m_2} \frac{\int_0^1 p^m q^{n-m}\, dp}{\int_0^1 p^{m_1} q^{n_1-m_1}\, dp} = \binom{n_2}{m_2} \frac{B(m+1, n+1)}{B(m_1+1, n_1+1)}, \quad (3.7a), (3.8a)$$

$$P(X_2 = m_2 | X_1 = m_1) =$$

$$= \binom{n_2}{m_2} \frac{\vartheta^{[n,-\beta]} (1-\vartheta)^{[n-m,-\beta]} / 1^{[n,-\beta]}}{\vartheta^{[m_1,-\beta]} (1-\vartheta)^{[n_1-m_1,-\beta]} / 1^{[n_1,-\beta]}} = \quad (3.9a)$$

$$= \binom{n_2}{m_2} \frac{(\vartheta+m_1\beta)^{[m_2,-\beta]} \quad (1-\vartheta+(n_1-m_1)\beta)^{[n_2-m_2,-\beta]}}{(1+n_1\beta)^{[n_2,-\beta]}} \quad .$$

Formula (3.9a) implies the following. If the continuous Polya scheme appearing in the experiment is replaced by a Bernoullie scheme i.e. if $\alpha = 0$, and the random variable P has distribution (1.8), then, the random variable $X_2|X_1$ has Polya distribution (3.9a).

§ 4. THE MOMENTS OF THE RANDOM VARIABLE $X_2|X_1$

An ordinary factorial moment of the random variable $X_2|X_1$ is given by the formula

$$\alpha_{[r]} = \frac{n_2^{[r]}}{(1+n_1\alpha)^{[r,-\alpha]}} \frac{\int_0^1 p^{[m_1+r,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp}{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp} \qquad (4.1)$$

or in a more extended form

$$\alpha_{[r]} = \frac{n_2^{[r]}}{(1+n_1\alpha)^{[r,-\alpha]}} \frac{\sum_{i,j=0}^{m_1+r,n_1-m_1} s_i^{m_1+r_1} s_j^{n_1-m_1} (-\alpha)^{n_1+r-i-j} \int_0^1 p^i q^j f(p)dp}{\sum_{i,j=0}^{m_1,n_1-m_1} s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} \int_0^1 p^i q^j f(p)dp} . \qquad (4.1)$$

In fact, emploing in succession the definition of the r-th ordinary factorial moment, the distribution (3.5) of the random variable $X_2|X_1$ and the formula for the r-th ordinary moment of the Polya distribution we obtain

$$\alpha_{[r]} = \sum_{m_2=0}^{n_2} m_2^{[r]} P\left(X_2 = m_2 \middle| X_1 = m_1\right) =$$

$$= \frac{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p) \sum_{m_2=0}^{n_2} m_2^{[r]} \binom{n_2}{m_2} \frac{(p+m_1\alpha)^{[m_2,-\alpha]}\left(q+(n_1-m_1)\alpha\right)^{[n_2-m_2,-\alpha]}}{(1+n_1\alpha)^{[n_2,-\alpha]}} dp}{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp} =$$

$$= \frac{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} \left(n_2^{[r]}(p+m_1\alpha)^{[r,-\alpha]}/(1+n_1\alpha)^{[r,-\alpha]}\right) f(p)dp}{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp}$$

$$= \frac{n_2^{[r]}}{(1+n_1\alpha)^{[r,-\alpha]}} \frac{\int_0^1 p^{[m_1+r,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)dp}{\int_0^1 p^{[m_1,-\alpha]} q^{[n_1-m_1,-\alpha]} f(p)\, dp} .$$

On replacing the product of factorial polynomials appearing under the integral in the numerator and in the denominator by the corresponding sum we get (4.1').

For P with uniform distribution (1.7) formulas (4.1) and (4.1') become

$$\alpha_{[r]} = \frac{n_2^{[r]}}{(1 + n_1\alpha)^{[r,-\alpha]}} \frac{\int_0^1 p^{[m_1+r,-\alpha]} \cdot q^{[n_1-m_1,-\alpha]} dp}{\int_0^1 p^{[m_1,-\alpha]} \cdot q^{[n_1-m_1,-\alpha]} dp} , \qquad (4.2)$$

$$\alpha_{[r]} = \frac{n_2^{[r]}}{(1+n_1\alpha)^{[r,-\alpha]}} \frac{\sum_{i,j=0}^{m_1+r,\, n_1-m_1} s_i^{m_1+r} s_j^{n_1+r-i-j} (-\alpha)^{n_1+r-i-j} B(i+1,j+1)}{\sum_{i,j=0}^{m_1,\, n_1-m_1} s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} B(i+1,j+1)} . \qquad (4.2')$$

