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Agnieszka Plucińska

SOME PROBLEMS CONCERNING THE PROSPECTIVE AND THE RETROSPECTIVE EQUATIONS FOR NON-MARKOVIAN PROCESSES

Let $Y_t = Y_t(\omega)$ $(t \in \langle 0,T\rangle,\ \omega \in \Omega)$ be a stochastic process defined in a probability space (Ω, S, P). We shall assume that Y_t is a stochastic process with real values from some interval I, by A we shall denote a Borel subset of I. Let $0 \leqslant t_0 \leqslant \leqslant t_1 \leqslant \ldots \leqslant t_{n-1} < t_n \leqslant T$, $y_i \in J$ for $i = 1,2,\ldots,n$.

Let $F_{Y,n}$ denote a σ-algebra generated by the family of sets $\left\{\omega : Y_{t_1}(\omega) < y_1,\ Y_{t_2}(\omega) < y_2,\ldots,Y_{t_n}(\omega) < y_n\right\}$.

We shall use the following denotations for the conditional probabilities with respect to σ-algebra $F_{Y,n}$

$$P(Y_t \in A | F_{Y,1}) = P(Y_t \in A | Y_{t_1} = y_1) = P(t_1, y_1, t, A) \tag{1}$$

$$P(Y_{t_2} < y_2 | F_{Y,1}) = P(Y_{t_2} < y_2 | Y_{t_1} = y_1) = F(t_1, y_1, t_2, y_2) \tag{2}$$

$$P(Y_{t_3} < y_3 | F_{Y,2}) = P(Y_{t_3} < y_3 | Y_{t_1} = y_1,\ Y_{t_2} = y_2) = H(t_1, y_1, t_2, y_2, t_3, y_3) \tag{3}$$

$$P(Y_{t_2} < y_2, Y_{t_3} < y_3 | F_{Y,1}) = P(Y_{t_2} < y_2, Y_{t_3} < y_3 | Y_{t_1} = y_1) = G(t_1, y_1, t_2, y_2, t_3, y_3) \tag{4}$$

and the following denotations for the transition probability densities

$$\frac{\partial}{\partial y_2} F(t_1,y_1,t_2,y_2) = f(t_1,y_1,t_2,y_2)$$

$$\frac{\partial}{\partial y_3} H(t_1,y_1,t_2,y_2,t_3,y_3) = h(t_1,y_1,t_2,y_2,t_3,y_3)$$

$$\frac{\partial^2}{\partial y_2\,\partial y_3} G(t_1,y_1,t_2,y_2,t_3,y_3) = g(t_1,y_1,t_2,y_2,t_3,y_3)$$

If Y_t is a Markov process and some regularity conditions are satisfied then it is well known that the following Kołmogorov equations hold

$$\frac{\partial}{\partial t_1} F(t_1,y_1,t_2,y_2) + a_1(t_1,y_1)\frac{\partial}{\partial y_1} F(t_1,y_1,t_2,y_2) + \frac{1}{2}a_2(t_1,y_1)\frac{\partial^2}{\partial y_1^2} F(t_1,y_1,t_2,y_2) = 0\ , \tag{5}$$

$$\frac{\partial}{\partial t_1} f(t_1,y_1,t_2,y_2)+a_1(t_1,y_1)\frac{\partial}{\partial y_1} f(t_1,y_1,t_2,y_2) + \frac{1}{2}a_2(t_1,y_1)\frac{\partial^2}{\partial y_1^2} f(t_1,y_1,t_2,y_2) = 0\ , \tag{5'}$$

$$\frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_2) + \frac{\partial}{\partial y_2}\left[a_1(t_2,y_2)f(t_1,y_1,t_2,y_2)\right] - \frac{1}{2}\frac{\partial^2}{\partial y_2^2}\left[a_2(t_2,y_2)f(t_1,y_1,t_2,y_2)\right] = 0\ , \tag{6}$$

where a_i are infinitesimal moments given by formulas

$$a_i(t,y) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y_1-y|<\delta} (y_1-y)^i \, d_{y_1} F(t,y,t+\Delta t,y_1) , \qquad (7)$$

$i = 1,2; \quad \delta > 0.$

It is also assumed that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y_1-y| \geq \delta} d_{y_1} F(t,y,t+\Delta t,y_1) = 0 . \qquad (8)$$

In paper [1] a generalization of equation (6) to the case of non-markovian processes was given. Namely it was proved the following.

T h e o r e m 1. If for any $\delta > 0$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y_3-y_2|<\delta} (y_3-y_2)^i \, h(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3) dy_3 =$$

$$= a_i^h(t_1,y_1,t_2,y_2) , \quad i = 1,2 , \qquad (9)$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y_3-y_2| \geq \delta} h(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3) dy_3 = 0 \qquad (10)$$

the convergence in (9), (10) is unform with respect to y_2, functions

$h(t_1,y_1,t_2,y_2,t_3,y_3)$,

$$\frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_2), \frac{\partial^i}{\partial y_2^i}\left[a_i^h(t_1,y_1,t_2,y_2) f(t_1,y_1,t_2,y_2)\right] , \qquad (11)$$

$i = 1,2$

are continuous, then the following partial differential equation holds

$$\frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_2)+\frac{\partial}{\partial y_2}\left[a_1^h(t_1,y_1,t_2,y_2)f(t_1,y_1,t_2,y_2)\right] + \tag{12}$$

$$- \frac{1}{2} \frac{\partial^2}{\partial y_2^2} \left[a_2^h(t_1,y_1,t_2,y_2)f(t_1,y_1,t_2,y_2)\right] = 0 .$$

In [2] was proved the following

Theorem 2. If conditions (7), (8) are satisfied uniformly in y_2, there exist continuous derivatives

$$\frac{\partial}{\partial t_2} H(t_1,y_1,t_2,y_2,t_3,y_3) , \tag{13}$$

$$\frac{\partial^i}{\partial y_2^i} H(t_1,y_1,t_2,y_2,t_3,y_3) , \quad i = 1,2$$

then the following partial differential equation holds

$$\frac{\partial}{\partial t_2} H(t_1,y_1,t_2,y_1,t_3,y_3)\Big|_{t_2=t_1} +$$

$$+ a_1(t_1,y_1) \frac{\partial}{\partial y_2} H(t_1y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1} + \tag{14}$$

$$+ \frac{1}{2} a_2(t_1,y_1) \frac{\partial^2}{\partial y_2^2} H(t_1,y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1} = 0 ,$$

where

$$\frac{\partial^i}{\partial y_2^i} H(t_1,y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1} =$$

$$= \lim_{\Delta t \to 0} \left[\frac{\partial^i}{\partial y_2^i} H(t_1 - \Delta t,y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1}\right]; \quad i = 1,2. \tag{15}$$

The explanation (15) is connected with the fact that the function $H(t_1,y_1,t_2,y_2,t_3,y_3)$ is undefined on the set S: $t_1=t_2$, $y_1 \neq y_2$.

