DEMONSTRATIO MATHEMATICA
Vol. IV No 3 1972

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ON SOME CLASSES OF NON-BOUNDED FUNCTIONS

I n t r o d u c t i o n. Recent investigations in the theory of functional equations have singled out some classes of non-bounded functions. In [1] a class $\bar{K}(M, \mu, \varphi)$ of functions defined on an open arc L_0 has been introduced, and in [2] - a class $\bar{K}(\alpha, \beta, \omega)$ of functions defined on a set $\Omega < E^n$. In the sequel we shall denote the first class by W, and the second by H. Our aim is to show that in case $L_0 = \Omega\,(n=2)$ the class H^* is included in the class H.

T h e c l a s s W. Let $L = \breve{ab}$ be a non-closed smooth arc with ends a and b. For every $t \in L$ let t^* denote that end of L for which $le.\breve{tt^*} = \min(le.\breve{at}, le.\breve{at})$, where $le.s$ denotes the arc s. Analogously, let t_1^* denote that end of L for which $le.\breve{t_1t_1^*} = \min(le.\breve{at_1}, le.\breve{t_1b})$. Let $L_0 = = L - \{a,b\}$. For every pair of points of L_0 we denote one point by t and the other by t_1 according to

$$le.\breve{tt^*} \leqslant le.\breve{t_1t_1^*}. \tag{1W}$$

Consider a function $\varphi = \varphi(\sigma)$ increasing and continuous in the interval $\langle 0,+\infty\rangle$ satisfying the following conditions

$$\varphi(0) = 0 \tag{2W}$$

$$\bigwedge_{\nu\geqslant 1}\ \bigwedge_{\sigma\geqslant 0} \varphi(\nu\sigma) \leqslant \varphi(\sigma) \tag{3W}$$

$$\bigwedge_{\sigma\in\langle 0,1\rangle} \varphi(\sigma) \geqslant \sigma. \tag{4W}$$

D e f i n i t i o n. The class W is defined as the set of all complexvalued functions $f(t)$ of a complex variable t that are defined and continuous in L_0 and satisfy the inequalities

$$|f(t)| \leqslant \frac{M}{|t - t^*|}, \tag{5W}$$

$$|f(t) - f(t_1)| \leqslant \frac{M}{|t - t^*|^{\mu}} \varphi\left(\left|\frac{t - t_1}{t - t^*}\right|\right), \tag{6W}$$

where $M > 0$ and $0 \leqslant \mu < 1$ are fixed constants.

T h e c l a s s H. In [2] Adamczyk introduced a class of functions defined in a domain $\Omega \subset E^n$. We shall restrict Ω to L_0, where as previously L_0 denotes the interior of a non-closed smooth arc L with ends a and b whose length is l.

Consider real functions $\alpha(t)$ and $\beta(t)$ defined and continuous in the set L_0 and assuming only positive values. For every pair of points of L_0 let t denote one point and t_1 the other according to

$$\beta(t) \geqslant \beta(t_1). \tag{1H}$$

Next consider a function $\omega = \omega(\delta)$ increasing and continuos in the interval $\langle 0,1 \rangle$ satisfying the condition

$$\omega(0) = 0. \tag{2H}$$

We assume that for arbitrarily close points $t \in L_0$, $t^* \in \{a,b\}$ the following inequality holds

$$\frac{\alpha(t)}{\beta(t)} \leqslant K_{\omega}\,\omega(|t - t^*|), \tag{3H}$$

and for every pair of arbitrarily close points $t, t_1 \in L_0$ the following condition holds

$$\left|\beta(t) - \beta(t_1)\right| \leqslant K_\beta \frac{\beta(t) - \beta(t_1)}{\alpha(t_1)} \omega\left(\left|t - t_1\right|\right) \tag{4H}$$

where K_ω and K_β denote some positive constants.

D e f i n i t i o n. The class H is defined as the set of all complexvalued functions of a complex variable t, defined and continuous in L_o, such that the following inequalities hold

$$\left|f(t)\right| \leqslant M_f\, \alpha(t), \tag{5H}$$

$$\left|f(t) - f(t_1)\right| \leqslant K_f\, \beta(t)\, \omega\left(\left|t - t_1\right|\right), \tag{6H}$$

where M_f and K_f are some positive constants.