If P is subject to distribution beta (1.8), formula (4.1') is reduced to

$$\alpha_{[r]} = \frac{n_2^{[r]}}{(1+n_1\alpha)^{[r,-\alpha]}} \frac{\sum\limits_{i,j=0}^{m_1+r,\, n_1-m_1} s_i^{m_1+r} s_j^{n_1-m_1} (-\alpha)^{n_1+r-i-j} \left(\vartheta^{[i,-\beta]} (1-\vartheta)^{[j,-\beta]} / 1^{[i+j,-\beta]}\right)}{\sum\limits_{i,j=0}^{m_1,\, n_1-m_1} s_i^{m_1} s_j^{n_1-m_1} (-\alpha)^{n_1-i-j} \left(\vartheta^{[i,-\beta]} (1-\vartheta)^{[j,-\beta]} / 1^{[i+j,-\beta]}\right)} \cdot \qquad (4.3)$$

If in formulas (4.1), (4.2), (4.2'), (4.3) we put $\alpha = 0$ we obtain respectively the formulas

$$\alpha_{[r]} = n_2^{[r]} \frac{\int_0^1 p^{m_1+r} q^{n_1-m_1} f(p)\, dp}{\int_0^1 p^{m_1} q^{n_1-m_1} f(p)\, dp}, \qquad (4.1a)$$

$$\alpha_{[r]} = n_2^{[r]} \frac{\int_0^1 p^{m_1+r} q^{n_1-m_1} f(p) dp}{\int_0^1 p^{m_1} q^{n_1-m_1} dp} =$$

$$= n_2^{[r]} \frac{B(m_1+r+1,\ n_1-m_1+1)}{B(m_1+1,\ n_1-m_1+1)} = \frac{n_2^{[r]} (m_1+r)!}{(n_1+3)^{[r,-1]}}. \qquad (4.2a)$$

$$\alpha_{[r]} = n_2^{[r]} \frac{\vartheta^{[m_1+r,-\alpha]} (1-\vartheta)^{[n_1-m_1,-\beta]} / 1^{[n_1+r,-\beta]}}{\vartheta^{[m_1,-\beta]} (1-\vartheta)^{[n_1-m_1,-\beta]} / 1^{[n_1,-\beta]}} =$$

$$= \frac{n_2^{[r]} (\vartheta + m_1\beta)^{[r,-\beta]}}{(1+n_1\beta)^{[r,-\beta]}} \cdot \qquad (4.3a)$$

The ordinary moments of the random variable $X_2|X_1$ can easily by obtained using the factorial moments found above and the quoted relationship expressing the ordinary moments in terms of the factorial ones.

§ 5. EXAMPLES

Suppose that a large number of industrial enterprises form a union. We may accept that 1° the occurence of an accident in one of them is a random event, 2^o the occurence of an accident in some enterprise reduces the probability of another accident in the same enterprise (because the care for security measures is then increased), 3^o the probability of an accident in different enterprsees in the unit of time is different. We observe the enterprise in which the next accident happens; what is the probability of m_1 accidents within n_1 units of time in an enterprise in which an accident has accured (if we assume that only one accident is possible in a unit of time, this is the case if for instance the accident makes the work impossible during the whole unit of time).

Thus in this problem it may be assumed that the parameter p is a realization of the random variable P in the meaning that the observed enterprise has been chosen at random from all the enterprises in the union. If moreover we take into account assumption 2^o, then it is seen that m_1 is a realization of the random variable X_1 considered in the paper and thus the required probability is given by formula (1.6), which unfortunately involves an unknown density f(p) and an unknown parameter α; they should be taken from the preceding observation of the problem. The method of estimating the parameters ϑ and β appearing in the density f(p) defined by formula (1.8) is given in paper [5].

There is a number of similar examples. It suffices to replace the industrial enterprises in the above example by various populations (e.g. by professions if human populations are concerned), and the accident by contraction of an infections disease (every infection increases the chances of a next one).

If seems that the distribution of the random variable $X_2|X_1$ may also be employed in such situations as those quoted above if the present data are used to estimate the future (and the past data to estimate the present).

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