If there exists the density h then it follows from (14) that the following equation holds

$$\frac{\partial}{\partial t_2} h(t_1,y_1,t_2,y_1,t_3,y_3)\Big|_{t_2=t_1} +$$

$$+a_1(t_1,y_1) \frac{\partial}{\partial y_2} h(t_1,y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1} + \tag{14'}$$

$$+ \frac{1}{2}a_2(t_1,y_1) \frac{\partial^2}{\partial y_2^2} h(t_1,y_1,t_1,y_2,t_3,y_3)\Big|_{y_2=y_1} = 0 .$$

In this paper the following questions will be considered:

I. A generalization of theorem 1.

II. Some conditions that a process will be a Markov process in the wide sense.

III. The unique solution of some differential partial equations.

IV. A correction concerning paper [9].

Equations (5), (5'), (6) for Markov processes are proved under different conditions. They are given for example in [3]

- [6]. In [3] the proof is based on Lebesgue integral, in [4] - [6] on Rieman integral.

In this paper all the integral are Lebesgue integrals. In the theorem based on Lebesgue integral the assumption of uniformity of convergence may be omitted.

By $\bar{I}$ we shall denote the closure of I on the straight line.

I. In this part we shall assume that the function h exists and is continuous in $\bar{I}$. Next we shall assume that for an arbitrary $\delta > 0$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_I \int_{|y_3-y_2| \geq \delta} g(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3)dy_3dy_2 = 0 \qquad (16)$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y_3-y_2|<\delta} (y_3-y_2)^i \, g(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3)dy_3 = $$
$$= b_i(t_1,y_1,t_2,y_2); \qquad i = 1,2 .$$

It is evident that if there exist limits (9), (10) then there exist limits (16), (17) (on condition that h exists) and

$$b_i(t_1,y_1,t_2,y_2)=a_i^h(t_1,y_1,t_2,y_2)f(t_1,y_1,t_2,y_2) .$$

We shall denote

$$\frac{1}{\Delta t} \int_{|y_3-y_2|<\delta} (y_3-y_2)^i g(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3)dy_3 =$$

$$= B_i(t_1,y_1,t_2,y_2,\Delta t); \qquad i = 1,2.$$

We shall prove the following

T h e o r e m 3. If for some Δ_0 and for an arbitrary interval (a,b) there exist functions $C_i(t_1, y_1, t_2, y_2)$ integrable with respect to y_2 for $a < y_2 < b$ such that

$$\left|B_i(t_1,y_1,t_2,y_2,\Delta t)\right| < C_i(t_1,y_1,t_2,y_2)$$

almost everywhere in (a,b) for $\Delta t < \Delta_0$, relations (16), (17) hold, functions

$$\frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_2), \quad \frac{\partial^i}{\partial y_2^i} b_i(t_1,y_1,t_2,y_2) \ , \quad i = 1,2 \qquad (18)$$

are continuous in $\bar{I}$, then equation (12) holds.

P r o o f. Let a and b be arbitrary real numbers such that $(a,b) \subset I$. Let R(y) denote an arbitrary non-negative function from class C^2 and let

$$R(y) = 0 \quad \text{for} \quad y < a \quad \text{and for} \quad y > b.$$

In virtue of the relation

$$f(t_1,y_1,t_2+\Delta t,y_3) = \int_I g(t_1,y_1,t_2,y_2,t_2+\Delta t_1,y_3)dy_2$$

we have

$$\frac{1}{\Delta t}\int_a^b \left[f(t_1,y_1,t_2+\Delta t,y_3)-f(t_1,y_1,t_2,y_3)\right]R(y_3)dy_3 =$$

$$(19)$$

$$= \frac{1}{\Delta t}\int_a^b \left[\int_I g(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3)dy_2-f(t_1,y_1,t_2,y_3)\right]R(y_3)dy_3 \ .$$

In virtue of the properties of the function R(y), the continuity of the functions $\frac{\partial f}{\partial t_2}$, h and relation (19) we have

$$\int_a^b \frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_3)R(y_3)dy_3 = \tag{20}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[\iint_{I\,I} g(t_1,y_1,t_2,y_2,t_2+\Delta t,y_3)R(y_3)dy_2dy_3 +$$

$$- \int_I f(t_1,y_1,t_2,y_3)R(y_3)dy_3\Big] =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[\int_I\int_I g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)R(y_2)dy_2dy_3 +$$

$$- \int_I f(t_1,y_1,t_2,y_3)R(y_3)dy_3\Big] =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_I \int_{|y_3-y_2|\geqslant\sigma} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)R(y_2)dy_2dy_3 +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[\int_I \int_{|y_3-y_2|<\sigma} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)R(y_2)dy_2dy_3 +$$

$$- \int_I f(t_1,y_1,t_2,y_3)R(y_3)dy_3\Big] = \lim_{\Delta t \to 0} J_1 + \lim_{\Delta t \to 0} J_2 .$$

From the properties of the function $R(y)$ it follows that there exists a constant M such that $|R(y)| < M$. Taking into account, (10) and (20) we get

$$\lim_{\Delta t \to 0} J_1 = \lim_{\Delta t \to 0} \left| \frac{1}{\Delta t}\int_I \int_{|y_3-y_2|\geqslant\sigma} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)R(y_2)dy_2dy_3\right| \tag{21}$$

$$\leqslant M \lim_{\Delta t \to 0} \int_I \int_{|y_3-y_2|\geqslant\delta} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)dy_2dy_3 = 0 .$$

Expanding function R into Taylors series in virtue of (20), (21) we have

$$\int_a^b \frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_3)R(y_3)dy_3 = \tag{22}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Big\{ \int_a^b dy_3 \int_{|y_3-y_2|<\delta} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)\Big[R(y_3) +$$

$$+(y_2-y_3)R'(y_3)+\frac{(y_2-y_3)^2}{2}R''(y_3)+o(y_2-y_3)^2\Big]dy_2 +$$

$$- \int_a^b f(t_1,y_1,t_2,y_3)R(y_3)dy_3\Big\} =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^b dy_3 \int_{|y_3-y_2|<\delta} g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)\Big[(y_2-y_3)R'(y_3) +$$

$$+ \frac{1}{2}(y_2-y_3)^2R''(y_3)\Big]dy_2 +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^b dy_3 \int_{|y_3-y_2|<\delta} o(y_2-y_3)^2 g(t_1,y_1,t_2,y_3,t_2+\Delta t,y_2)dy_2 +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^b dy_3 \int_{|y_3-y_2| \geq \delta} g(t_1, y_1, t_2, y_3, t_2+\Delta t, y_2) R(y_3) dy_2 =$$

$$= \lim_{\Delta t \to 0} J_3 + \lim_{\Delta t \to 0} J_4 + \lim_{\Delta t \to 0} J_5 .$$

