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ON SOME CLASS $\mathfrak{H}$ (M, α, φ) OF FUNCTIONS DEFINED ON A NON-CLOSED ARC

Introduction. We shall give the definition of a class $\mathfrak{H}(M,\alpha,\varphi)$ of functions defined on a non-closed arc. This class contains, among others, the class $\mathfrak{H}^{\mu}_{\alpha}$ defined by W. Pogorzelski ([1], [2]). A modified definition of the class $\mathfrak{H}^{\mu}_{\alpha}$ was given by author in [3].

Let M and α be fixed real numbers, $M \in (0,+\infty)$ and $\alpha \in \langle 0,1)$. Let $\varphi=\varphi(6)$ be a real function, continuous and increasing in the interval $\langle 0,+\infty)$, which satisfies the following conditions:

1° $\varphi(0) = 0$,

2° $\bigwedge\limits_{6\in\langle 0,1\rangle} \varphi(6) \geqslant 6$

3° $\bigwedge\limits_{\nu\geqslant 1} \bigwedge\limits_{6\geqslant 0} \varphi(\nu 6) \leqslant \nu\varphi(6)$

Consider a non-closed smooth arc $L = \breve{ab}$, with ends a and b. Let $L_0 = L-\{a,b\}$. For any $t \in L_0$ let t^* denote this element of the set $\{a,b\}$ for which length $\breve{tt^*}$ = min (lenght $\breve{at}$, length $\breve{tb}$). Analogously, for any $t_1 \in L_0$ let t_1^* denote this member of the set $\{a,b\}$ for which length $\breve{t_1t_1^*}$ = min (length $\breve{at_1}$, length $\breve{t_1b}$). We assume that length $\breve{tt^*} \leqslant$ length $\breve{t_1t_1^*}$.

Definition. By the class $\mathfrak{H}$ (M,α,φ) we understand the set of all functions f(t) of a complex variable t, defined and continuous in the set L_0, which for every pair of points $t,t_1 \in L_0$ satisfy the following inequalities:

$$|f(t)| \leqslant \frac{M}{|t-t^*|^{\alpha}} \tag{1}$$

$$\left|f(t) - f(t_1)\right| \leqslant \frac{M}{|t-t^*|^\alpha}\varphi\left(\left|\frac{t-t_1}{t-t^*}\right|\right) . \qquad (2)$$

In case $\varphi(\sigma) = \sigma^\mu$ $(0<\mu<1,\ \alpha+\mu<1)$ $\mathcal{H}(M,\alpha,\varphi)$ is the class $\mathcal{H}_\alpha^\mu$ of Pogorzelski with the constant M.

We shall give an example of a function from the class $\mathcal{H}(M,\alpha,\varphi)$ which does not belong to $\mathcal{H}_\alpha^\mu$. This will show that the class $\mathcal{H}(M,\alpha,\varphi)$ is an essential extension of the class $\mathcal{H}_\alpha^\mu$.

Let

$$\varphi(\sigma) = \begin{cases} 0, & \text{if } \sigma = 0 \\ -\dfrac{1}{\ln\sigma}, & \text{if } 0<\sigma\leqslant e^{-4} \\ \sigma^{\frac{\ln 2}{2}}, & \text{if } \sigma \geqslant e^{-4} . \end{cases}$$

The so-defined function $\varphi(\sigma)$ is continuous and increasing in the interval $\langle 0,+\infty)$ and satisfies conditions 1^o and 2^o. To show that φ satisfies also condition 3^o we consider three cases:

1) $\nu\sigma\leqslant e^{-4}$

$$\varphi(\nu\sigma) = -\frac{1}{\ln\nu\sigma} \quad \text{and} \quad \varphi(\sigma) = -\frac{1}{\ln\sigma} .$$

Since $\varphi'(\sigma) = \dfrac{1}{\sigma\ln^2\sigma} > 0$ and $\varphi''(\sigma) = -\dfrac{2+\ln\sigma}{\sigma^2\ln^3\sigma} < 0$,

we have

$$\frac{\varphi(\nu\sigma)}{\varphi(\sigma)} \leqslant \frac{\nu\sigma}{\sigma} \quad \text{i.e.} \quad \varphi(\nu\sigma)\leqslant\nu\varphi(\sigma)$$