T h e o r e m. $W \subset H$.

P r o o f. It is to be shown that

$$f(t) \in W \Longrightarrow f(t) \in H.$$

Accordingly, assume that conditions (1W) - (6W) hold. It is known that there exists a positive constant χ such that for every point $t \in L_o$

$$\chi\, \mathrm{le}.\breve{tt}^* \leqslant \left|t - t^*\right|. \tag{1}$$

From (1) and (3W) we infer that

$$\varphi\left(\left|\frac{t - t_1}{t - t^*}\right|\right) \leqslant \varphi\left(\frac{|t - t_1|}{\chi \mathrm{le}.\breve{tt}^*}\right) \leqslant \begin{cases} \dfrac{1}{\chi \mathrm{le}.\breve{tt}^*}\, \varphi\left(\left|t - t_1\right|\right) & \text{if } \mathrm{le}.\breve{tt}^* \leqslant 1 \\[2ex] \dfrac{l\chi}{\chi\, \mathrm{le}.\breve{tt}^*}\, \varphi\left(\left|t - t_1\right|\right) & \text{if } \mathrm{le}.\breve{tt}^* > 1 \end{cases}$$

that is,

$$\varphi\left(\left|\frac{t - t_1}{t - t^*}\right|\right) \leqslant \frac{\max\,(1, \chi l)}{\chi\, \mathrm{le}.tt^*}\, \varphi\left(\left|t - t_1\right|\right). \tag{2}$$

Since

$$\varphi\left(|t - t_1|\right) \leqslant \begin{cases} \varphi\left(|t - t_1|^{\mu}\right) & \text{if } |t - t_1| \leqslant 1 \\ \frac{\varphi(1)}{\varphi(1)}\, \varphi\left(|t - t_1|^{\mu}\right) & \text{if } |t - t_1| > 1, \end{cases}$$

that is,

$$\varphi\left(|t - t_1|\right) \leqslant \max\left(1, \frac{\varphi(1)}{\varphi(1)}\right) \varphi(|t - t_1|^{\mu}),$$

we infer form (2) that

$$\varphi\left(\left|\frac{t - t_1}{t - t^*}\right|\right) \leqslant \frac{M_o}{\chi\, \mathrm{le}.\overset{\smile}{tt^*}}\, \varphi(|t - t_1|^{\mu}), \tag{3}$$

where $M_o = \max(1, \chi 1).\ \max\left(1, \frac{\varphi(1)}{\varphi(1)}\right)$.

In view of (5W) and (6W), and using (1) and (3) we infer that

$$f(t) \leqslant \frac{M}{\chi^{\mu}(\mathrm{le}.\overset{\smile}{tt^*})^{\mu}}, \tag{4}$$

$$|f(t) - f(t_1)| \leqslant \frac{MM_o}{\chi^{1+\mu}\, (\mathrm{le}.\overset{\smile}{tt^*})^{1+\mu}}\, \varphi(|t - t_1|^{\mu}). \tag{5}$$

Letting

$$M_f = \frac{M}{\chi^{\mu}}, \qquad K_f = \frac{MM_o}{\chi^{1+\mu}},$$

and

$$\alpha(t) = \frac{1}{(\mathrm{le}.\overset{\smile}{tt^*})^{\mu}}, \quad \beta(t) = \frac{1}{(\mathrm{le}.\overset{\smile}{tt^*})^{1+\mu}}, \quad \omega(\delta) = \varphi(\delta^{\mu}), \tag{6}$$

we see that the function $f(t)$ satisfies conditions (5H) and (6H). In view of (1W), (2W) and (6), conditions (1H) and (2H) also hold.