It is evident that

$$\lim J_5 = 0 .$$

Now let us notice that

$$\lim_{\Delta t \to 0} |J_4| = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left| \int_a^b dy_3 \int_{|y_2-y_3|<\delta} (y_2-y_3)^2 o(y_2-y_3)^2 \times \right. \tag{23}$$

$$\left. \times g(t_1, y_1, t_2, y_3, t_2+\Delta t, y_2) dy_2 \right| \leq$$

$$\leq \delta^2 \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_a^b dy_3 \int_{|y_2-y_3|<\delta} (y_2-y_3)^2 g(t_1, y_1, t_2, y_3, t_2+\Delta t, y_2) dy_2 =$$

$$= \delta^2 \int_a^b \lim_{\Delta t \to 0} B_2(t_1, y_1, t_2, y_3, \Delta t) dy_3 = \delta^2 \int_a^b b_2(t_1, y_1, t_2, y_3) dy_3 \to 0$$

when $\delta \to 0$
as the function $b_2(t_1, y_1, t_2, y_3)$ is continuous and consequently for $a \leq y_3 \leq b$ is bounded.

It follows from (17), (22) and (23) that

$$\int_a^b \frac{\partial}{\partial t_2} f(t_1, y_1, t_2, y_3) R(y_3) dy_3 = \tag{24}$$

$$= \int_a^b \left[b_1(t_1, y_1, t_2, y_3) R'(y_3) + \frac{1}{2} b_2(t_1, y_1, t_2, y_3) R''(y_3) \right] dy_3 .$$

Integrating the right side of (24) by parts we have

$$\int_a^b \left[\frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_3) + \frac{\partial}{\partial y_3} b_1(t_1,y_1,t_2,y_3) + \right. \tag{25}$$

$$\left. - \frac{1}{2}\frac{\partial^2}{\partial y_3^2} b_2(t_1,y_1,t_2,y_3)\right] R(y_3)dy_3 = 0 \ .$$

Formula (10) follows immediately from (25) and (17) as the function $R(y)$ is arbitrary.

R e m a r k. It is evident that theorem 3 is true if we replace the assumption: "functions (18) are continuous" by the assumption: "there exist continuous functions equal almost everywhere to functions (18)".

In theorem 1 the convergence must to be uniform. In theorem 3 the condition of uniform convergence can be omitted. It is also evident that the uniformity of convergence is not necessary in theorem 2.

II. Now we are going to give conditions that equations (5) (5'), (6) for Markov processes and equations (14), (14'),(12) for non-markovian processes have the same form. This will permit us to give some conditions that a process is a Markov process in the wide sense. The definition of the Markov process in the wide sense is given for example in [4].

It is evident that if

$$a_i^h(t_1,y_1,t_2,y_2) = a_i(t_2,y_2), \quad i = 1,2, \tag{26}$$

then equation (6) and (12) are identical.

Next note that the function $H(t_1,y_1,t_2,y_2,t_3,y_3)$ is undefined on the set

$$S: t_1 = t_2 \ , \quad y_1 \neq y_2 \ ,$$

since by putting $t_1 = t_2$, $y_1 \neq y_2$ we obtain a contradicting condition in (2). Thus all considerations must concern only the values outside S. It is also evident that the limit

$$\lim_{\Delta t \to 0} H(t_1 - \Delta t, y_1, t_1, y_2, t_3, y_3)$$

can bè considered only in the case $|y_2 - y_1| \to 0$ and it ought to be

$$\lim_{\substack{\Delta t \to 0 \\ \delta \to 0}} H(t_1 - \Delta t, y_2, t_2, y_2, t_3, y_3) = F(t_2, y_2, t_3, y_3) .$$

From the definitions of functions F and H it follows that

$$H(t_2, y_2, t_2, y_2, t_3, y_3) = F(t_2, y_2, t_3, y_3) .$$

Moreover we shall assume that for $|y_2 - y_1| < \delta$

$$\lim_{\substack{\Delta t \to 0 \\ \delta \to 0}} \frac{1}{\Delta t} \left[H(t_2 - \Delta t, y_1, t_2, y_2, t_3, y_3) - F(t_2, y_2, t_3, y_3) \right] = 0 . \qquad (27)$$

We are going to prove the following

T h e o r e m 4. If condition (27) is satisfied then equations (14), (14') have respectively forms (5), (5').

P r o o f. Note that

$$\frac{1}{\Delta t} F(t_1 - \Delta t, y_1, t_3, y_3) =$$

$$= \frac{1}{\Delta t} \int_I H(t_1 - \Delta t, y_1, t_1, y_2, t_3, y_3) d_{y_2} F(t_1 - \Delta t, y_1, t_1, y_2) =$$

$$= \frac{1}{\Delta t} \int_I F(t_1, y_2, t_3, y_3) d_{y_2} F(t_1 - \Delta t, y_1, t_1, y_2) +$$

$$+ \frac{1}{\Delta t} \int_I H(t_1 - \Delta t, y_1, t_1, y_2, t_3, y_3) +$$

$$- F(t_1,y_2,t_3,y_3)\Big] d_{y_2}F(t_1-\Delta t,y_1,t_1,y_2) = J_1+J_2 .$$

Now we shall evaluate J_2. From the properties of functions F and H it follows that

$$\Big| \frac{1}{\Delta t} \int\limits_{|y_2-y_1|\geqslant\delta} \Big[H(t_1-\Delta t,y_1,t_1,y_2,t_3,y_3) + \tag{29}$$

$$- F(t_1,y_2,t_3,y_3)\Big] d_{y_2}F(t_1-\Delta t,y_1,t_1,y_2)$$

$$\leqslant \frac{2}{\Delta t} \int\limits_{|y_2-y_1|\geqslant\delta} d_{y_2}F(t_1-\Delta t,y_1,t_1,y_2) \to 0 \quad \text{when} \quad \Delta t \to 0 ,$$

$$\lim_{\substack{\Delta t\to 0\\ \delta\to 0}} \frac{1}{\Delta t} \int\limits_{|y_2-y_1|<\delta} \Big[H(t_1-\Delta t,y_1,t_1,y_2,t_3,y_3) + \tag{30}$$