2) $\sigma\geqslant e^{-4}$

$$\varphi(\nu\sigma) = \nu^{\frac{\ln 2}{2}}\sigma^{\frac{\ln 2}{2}} \leqslant \nu\sigma^{\frac{\ln 2}{2}} = \nu\varphi(\sigma)$$

3) $\nu\sigma > e^{-4}$ and $\sigma < e^{-4}$

$$\varphi(\nu\sigma) = \nu^{\frac{\ln 2}{2}} \sigma^{\frac{\ln 2}{2}} \leqslant \nu\sigma^{\frac{\ln 2}{2}} \leqslant \nu\left(-\frac{1}{\ln\sigma}\right) = \nu\varphi(\sigma)\,,$$ because

$$\bigwedge_{\sigma\in(0,e^{-4})} \sigma^{\frac{\ln 2}{2}} \leqslant -\frac{1}{\ln\sigma}\,.$$

Moreover it is evident that

$$\bigwedge_{A>0}\ \bigwedge_{0<\mu<1}\ \bigwedge_{0<\alpha<1}\ \bigvee_{\delta>0}\ \bigwedge_{0<\sigma<\delta} \varphi(\sigma) > (4e^{4})^{\alpha+\mu} A\sigma^{\mu}\,. \tag{3}$$

Consider now the following function

$$f(t) = \begin{cases} 0, & \frac{e^{-4}}{2} < t < 0 \\ \varphi(t), & 0 \leqslant t < \frac{e^{-4}}{2}\,. \end{cases} \tag{4}$$

We shall show that $f(t) \in \mathfrak{K}(1, \alpha, \varphi)$.
The function $f(t)$ satisfies inequality (1).
In fact, we have

$$|f(t)| \leqslant \varphi(t) \leqslant \frac{\varphi(t)}{|t-t^{*}|^{\alpha}} \leqslant \frac{1}{|t-t^{*}|^{\alpha}}\,.$$

To show that $f(t)$ satisfies inequality (2) we consider four cases.

1) $-\frac{e^{-4}}{2} < t \leqslant t_1 \leqslant 0$ - obvious.

2) $-\frac{e^{-4}}{2} < t < 0 \wedge 0 < t_1 < \frac{e^{-4}}{2}$

$$\left|f(t)-f(t_1)\right| = -\frac{1}{\ln t_1} \leq -\frac{1}{\ln(t_1-t)} = \varphi(|t-t_1|) \leq \frac{\varphi\left(\left|\frac{t-t_1}{t-t^*}\right|\right)}{|t-t^*|^{\alpha}}.$$

3) $0 < t < \frac{e^{-4}}{2} \wedge \frac{-e^{-4}}{2} < t_1 < 0$ - analogously as in 2).

4) $0 \leq t_1 < t < \frac{e^{-4}}{2}$

$$\left|f(t) - f(t_1)\right| = \varphi(t) - \varphi(t_1) = \frac{1}{\ln t_1} - \frac{1}{\ln t} = \frac{\ln t - \ln t_1}{\ln t \cdot \ln t_1}.$$

Let

$$g(t_1) = \frac{\ln t - \ln t_1}{\ln t . \ln t_1} + \frac{1}{\ln(t-t_1)}, \quad t_1 \in (0,t) .$$

The derivative of this function is

$$g'(t_1) = \frac{1}{(t-t_1)\ln^2(t-t_1)} - \frac{1}{t_1 \ln^2 t_1}, \quad t_1 \in (0,t).$$

The function $\frac{1}{r(x)}$, where $r(x) = x\ln^2 x$, is decreasing in the interval $(0,t)$. Consequently,

$$g'(t_1) \begin{cases} <0, & \text{if} \quad t_1 < t-t_1 \\ >0, & \text{if} \quad t_1 > t-t_1 . \end{cases}$$