According to (6) we have

$$\frac{\alpha(t)}{\beta(t)} = \text{le.}\overset{\smile}{tt^*}. \tag{7}$$

Using (1) and (4W) we see that

$$\text{le.}\overset{\smile}{tt^*} \leqslant \frac{1}{\chi}\,|t - t^*| \leqslant \begin{cases} \dfrac{1}{\chi}\;\varphi(|t - t^*|^\mu) \text{ if } |t - t^*| \leqslant 1 \\ \dfrac{1}{2\chi\varphi(1)}\;\varphi(|t - t^*|^\mu) \text{ if } |t - t^*| > 1. \end{cases}$$

Hence by virtue of (7) we infer that

$$\frac{\alpha(t)}{\beta(t)} \leqslant \frac{1}{\chi} \max\left(1, \frac{1}{2\varphi(1)}\right) \varphi(|t - t^*|^\mu). \tag{8}$$

Now letting

$$K_\omega = \frac{1}{\chi} \max\left(1, \frac{1}{2\varphi(1)}\right)$$

and using (6) we get

$$\omega(|t - t^*|) = \varphi(|t - t^*|^\mu), \tag{9}$$

which shows that condition (3H) holds.

It remains to show that condition (4H) holds.
In view of (6) we infer that

$$|\beta(t) - \beta(t_1)| = \frac{(\text{le.}\overset{\smile}{t_1t_1^*})^{1+\mu} - (\text{le.}\overset{\smile}{tt^*})^{1+\mu}}{(\text{le.}\overset{\smile}{tt^*})^{1+\mu}\,(\text{le.}\overset{\smile}{t_1t_1^*})^{1+\mu}} =$$

$$= \beta(t)\,\beta(t_1)\,\left|(\text{le.}\overset{\smile}{t_1t_1^*})^{1+\mu} - (\text{le.}\overset{\smile}{tt^*})^{1+\mu}\right| =$$

$$= \beta(t)\,\beta(t_1)\,\Big|(\text{le.}\overset{\smile}{t_1t_1^*})^{\mu}\left[\text{le.}\overset{\smile}{t_1t_1^*} - \text{le.}\overset{\smile}{tt^*}\right] +$$

$$+ \text{le.}\overset{\smile}{tt^*}\left[(\text{le.}\overset{\smile}{t_1t_1^*})^{\mu} - (\text{le.}\overset{\smile}{tt^*})^{\mu}\right]\Big| \leqslant$$

$$\leqslant \beta(t)\,\beta(t_1)\left[\frac{1}{\alpha(t_1)}\,le.\widetilde{t_1 t} + \frac{1}{\alpha(t)}\,(le.\widetilde{tt^*})^{1-\mu}\,(le.\widetilde{t_1 t})^{\mu}\right].$$

Hence according to (1), (1W) and (6) we get

$$\left|\beta(t) - \beta(t_1)\right| \leqslant \frac{\beta(t)\,\beta(t_1)}{\alpha(t_1)}\left[(le.\widetilde{t_1 t})^{1-\mu}\,(le.\widetilde{t_1 t})^{\mu} + \right.$$

$$\left. + (le.\widetilde{tt^*})^{1-\mu}\,(le.\widetilde{t_1 t})^{\mu}\right] \leqslant \frac{\beta(t)\beta(t_1)}{\alpha(t_1)}\;\frac{2}{\chi^{\mu}}\,l^{1-\mu}\left|t - t_1\right|^{\mu}. \qquad (10)$$

Applying (4W) we obtain

$$\left|t - t_1\right|^{\mu} \leqslant \begin{cases} \varphi(\left|t - t_1\right|^{\mu}) \text{ if } \left|t - t_1\right| \leqslant 1 \\ \frac{l^{\mu}}{\varphi(1)}\,\varphi(\left|t - t_1\right|^{\mu}) \text{ if } \left|t - t_1\right| > 1. \end{cases} \qquad (11)$$

Formulas (10) and (11) imply that

$$\left|\beta(t) - \beta(t_1)\right| \leqslant \max\left(\frac{2}{\chi^{\mu}}\;l^{1-\mu},\;\frac{2l}{\chi^{\mu}\varphi(1)}\right)\frac{\beta(t)\beta(t_1)}{\alpha(t_1)}\;\varphi(\left|t-t_1\right|^{\mu}).$$

Finally putting

$$K = \max\left(\frac{2}{\chi^{\mu}}\,l^{1-\mu},\;\frac{2l}{\varphi(1)\,\chi^{\mu}}\right),$$

and considering (9) we see that condition (4H) holds.

Hence we have shown that (1H) - (6H) hold, which means that f(t) H. This ends the proof of the theorem.

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Received January 29th, 1972.

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