$$- F(t_1,y_2,t_3,y_3)\Big] d_{y_2}F(t_1-\Delta t,y_1,t_1,y_2) =$$

$$= \lim_{\delta\to 0} \int\limits_{|y_2-y_1|<\delta} \lim_{\Delta t\to 0} \frac{1}{\Delta t}\Big[H(t_1-\Delta t,y_1,t_1,y_2,t_3,y_3) +$$

$$- F(t_1,y_2,t_3,y_3)\Big] d_{y_2}F(t_1-\Delta t,y_1,t_1,y_2) = 0 .$$

In virtue of (29), (30) we have

$$\lim_{\substack{t\to 0\\ \delta\to 0}} J_2 = 0 .$$

From (28) - (30) it follows that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[F(t_1-\Delta t, y_1, t_1, y_3) + \qquad (31)$$

$$- \int_I F(t_1, y_2, t_3, y_3) d_{y_2} F(t_1-\Delta t, y_1, t_1, y_2)\Big] = 0.$$

Relation (31) has the same form as in the case of Markov process. Then repeating the reasoning used for Markov process in virtue of (31) we get the thesis of theorem 4.

Now we are going to give some conditions that a process is a Markov process in the wide sense.

We assume that there exists a transition density f, equation (5') holds and functions f, a_i (i = 1,2) satisfy some additional regularity conditions, which will be specified in progress of succesive considerations.

Let us consider equation (5'). Let the coefficients a_i be such that this equation has a unique solution (some additional marginal conditions may be added).

Multiplying (5') by $f(t_2, y_2, t_3, y_3)$ and integrating we get

$$\int_I \frac{\partial}{\partial t_1} f(t_1, y_1, t_2, y_2) f(t_2, y_2, t_3, y_3) dy_2 +$$

$$+ a_1(t_1, y_1) \int_I \frac{\partial}{\partial y_1} f(t_1, y_1, t_2, y_2) f(t_2, y_2, t_3, y_3) dy_2 +$$

$$+ \frac{1}{2} a_2(t_1, y_1) \int_I \frac{\partial^2}{\partial y_1^2} f(t_1, y_1, t_2, y_2) f(t_2, y_2, t_3, y_3) dy_2 = 0 .$$

If the function f is sufficiently regular and the order of integration and differentiation can be interchanged then we get

$$\frac{\partial}{\partial t_1} \int_I f(t_1,y_1,t_2,y_2)f(t_2,y_2,t_3,y_3)dy_2 +$$

$$+ a_1(t_1,y_1) \int_I f(t_1,y_1,t_2,y_2)f(t_2,y_2,t_3,y_3)dy_2 +$$

$$+ \frac{1}{2} a_2(t_1,y_1) \frac{\partial^2}{\partial y_1^2} \int_I f(t_1,y_1,t_2,y_2)f(t_2,y_2,t_3,y_3)dy_2 = 0 .$$

Let us put in (5') $t_2 = t_3$, $y_2 = y_3$.
If the solution of (5') is unique then it must be

$$f(t_1,y_1,t_3,y_3) = \int_I f(t_1,y_1,t_2,y_2)f(t_2,y_2,t_3,y_3)dy_2 . \quad (32)$$

In virtue of the relation

$$P(t_1,y_1,t_2,A) = \int_A d_{y_2}F(t_1,y_1,t_2,y_2) = \int_A f(t_1,y_1,t_2,y_2)dy_2$$

equation (32) can be written in the following form

$$f(t_1,y_1,t_3,y_3) = \int_I f(t_2,y_2,t_3,y_3)P(t_1,y_1,t_2,dy_2). \quad (33)$$

If the order of integration is interchangeable then in virtue of (33) we get

$$P(t_1,y_1,t_3,A) = \int_A f(t_1,y_1,t_3,y_3)dy_3 =$$

$$= \int_A \Big[\int_I f(t_2,y_2,t_3,y_3)P(t_1,y_1,t_2,dy_2)\Big] dy_3 =$$

$$= \int_I \left[\int_A f(t_2,y_2,t_3,y_3)dy_3 \right] P(t_1,y_1,t_2,dy_2) =$$

$$= \int_I P(t_2,y_2,t_3,A)P(t_1,y_1,t_2,dy_2).$$

Then the considered process is a Markov process in the wide sense.

The analogical reasoning can be given with respect to equations (5) and (6).

III. Let us denote by K the class of probability density functions f such that:

1° $I = (0, +\infty)$

2° $f(t_1,y_1,t_2,y_2) = f_1(\Delta t)f_2\left(\frac{y_1}{\Delta t^{1/p}}, \frac{y_2}{\Delta t^{1/p}}\right)$

where p - natural number, $\Delta t = t_2-t_1$.

3° there exists a constant $d \geqslant 0$ such that

$$f_2\left(\frac{y_1}{\Delta t^{1/p}}, \frac{y_2}{\Delta t^{1/p}}\right) = \left(\frac{y_2}{\Delta t^{1/p}}\right)^d f_3\left(\frac{y_1}{\Delta t^{1/p}}, \frac{y_2}{\Delta t^{1/p}}\right)$$

and f_3 is analitical in $\bar{J}$.

In virtue of

$$\int_I f(t_1,y_1,t_2,y_2)dy_2 = 1 ,$$

it follows that

$$f_1(\Delta t) = \Delta t^{-1/p} . \qquad (34)$$

We shall find a solution of equations (5'), (6) in the class K.

The assumption that the function is analytic is one of the fundamental assumption in the theory of partial differential equations [7].

In the paper [9] a stochastic process was considered for which the probability density function f was given by the formula

$$f(t_1,y_1,t_2,y_2) = f(\Delta t,y_1,y_2) = \sum_{k=0}^{\infty} c_k(z_1)\tilde{f}_{d+kp}(z_2) \quad (35)$$

where

$$z_i = \frac{y_i}{(p\ \Delta t)^{1/p}}, \quad i = 1,2$$

$$c_k(z_1) = \frac{1}{k!}\ z_1^{kp}\ e^{-z_1^p} \quad (36)$$

$$\tilde{f}_{d+kp}(z_2) = \frac{p}{\Gamma(\frac{d+kp+1}{p})}\ z_2^{d+kp}\ e^{-z_2^p}. \quad (37)$$

The functions f_{d+kp} are general gamma densities, c_k are Poisson coefficients, function (35) is a randomized general gamma density. It was shown that for $p=1$, $d \geqslant 0$ and for $p=2$, $d=2n$ (n - natural number or 0) the infinitesimal moments a_i are given by formulas

$$a_1(t,y) = a_1(y) = \frac{2+d-p}{p}\ y^{1-p} \quad (38)$$

$$a_2(t,y) = a_2(y) = \frac{2}{p}\ y^{2-p}. \quad (38')$$

For $p=2$, $d=0$ the coefficients a_i have the same form as for the Wiener process defined for $I = (-\infty, +\infty)$.