Since $\lim\limits_{t_1 \to 0} g(t_1) = 0$ and $\lim\limits_{t_1 \to t} g(t_1) = 0$,

we see that $\bigwedge\limits_{t_1 \in (0,t)} g(t_1) < 0$, hence in the considered case:

$$\frac{\ln t-\ln t_1}{\ln t.\ln t_1} < -\frac{1}{\ln(t-t_1)} .$$

Applying this inequality we obtain

$$|f(t) - f(t_1)| < -\frac{1}{\ln(t-t_1)} = \varphi(t-t_1) \leqslant \frac{\varphi\left(\left|\frac{t-t_1}{t-t^*}\right|\right)}{|t-t^*|^\alpha} .$$

Since the function f(t) satisfies inequalities (1) and (2) with M=1, we infer that $f(t) \in \mathfrak{H}(1,\alpha,\varphi)$.

We shall show that the function f(t) defined in (4) does not belong to Pogorzelski's class $\mathfrak{H}_\alpha^\mu$.

Let $t_1=0$ and $t \in (0, \frac{e^{-4}}{4})$.

We have:

$$|f(t) - f(t_1)| = -\frac{1}{\ln t} = -\frac{1}{\ln(t-t_1)} = \varphi(|t-t_1|) =$$

$$= \frac{|t-t^*|^{\alpha+\mu}\varphi(|t-t_1|)}{|t-t^*|^{\alpha+\mu}} > \frac{\left(\frac{e^{-4}}{4}\right)^{\alpha+\mu}}{|t-t^*|^{\alpha+\mu}} \varphi(|t-t_1|) .$$

According to (3) we see that

$$\bigwedge_{A>0} \bigwedge_{0<\mu<1} \bigwedge_{0\leqslant\alpha<1} \bigvee_{t\in(0,\frac{e^{-4}}{4})} |f(t)-f(t_1)| > \frac{A|t-t_1|^\mu}{|t-t^*|^{\alpha+\mu}} ,$$

which means that $f(t) \notin \mathfrak{H}_\alpha^\mu$.

THE COMPACTNESS OF THE SET $\mathfrak{H}(M,\alpha,\varphi)$

It is easy to see that the set $\mathfrak{H}(M,\alpha,\varphi)$ is closed and convex.

Consider the functional space Λ consisting of all complex functions

$$U = \left[f(t)\right] \tag{5}$$

which are continuous in the set L_0 and satisfy the condition

$$\sup_{t\in L_0} \left[|t-t^*|^{1+\alpha}|f(t)|\right] < \infty \tag{6}$$

These functions are the points of Λ.

We define the sum of two points $U = \left[f(t)\right]$ and $V = \left[g(t)\right]$, and the product of a point by a number as follows:

$$U+V = \left[f(t) + g(t)\right], \quad \lambda U = \left[\lambda f(t)\right] .$$

The norm $\|U\|$ of the point (5) is defined by the formula

$$\|U\| = \sup_{t\in L_0} \left[|t-t^*|^{1+\alpha}|f(t)|\right] . \tag{7}$$

The distance between two points U and V can then be defined as the norm of the difference U-V, i.e. $\varrho(U,V) = \|U-V\|$.

In this way the space Λ becomes a Banach space.

Let $\{U_n\} = \{[f_n(t)]\}$ be an arbitrary infinite sequence of elements of the set $\mathfrak{H}(M,\alpha,\varphi)$.

According to the definition of the set $\mathfrak{H}(M,\alpha,\varphi)$, for every natural number n the following inequalities are satisfied

$$|f_n(t)| \leqslant \frac{M}{|t-t^*|^{\alpha}} \tag{8}$$

and

$$|f_n(t)-f_n(t_1)| \leqslant \frac{M}{|t-t^*|^{\alpha}}\,\varphi\left(\left|\frac{t-t_1}{t-t^*}\right|\right) . \tag{9}$$

Consider now the functional sequence with a general term

$$\Phi_n(t) = \begin{cases} (\text{le}.\breve{tt^*})^{1+\alpha} \quad f_n(t), & \text{for} \quad t\in L_0 \\ \\ 0 \qquad , & \text{for } t=a \text{ or } t=b \end{cases} \tag{10}$$

where le. denotes-length.

We have the following lemmas.