In this paper we shall show that (35) is the unique solution in the class K of equations (5'), (6) if a_i are given by (38), (38') for $d \geqslant 0$, $p=1$ and for $p = 2$, $d=2n$ $(n=0,1,2,\ldots)$.

First we must show that functions a_i, f satisfy the assumptions of theorems 3, 4.

Let us put $\delta = +\infty$. In paper [9] it was shown that: for $p=1$, $d \geqslant 0$

$$m_r(y_1,\Delta t) = \int_0^\infty y_2^r f(t_1,y_1,t_2,y_2)dy_2 = \sum_{k=0}^{r} \binom{r}{k} y_1^k (\Delta t)^{r-k} \frac{\Gamma(d+1+r)}{\Gamma(d+1+k)}$$

and therefore in virtue of (17)

$$B_1(t_1,y_1,t_2,y_2,\Delta t) = \frac{1}{\Delta t}\left[m_1(y_2,\Delta t)-y_2\right] f(t_1,y_1,t_2,y_2) = \tag{39}$$

$$= (d+1) f(t_1,y_1,t_2,y_2)$$

$$B_2(t_1,y_1,t_2,y_2,\Delta t) = \tag{39'}$$

$$= \frac{1}{\Delta t}\left[m_2(y_2,\Delta t)-2y_2 m_1(y_2,\Delta t)+y_2^2\right] f(t_1,y_1,t_2,y_2) =$$

$$= \left[2y_1+(d+1)(d+2)\Delta t\right] f(t_1,y_1,t_2,y_2)$$

for $p=2$, $d=2n$ $(n=0,1,2,\ldots)$.

$$m_1(y_1,\Delta t) = e^{-u}\sqrt{2\Delta t}\,\frac{\partial^n}{\partial u^n}\left[\frac{u^n}{\Gamma(n+\frac{1}{2})} + \sqrt{u}\, e^u\, \Phi(\sqrt{u}) + \right.$$

$$\left. + \sqrt{\frac{u}{\pi}} \sum_{k=0}^{n-1} \frac{u^{k+\frac{1}{2}}\, 2^{k+1}}{(2k+1)!!}\right],$$

for $n \geqslant 1$.

where

$$\Phi(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}dx, \quad u = \frac{1}{2\Delta t} y_1^2, \quad d = 2n \quad (n=0,1,2,\ldots);$$

$$m_2(y_1,\Delta t) = \sum_{k=0}^{1} \binom{1}{k} y_1^{2k} (2\Delta t)^{1-k} \frac{\Gamma(\frac{d+1}{2}+1)}{\Gamma(\frac{d+1}{2}+k)}$$

and therefore in virtue of [9]

$$B_1(t_1,y_1,t_2,y_2,\Delta t) = \frac{1}{\Delta t}\left[m_1(y_2,\Delta t)-y_2\right]f(t_1,y_1,t_2,y_2) = \tag{40}$$

$$= \left\{\frac{1}{\Delta t}\left[y_1\ \Phi\left(\frac{y_1}{(2\Delta t)^{1/2}}\right) - y_1 + o(\Delta t)\right] + \right.$$

$$\left. + \frac{n}{y_1}\ \Phi\left(\frac{y_1}{(2\Delta t)^{1/2}}\right)\right\} f(t_1,y_1,t_2,y_2) =$$

$$= f(t_1,y_1,t_2,y_2)\left[\frac{n}{y_1}\ \Phi\left(\frac{y_1}{(2\Delta t)^{1/2}}\right) - \frac{1}{\Delta t}\ y_1\ \frac{2}{\sqrt{\pi}} \int\limits_{\frac{y_1}{(2\Delta t)^{1/2}}}^{\infty} e^{-x^2} dx\right] +$$

$$+ O(\Delta t) = \frac{n}{y_1}\ \Phi\left(\frac{y_1}{(2\Delta t)^{1/2}}\right) f(t_1,y_1,t_2,y_2) + O_1(\Delta t) ,$$

$$B_2(t_1,y_1,t_2,y_2,\Delta t) = \tag{40'}$$

$$= \frac{1}{\Delta t}\left[m_2(y_2,\Delta t) - 2y_2 m_1(y_2,\Delta t) + y_2^2\right] f(t_1,y_1,t_2,y_2) =$$

$$= f(t_1,y_1,t_2,y_2) + \frac{o(\Delta t)}{\Delta t} = f(t_1,y_1,t_2,y_2) + o(\Delta t) .$$

Taking into account (39), (39'), (40), (40') and remark 1 it is easy to verify that all the assumptions of the theorem 3 are satisfied.

Finding the partial derivatives one can verify that (35) satisfies equations (5) and (6).

Now we are going to show that (35) is the unique solution in the class K of equations (5) and (6).

In virtue of (34) and assumption 3^{o} we can write

$$f(t_1,y_1,t_2,y_2) = (p\Delta t)^{-1/p} f_2(z_1,z_2) .$$

According to assumption 3^{o} we substitute

$$f(t_1,y_1,t_2,y_2) = (p\Delta t)^{-1/p} z_2^d f_3(z_1,z_2) = \tag{41}$$

$$= (p\Delta t)^{-1/p} z_2^d e^{-z_2^p} f_4(z_1,z_2) .$$

Taking into account (38), (38') and (41) we have

$$-\frac{\partial}{\partial t_1} f(t_1,y_1,t_2,y_2) = \frac{\partial}{\partial t_2} f(t_1,y_1,t_2,y_2) \tag{42}$$

$$= -(p\ \Delta t)^{-\frac{1+p}{p}}\left[f_2+z_1\frac{\partial f_2}{\partial z_1} + z_2\frac{\partial f_2}{\partial z_2}\right] =$$

$$= -(p\Delta t)^{-\frac{1+p}{p}}\left[(d+1)f_4+z_1\frac{\partial f_4}{\partial z_1} - pz_2^p f_4+z_2\frac{\partial f_4}{\partial z_2}\right]z_2^d e^{-z_2^p} ,$$

$$a_1(y_1)\frac{\partial}{\partial y_1} f(t_1,y_1,t_2,y_2) + \frac{1}{2}a_2(y_1)\frac{\partial^2}{\partial y_1^2} f(t_1,y_1,t_2,y_2) = \tag{43}$$

$$= \frac{1}{p}(p\ \Delta t)^{-\frac{1+p}{p}}\left[(2+d-p)z_1^{1-p}\frac{\partial f_2}{\partial z_1} + z_1^{2-p}\frac{\partial^2 f_2}{\partial z_1}\right] =$$

$$= \frac{1}{p}(p\ \Delta t)^{-\frac{1+p}{p}}\left[(2+d-p)z_1^{1-p}\frac{\partial f_4}{\partial z_1} + z_1^{2-p}\frac{\partial^2 f_4}{\partial z_1}\right]z_2^d e^{-z_2^p} ,$$

$$\frac{\partial}{\partial y_2}\left[a_1(y_2)f(t_1,y_1,t_2,y_2)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_2^2}\left[a_2(y_2)f(t_1,y_1,t_2,y_2)\right]= \tag{44}$$

$$=\frac{1}{p}(p\ \Delta t)^{\frac{1+p}{p}}\left\{\frac{\partial}{\partial z_2}\left[(2+d-p)z_2^{1-p}\ f_2\right]-\frac{\partial^2}{\partial z_2^2}\left[z_2^{2-p}\ f_2\right]\right\}=$$

$$=\frac{1}{p}(p\ \Delta t)^{-\frac{1+p}{p}}\left[p(d+1)f_4+2pz_2\frac{\partial f_4}{\partial z_2}-(2+d-p)z_2^{1-p}\frac{\partial f_4}{\partial z_2}-z_2^{2-p}\frac{\partial^2 f_4}{\partial z_2^2}+\right.$$

$$\left.-p^2z_2^p\ f_4\right]z_2^d\ e^{-z_2^p}\ .$$

It follows from (42) - (44) that equations (5') and (6) can now be written in the following form

$$\left\{p\left[(d+1)f_4+z_1\frac{\partial f_4}{\partial z_1}-pz_2^pf_4+z_2\frac{\partial f_4}{\partial z_2}\right]+\right. \tag{45}$$

$$\left.-(2+d-p)z_1^{1-p}\frac{\partial f_4}{\partial z_1}-z_1^{2-p}\frac{\partial^2 f_4}{\partial z_1^2}\right\}z_2^d=0\ .$$

$$\left\{p\left[(d+1-pz_2^p)f_4+z_1\frac{\partial f_4}{\partial z_1}+z_2\frac{\partial f_4}{\partial z_2}\right]+\right. \tag{46}$$

$$\left.-p(d+1+pz_2^p)f_4+\left[2pz_2-(2+d-p)z_2^{1-p}\right]\frac{\partial f_4}{\partial z_2}-z_2^{2-p}\frac{\partial^2 f_4}{\partial z_2^2}\right\}z_2^d=0.$$

The function f_3 is analytical, hence the function f_4 is also analytical. Therefore we can write

$$f_4(z_1,z_2) = \sum_{k=0}^{\infty} c_k(z_1)z_2^k \, . \tag{47}$$

Now we shall find conditions, which the functions $c_k(z_1)$ must satisfy.

Substituting (47) into (45) and (46) we get

$$\sum_{k=0}^{\infty}\left\{z_2^{d+k}\left[p(d+1+k)c_k+pz_1c_k'+(2+d-p)z_1^{1-p}c_k'+z_1^{2-p}c_k''\right] + \right. \tag{48}$$

$$\left. - p^2\, z_2^{d+k+p}\, c_k\right\} = 0 \, .$$

$$\sum_{k=0}^{\infty}\left\{z_2^{d+k}(pz_1c_k' - pkc_k) + k(1+d-p+k)z_2^{d+k-p}\, c_k\right\} = 0 \, . \tag{49}$$

First we shall consider equation (49). This equation can be written in the following form

$$\sum_{k=0}^{p-1} k(1+d-p+k)z_2^{d+k-p}\, c_k + \tag{50}$$

$$+\sum_{k=0}^{\infty}\left[pz_1c_k' - pkc_k + (k+p)(1+d+k)c_{k+p}\right]z_2^{d+k} = 0 \, .$$

Equation (50) is satisfied for all $z_2 > 0$.Then the coefficients must be zero i.e.

$$pz_1c_k' - pkc_k + (k+p)(1+d+k)c_{k+p} = 0 \quad \text{for} \quad k \geqslant 0 \tag{51}$$

$$k(1+d-p+k)c_k = 0 \quad \text{for} \quad 0 \leqslant k \leqslant p-1 \, . \tag{51'}$$

Now we shall consider two cases:

case A ... $p-1-d$ is not a natural number

case B ... $p-1-d$ is a natural number.

In the case A in virtue of (51') it must be

$$c_k = 0 \quad \text{for} \quad 1 \leqslant k \leqslant p-1 \, ,$$

therefore in virtue of (51)

$$c_k = 0 \quad \text{if} \quad k \neq rp \,, \quad r = 0,1,2,\dots \tag{52}$$

Differentiating (51) we get

$$c'_{k+p} = \frac{1}{(k+p)(1+d+k)}\left[p(k-1)c'_k - pz_1 c''_k\right] . \tag{53}$$

Equation (48) can be written in the following form

$$\sum_{k=0}^{p-1}\left[p(d+1+k)c_k+pz_1c'_k+(2+d-p)z_1^{1-p}\; c'_k+z_1^{2-p}\; c''_k\right] z_2^{d+k}+ \tag{54}$$

$$+\sum_{k=0}^{\infty}\left[-p^2c_k+p(d+1+k+p)c_{k+p}+pz_1c'_{k+p}+(2+d-p)z_1^{1-p}\; c'_{k+p} + \right.$$

$$\left. + z_1^{2-p}\; c''_{k+p}\right] z_2^{d+k+p} = 0 .$$

Equation (54) is satisfied for all $z_2 > 0$, hence the coefficients must be zero, i.e.

$$-p^2c_k+p(d+1+k+p)c_{k+p}+pz_1c'_{k+p}+(2+d-p)z_1^{1-p}c'_{k+p}+z_1^{2-p}c''_{k+p}=0, \tag{55}$$

for $k \geqslant 0$

$$p(d+1+k)c_k+pz_1c'_k+(2+d-p)z_1^{1-p}c'_k+z_1^{2-p}c''_k = 0 \tag{55'}$$

for $0 \leqslant k \leqslant p-1$.