L e m m a 1. The functions in the sequence (10) are uniformly bounded in the set L.

P r o o f. In view of (8) and (10), for every natural n and every $t\in L_0$ we have

$$|\Phi_n(t)| \leqslant \frac{M}{|t-t^*|^\alpha}\cdot\frac{1}{\chi^{1+\alpha}}\cdot|t-t^*|^{1+\alpha}\leqslant\frac{Ml}{2\chi^{1+\alpha}}$$

where l is the length of the arc L, and $0<\chi\leqslant 1$ is the greatest lower bound of the quotient of the chord and the arc corresponding to this chord.

Since $\Phi_n(a)=\Phi_n(b)=0$, we see that

$$\bigwedge_n \bigwedge_{t\in L} |\Phi_n(t)| \leqslant \frac{Ml}{2\chi^{1+\alpha}} \tag{11}$$

which ends the proof of the lemma.

L e m m a 2. The functions in the sequence (10) are equicontinuous in the set L.

P r o o f. We consider two cases.

1° Let $|t-t_1|\geqslant 1$. Then it follows from (11) that

$$\bigwedge_n \bigwedge_{t,t_1\in L} |\Phi_n(t)-\Phi_n(t_1)| \leqslant \frac{Ml}{\chi^{1+\alpha}\,\varphi(1)}\,\varphi(|t-t_1|^\alpha) \tag{12}$$

2° Let $|t-t_1|<1$. In view of (8), (9), (10) we obtain

$$|\Phi_n(t)-\Phi_n(t_1)| = |(\text{le}.\breve{tt^*})^{1+\alpha}f_n(t)-(\text{le}.\breve{t_1t_1^*})^{1+\alpha}f_n(t_1)| \leqslant \tag{13}$$

$$\leqslant \frac{M}{|t_1-t_1^*|^{\alpha}}\left|(\text{le.}\,\breve{t_1t_1^*})^{1+\alpha}-(\text{le.}\,\breve{tt^*})^{1+\alpha}\right|+$$

$$+\frac{M(\text{le.}\,\breve{tt^*})^{1+\alpha}}{|t-t^*|^{\alpha}}\varphi\left(\left|\frac{t-t_1}{t-t^*}\right|\right)=\Delta_1+\Delta_2$$

For every pair of points $t, t_1 \in L_0$ we have

$$\Delta_1=\frac{M}{|t_1-t_1^*|^{\alpha}}\left|(\text{le.}\,\breve{t_1t_1^*})^{\alpha}\left[\text{le.}\,\breve{t_1t_1^*}-\text{le.}\,\breve{tt^*}\right]+\right.$$

$$\left.+\,\text{le.}\,\breve{tt^*}\left[(\text{le.}\,\breve{t_1t_1^*})^{\alpha}-(\text{le.}\,\breve{tt^*})^{\alpha}\right]\right|\leqslant \tag{14}$$

$$\leqslant\frac{M(\text{le.}\,t_1t_1^*)^{\alpha}}{\chi^{1+\alpha}(\text{le.}\,t_1t_1^*)^{\alpha}}|t-t_1|^{\alpha}+\frac{M(\text{le.}\,\breve{t_1t_1^*})^{\alpha}\cdot(\text{le.}\,\breve{tt^*})^{1-\alpha}}{\chi^{\alpha}(\text{le.}\,\breve{t_1t_1^*})^{\alpha}}(\text{le.}\,\breve{t_1t})^{\alpha}\leqslant$$

$$\leqslant\frac{M}{\chi^{1+\alpha}}|t-t_1|^{\alpha}+\frac{M}{\chi^{2\alpha}}\left(\frac{1}{2}\right)^{1-\alpha}|t-t_1|^{\alpha}\leqslant\frac{M}{\chi^{1+\alpha}}\left[1+\left(\frac{1\chi}{2}\right)^{1-\alpha}\right]\varphi\left(|t-t_1|^{\alpha}\right).$$

We shall estimate the component Δ_2 in (13). Let $r\leqslant\min\left(1,\frac{|b-a|}{4}\right)$ be the radius of a circle $K(t^*,r)$ with the center t^*. We assume that this circle is sufficiently small in order that it intersect the arc $\breve{t^*t_n}$ exactly in one point (t_n is the end of the arc $L=\breve{ab}$ opposite to t^*). Let Q denote the middle point of the arc L.

If t is an interior point of the arc $\breve{t^*t_w}$, then $\frac{1}{|t-t^*|}\geqslant 1$.

In this case we have

$$\Delta_2\leqslant\frac{M}{\chi^{1+\alpha}}\varphi(|t-t_1|^{\alpha}). \tag{15}$$

If $t \in \breve{t_w Q}$, then $\frac{1}{|t-t^*|} \leqslant \frac{1}{r}$.

Moreover, we also have

$$\Delta_2 \leqslant M\left(\frac{1}{2r}\right)^{1+\alpha} \varphi(|t-t_1|^\alpha) . \qquad (16)$$

Observe that if $t=t^*$ or $t_1=t_1^*$, we can apply (10) to obtain

$$|\Phi_n(t)-\Phi_n(t_1)| \leqslant \begin{cases} \frac{M}{\chi^{1+\alpha}} \varphi(|t-t_1|^\alpha), & \text{if } |t-t_1| < 1 \\ \frac{Ml}{\chi^{1+\alpha}\varphi(1)} \varphi(|t-t_1|^\alpha), & \text{if } |t-t_1| \geqslant 1 \end{cases} \qquad (17)$$

In view of (12) - (17) we see that there exists a positive constant K_Φ which is independent of n and which depends on M, χ, α, r, l and on the function $\varphi = \varphi(6)$ such that for every pair of points $t, t_1 \in L$ we have

$$|\Phi_n(t)-\Phi_n(t_1)| \leqslant K_\Phi \varphi(|t-t_1|^\alpha). \qquad (18)$$

Moreover it can be shown that if $\alpha = 0$ then there exists a positive constant $\tilde{K}_\Phi$ independent of n and depending only on M, χ, r, l and $\varphi = \varphi(6)$ such that for every pair of points $t, t_1 \in L$ we have

$$|\Phi_n(t) - \Phi_n(t_1)| \leqslant \tilde{K}_\Phi \varphi(|t-t_1|) \qquad (19)$$

From (18) and (19) it follows that the functions of the sequence (10) are equicontinuous in the closed set L.

We can now prove the following theorem:

T h e o r e m. The subset $\hbar(M,\alpha,\varphi)$ of the Banach spape Λ is compact in itself.

P r o o f. In view of Lemmas 1 and 2 the assumptions of Arzeli's theorem are satisfied and we can infer that there exists a subsequence $\{\Phi_{k_n}(t)\}$ of the sequence (10) which is uniformy convergent in the set L. This subsequence must satisfy Cauchy's condition, so that

$$\bigwedge_{\varepsilon>0} \bigvee_{N_\varepsilon} \bigwedge_{r,s>N} \sup_{t\in L} \left|\Phi_{k_r}(t) - \Phi_{k_s}(t)\right| < \varepsilon . \qquad (20)$$

From (10), (20) and the obvious inequality $|t-t^*| \leqslant$ length $\breve{tt^*}$ it follows that

$$\bigwedge_{\varepsilon>0} \bigvee_{N_\varepsilon} \bigwedge_{r,s>N_\varepsilon} \sup_{t\in L_0} \left[\, |t-t^*|^{1+\alpha} \left|f_{k_r}(t) - f_{k_s}(t)\right|\right] < \varepsilon . \quad (21)$$

This means that a subsequence $\{f_{k_n}(t)\}$ of an arbitrary sequence of points from the set $\mathfrak{H}(M,\alpha,\varphi)$ satisfies Cauchy's condition in the norm (7). Since Λ is a complete space, this subsequence is convergent to a point $[f_*(t)]$ of Λ. On the other hand, all the elements of the sequence $\{f_n(t)\}$ belong to the closed set $\mathfrak{H}(M,\alpha,\varphi)$ hence $f_*(t) \in \mathfrak{H}(M,\alpha,\varphi)$. This shows that the set $\mathfrak{H}(M,\alpha,\varphi)$ is compact in itself.

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