Let us multiply (55) by z_1^d. Then we can write this equation in the following form

$$(z_1^{2+d-p}\; c'_{k+p})' +p(1+d+k+p)z_1^d\; c_{k+p}+pz_1^{1+d}\; c'_{k+p}-p^2z_1^dc_k=0 . \tag{56}$$

Substituting (53) into (56) and taking into account (51) we get after some transformations

$$(z_1^{2+d-p}\, c'_{k+p})' - \frac{p^2}{(k+p)(1+d+k)}\,(z_1^{2+d}\, c'_k + pz^{1+d}\, c_k)' = 0\ . \quad (57)$$

Equation (57) is equivalent to the equation

$$z_1^{3+d-p}\, c''_k + \left[(1-k)z_1^{2+d-p} + pz_1^{2+d}\right] c'_k + p^2 z_1^{1+d} c_k = \text{const} = D_{1,k}\ . \quad (58)$$

In order to find the solutions of equation (58) let us substitute

$$c_k = u_k z_1^k\, e^{-z_1^p}\ , \quad (59)$$

then after transformations equations (58) can be written in the following form

$$z_1^{3+d-p+k}\, u''_k + z_1^{2+d-p+k}\, u'_k = D_{1,k}\, e^{z_1^p}\ . \quad (60)$$

We can treat u'_k as an unknown function. Equation (60) is then a linear equation (ordinary) of the first order with respect to u'_k. It is evident that the general solution of (60) is

$$u_k = \int \frac{1}{z_1}\left[D_{1,k} \int z_1^{p-k-d-2}\, e^{z_1^p}\, dz_1 + D_{2,k}\right] dz_1 + D_{3,k}\ .$$

In virtue of (59) we have

$$c_k = z_1^k\, e^{-z_1^p}\left\{\int \frac{1}{z_1}\left[D_{1,k}\int z_1^{p-k-d-2}\, e^{z_1^p}\, dz_1 + D_{2,k}\right] dz_1 + D_{3,k}\right\} \quad (61)$$

$$c_{k+p}=z_1^{k+p}\, e^{-z_1^p}\left\{\int\frac{1}{z_1}\left[D_{1,k+p}\int z_1^{-k-d-2}\, e^{z_1^p}dz_1+D_{2,k+p}\right]dz_1+ + D_{3,k+p}\right\}. \tag{61'}$$

Solutions (61), (61') depend on constants $D_{i,k}$ and respectively $D_{i,k+p}$. Now we are going to show that it must be

$$D_{1,k} = D_{2,k} = 0 \quad \text{for} \quad k = 0,p,2p,\ldots \tag{62}$$

$$\left[pz_1c_k' - pkc_k + (k+p)(1+d+k)c_{k+p}\right]e^{z_1^p} = \tag{63}$$

$$=pz_1\left\{\left[kz_1^{k-1}-pz_1^{p+k-1}\right]\left[\int\frac{1}{z_1}(D_{1,k}\int z_1^{p-k-d-2}\, e^{z_1^p}\, dz_1+ + D_{2,k})dz_1 + +D_{3,k}\right] + z_1^k\left[\frac{1}{z_1}(D_{1,k}\int z_1^{p-k-d-2}\, e^{z_1^p}\, dz_1+D_{2,k})\right]\right\} +$$

$$-pkz^k\left[\int\frac{1}{z_1}(D_{1,k}\int z_1^{p-k-d-2}\, e^{z_1^p}\, dz_1+D_{2,k})dz_1+D_{3,k}\right] +$$

$$+(k+p)(1+d+k)z_1^{k+p}\left[\int\frac{1}{z_1}(D_{1,k+p}\int z_1^{-k-d-2}\, e^{z_1^p}\, dz_1 + +D_{2,k+p})dz_1+D_{3,k+p}\right] = 0 .$$

Multipying (63) by $z_1^{-(k+p)}$ and differentiating we get

$$-p^2 \frac{1}{z_1}(D_{1,k} \int z_1^{p-k-d-2} e^{z_1^p} dz_1 + D_{2,k}) + \tag{64}$$

$$-p^2 z_1^{-p-1}(D_{1,k} z_1^{p-k-d-2} e^{z_1^p} dz_1 + D_{2,k}) +$$

$$+pz_1^{-p} D_{1,k} z_1^{p-k-d-2} e^{z_1^p} + (k+p)(1+d+k) \frac{1}{z_1}(D_{1,k+p} \int z_1^{-k-d-2} e^{z_1^p} dz_1 +$$

$$+D_{2,k+p}) = 0 .$$

Multiplying (64) by z and differentiating we get

$$-p^2 D_{1,k} z_1^{p-k-d-2} e^{z_1^p} + p^3 z_1^{-p-1}\left[D_{1,k}\int z_1^{p-k-d-2} e^{z_1^p} dz_1 + D_{2,k}\right] +$$

$$- p^2 z_1^{-p} D_{1,k} z_1^{p-k-d-2} e^{z_1^p} + \tag{65}$$

$$+\left[p^2 z_1^{p-k-d-2} - p(k+d+1) z_1^{-d-k-2}\right] d_{1,k} e^{z_1^p} +$$

$$+ (k+p)(1+d+k) D_{1,k+p} z_1^{-k-d-2} e^{z_1^p} = 0 .$$

Multiplying (65) by z_1^{p+1}, differentiating and next multiplying by $e^{z_1^p} z_1^{k+d+2-p}$ we have

$$p^3 D_{1,k} + \left[(k+p)(1+d+k)D_{1,k+p} - p(p+1+d+k)D_{1,k}\right]\left[p-k-d-1+pz_1^p\right] = 0. \tag{66}$$

Relation (66) must be satisfied for all $z_1 > 0$, therefore the coefficient of z^p and the constant term must be zero i.e.

$$(k+p)(1+d+k)D_{1,k+p} = p(p+1+d+k)D_{1,k} , \tag{67}$$

$$(k+p)(1+d+k-p)D_{1,k+p} = p(1+d+k)D_{1,k} . \tag{67'}$$

In virtue of (67) and (67') it is evident that it must be

$$D_{1,k} = D_{1,k+p} = 0 .$$

In a similar way it can be shown that condition (62) is also satisfied for $D_{2,k}$.

Now we are going to find $D_{3,k}$. It fallows from (62) and (63) that

$$-p^2 D_{3,k} + (k+p)(1+d+k)D_{3,k+p} = 0 ,$$

hence

$$D_{3,k} = \frac{p^k}{k!(1+d)\cdot\ldots\cdot[1+d+(k-1)p]} D_{3,0} = \tag{68}$$

$$= \frac{\Gamma(\frac{1+d}{p})}{k!\,\Gamma(\frac{1+d+kp}{p})} D_{3,0} .$$

Now in virtue of (40), (41), (47), (61) and (62) we get

$$\int_I f_2(z_1,z_2)dz_2 =$$

$$= D_{3,0}\,\Gamma(\frac{1+d}{p}) \sum_{k=0}^{\infty} \frac{z_1^{kp}\, e^{-z_1^p}}{k!\,\Gamma(\frac{1+d+kp}{p})} \int_I z_2^{d+kp}\, e^{z_2^p}\, dz_2 =$$

$$= \frac{1}{p} D_{3,0} \Gamma\left(\frac{1+d}{p}\right) \sum_{k=0}^{\infty} \frac{z_1^{kp}}{k!} e^{-z_1^p} = \frac{1}{p} D_{3,0} \Gamma\left(\frac{1+d}{p}\right) = 1$$

therefore

$$D_{3,0} = \frac{p}{\Gamma\left(\frac{1+d}{p}\right)} .$$

Taking into account (40), (41), (47), (61), (62),(68) and (69) we get (35).

Thus the unique solution in the class K of equations(42), (43) is (35).

Now we shall consider the case B i.e. the case: $p-1-d$ is a natural number. Let us denote $N = p-1-d$. In virtue of (51) it must be

$$c_k=0 \quad \text{for} \quad 1 < k < p-1 \quad \text{except perhaps} \quad k=N$$

therefore instead of (52) we get

$$c_k=0 \quad \text{if} \quad k \neq rp \quad \text{or if} \quad k \neq N+rp, \quad r=0,1,2,\dots$$

If we repeat all the previous reasoning, we get for $k = N+rp$ function c_k of shape (61). It is easy to verify that (62), (63) must be satisfied. Now we are going to show that

$$D_{3,k} = 0 \quad \text{for} \quad k = N,N+p,N+2p,\dots \tag{70}$$

In virtue of (62) and (63) we have

$$D_{3,pr-1-d} = \frac{p^{2(r-1)}}{p^{r-1}(r-1)!(2p-1-d)\cdot\ldots\cdot(rp-1-d)} D_{3,N} =$$

$$= \frac{\Gamma\left(1+\frac{N}{p}\right)}{(r-1)!\,\Gamma\left(r+\frac{N}{p}\right)} D_{3,N}, \quad r = 1,2,\dots$$

In the considered case there can exist two sequences of coefficients c_k, which must not be zero, namelly $c_0, c_p, c_{2p}, \dots$ and $c_N, c_{N+p}, c_{N+2p}, \dots$. Then reapeating all the previous reasoning we get

$D_{1,N+rp} = D_{2,N+rp} = 0$ for $r=0,1,2,\dots$ and then

$$\int_I f_2(z_1,z_2)dz_2 = \int_I \sum_{k=0}^{\infty}\Big[c_{kp}(z_1)z_2^{d+kp} + \tag{71}$$

$$+ \sum_{k=0}^{\infty} c_{N+kp}(z_1)z_2^{d+N+kp}\Big] e^{-z_2^p}\, dz_2 =$$

$$= \frac{1}{p}D_{3,0}\,\Gamma\left(\frac{1+d}{p}\right) + \frac{1}{p}\, D_{3,N}\,\Gamma\left(1 + \frac{N}{p}\right)e^{-z_1^p} \sum_{k=0}^{\infty} \frac{z_1^{N+kp}}{\Gamma\left(\frac{N+kp+p}{p}\right)} =$$

$$= \frac{1}{p}\, D_{3,0}\,\Gamma\left(\frac{1+d}{p}\right) + \frac{1}{p}\Gamma\left(1 + \frac{N}{p}\right)\, \varphi(z_1^p) =$$

$$= \frac{1}{p}\, D_{3,0}\,\Gamma\left(\frac{1+d}{p}\right) + \frac{1}{p}\Gamma\left(1 + \frac{N}{p}\right)\, \varphi(u) .$$

Function (71) must be equal 1, i.e. function $\varphi(z_1^p)$ cannot depend on z_1. In other words $\varphi(z_1^p)$ must be constant. Now we are going to show that (70) is a necessary condition that (71) be constant. Indeed, let us notice that

$$\varphi'(u) = \sum_{k=0}^{\infty} \frac{D_{3,N}}{\Gamma\left(\frac{N+kp+p}{p}\right)} \left[(k + \frac{N}{p})u^{k+\frac{N}{p}-1} - u^{k+\frac{N}{p}}\right]e^{-u} =$$

$$= \frac{N}{p}\, D_{3,N}\, \frac{1}{\Gamma(1 + \frac{N}{p})}\, u^{-\frac{1+d}{p}}\, e^{-u} + \sum_{k=1}^{\infty} D_{3,N}\, \frac{k + \frac{N}{p}}{\Gamma(k + \frac{N}{p} + 1)}\, u^{k-1+\frac{N}{p}}\, e^{-u} +$$

$$-\sum_{k=0}^{\infty} D_{3,N} \frac{1}{\Gamma(k+\frac{N}{p}+1)} u^{k+\frac{N}{p}} e^{-u} = \frac{N}{p} D_{3,N} \frac{1}{\Gamma(1+\frac{N}{p})} u^{-\frac{1+d}{p}} e^{-u} .$$

Then the necessary condition that $\varphi(u) = 0$ is $N = 0$ or $D_{3,N} = 0$. The relation $N=0$ is contradictory with assumption B that N is a natural number. Therefore $D_{3,N} = 0$ and this means that condition (70) is satisfied.

Finally (35) is a unique solution of equations (5), (6) in both considered cases A and B.

Let us notice that if $P[Y_{t_1} = 0] = 1$ then

$$\frac{\partial}{\partial y_2} P\left[Y_{t_2} < y_2 \middle| Y_{t_1} = 0\right] = \frac{\partial}{\partial y_2} P\left[Y_{t_2} < y_2\right] =$$

$$= \frac{p}{\Gamma(\frac{d+1}{p})(p\Delta t)^{1/p}} \left[\frac{y_2}{(p\Delta t)^{1/p}}\right]^d e^{-\frac{y_2^p}{p\Delta t}}$$

and we get general gamma distribution.

In virtue of part II we can formulate the following statement:

If conditions (26), (27), (38), (38') are satisfied then the unique transition probability density f in the class K is given by (35) and f is the transition probability density in the Markov process in the wide sense.

IV. The correction concerning paper [8].

In paper [8] on pages 13, 14 examples of solutions of some partial differential equations were given. These examples should not be considered there [8], as the solutions do not satisfy the assumptions of theorems given in that paper.

This fact was observed by Prof.K. U r b a n i k.

The correct discussion of these examples for $I = (0,+\infty)$ is given in the present paper.

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Address of Author: dr Agnieszka Plucińska, Warszawa, ul. Polna 54 m 14.