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# A GENERALIZED NON-LINEAR PROBLEM OF HILBERT TYPE IN THE CLASS OF DISCONTINUOUS FUNCTIONS

## I. INTRODUCTION

Let $D^+$ denote a bounded domain in the complex plane $\Pi$, whose boundary is a Jordan-Lapunov curve. Let us denote by $C_1, C_2, \ldots, C_r$ $(r \geqslant 2)$ some points of the curve enumerated in the positive direction along the curve. Thus we have $L = \widehat{C_1C_2} + \widehat{C_2C_3} + \ldots + \widehat{C_rC_1}$ where the arcs $\widehat{C_\delta C_{\delta'}}$, $(\delta = 1,\ldots,r,\ \delta' = \delta + 1)$ are the Lapunov arcs ordered from $C_\delta$ to $C_{\delta'}$. The complement of the set $D^+ + L$ to $\Pi$ is denoted by $D^-$. Let us suppose that the origin of the coordinates is in the domain $D^+$.

## II. FORMULATION OF THE PROBLEM

The generalized mixed problem of Hilbert type consists in finding a generalized analytic function $w(z)$ fulfilling in $D^+$ the equation:

$$w(z) + A(z)\,w(z) + B(z)\,\overline{w(z)} = f(z) \tag{1}$$

and an analytic function $\Phi(z)$ in $D^-$ vanishing at infinity, whose boundary values satisfy at every point $t \in L_* = L - \sum_{k=1}^{r} C_k$ the non-linear boundary condition:

$$a_0(t)\, w^+(t) + \int_L A_0(t,\tau)\, w^+(\tau)\, d\tau =$$

$$= \sum_{k=0}^{n} \left[ b_k(t) \frac{d^k \Phi^-(t)}{dt^k} + B_k(t,\tau) \frac{d^k \Phi^-(\tau)}{d\tau^k} \right] +$$

$$+ g(t) + F\left[ t, w^+(t), \Phi^-(t), \frac{d\Phi^-(t)}{dt}, \dots, \frac{d^n \Phi^-(t)}{dt^n} \right] \qquad (2)$$

where $A(z)$, $B(z)$, $a_0(t)$, $A_0(t,\tau)$, $b_k(t)$, $B_k(t,\tau)$ $(k = 0,1,\dots,n)$ $g(t)$, $F(t,u,u_0,u_1,\dots,u_n)$ are given functions.

Moreover we require that the n-th derivatives of the function $\Phi(z)$ and $w(z)$ in a sufficiently small neighbourhood of the points $C_1,\dots,C_r$ satisfy for every $z \in D^+$, $z \in D^-$ the following inequalities:

$$\left| \frac{d^n \Phi^-(z)}{dz^n} \right| < \frac{\text{const}}{|z - C_\nu|^{\theta_0}}$$

$\nu = 1,2,\dots,r$; $\theta_0$ - a positive constant smaller than unity

$$|w^+(z)| < \frac{\text{const}}{|z - C_\nu|^{\theta_0}} .$$

The mixed problem of the Hilbert type was solved in the class of continuous functions by J.Wolska-Bochenek [6].

We make the following assumptions.

Assumptions.

$1^0$. The curve $L$ is a closed Jordan contour, with continuously varying tangent which forms with a constant direction an angle satisfying the Hölders condition with exponent $\alpha$ $(0 < \alpha \leqslant 1)$;

$2^0$. The complex functions $A(z)$, $B(z)$, $f(z)$ defined for every $z \in D^+ + L$, belong to the class $L_{p,2}$ $(p > 2)$ and $A(z) \equiv B(z) \equiv 0$ for $z \in D^-$.

$3^0$. The functions of a complex variable $a_0(t)$, $b_k(t)$ $(k = 0,1,\dots,n)$ are defined for every $t \in L$ and satisfy the inequalities:

$$|a_o(t) - a_o(t_1)| < k_a|t - t_1|^\beta$$
$$(3)$$
$$|b_k(t) - b_k(t_1)| < k_b|t - t_1|^\beta \quad (k=0,1,\ldots,n) , \qquad (0 < \beta \leqslant \alpha)$$

where $k_a$, $k_b$ are positive constants, moreover $a_o(t) \neq 0$, $b_n(t) \neq 0$ for every $t \in L$.

$4^o$. The functions $A_o(t,\tau)$, $B_k(t,\tau)$, $(k = 0,1,\ldots,n)$ of a complex variable are of the form

$$A_o(t,\tau) = \frac{A_o^o(t,\tau)}{|t - \tau|^\lambda} \qquad B_k(t,\tau) = \frac{B_k^o(t,\tau)}{|t - \tau|^\lambda} \qquad 0 \leqslant \lambda < 1 \qquad (4)$$

where the complex functions $A_o^o(t,\tau)$, $B_k(t,\tau)$ $(k=0,1,\ldots,n)$ are defined for $t \in L$, $\tau \in L$ and satisfy the conditions:

$$|A_o^o(t,\tau) - A_o^o(t_1, \tau_1)| < k_A\left[|t - t_1|^\beta + |\tau - \tau_1|^\beta\right]$$
$$(5)$$
$$|B_k^o(t,\tau) - B_k^o(t_1, \tau_1)| < k_B\left[|t - t_1|^\beta + |\tau - \tau_1|^\beta\right]$$

$k_A$, $k_B$ - positive constants.

$5^o$. The complex function $g(t)$ satisfies the inequalities

$$|g(t)| < \frac{M_g}{\prod_{\nu=1}^{r} |t - C'_\nu|^\alpha} ; \; |g(t) - g(t_1)| < \frac{k_g|t - t_1|^\beta}{W(t_1,t)} \qquad (6)$$

$$W(t,t_1) = \left[|t - C_\nu|\,|t_1 - C_{\nu'}|\right]^{\alpha+\beta}$$

for every $t \in L_*$, $t$, $t_1 \in \widehat{C_\delta\, C_{\delta'}} \in L$, $t_1 \in \widehat{tC_{\delta'}}$, $0 < \alpha < 1$, $\alpha + \beta < 1$, $k_g$, $M_g$ - positive constants.

$6^o$. The complex function $F(t,u,u_o,u_1,\ldots,u_n)$ defined for $t \in L_*$, $u \in \Pi$, $u_k \in \Pi$ $(k = 0,1,\ldots,n)$ satisfies the following inequalities

$$|F(t,u,u_0,\dots,u_n)| < \frac{M_F}{\prod_{\nu=1}^{r}|t - C_\nu|^\alpha} + M_F'\Big[|u| + \sum_{k=1}^{n}|u_k|\Big]$$

$$|F(t,u,u_0,\dots,u_n) - F(t_1',u',u_0',\dots,u_n')| < \frac{k_F\,|t - t_1'|^\beta}{W(t,t_1')} +$$
$$+ k_F\Big[\sum_{k=0}^{n-1}|u_k - u_k'|^\beta + |u - u'| + |u_n - u_n'|\Big] \qquad (7)$$

$$t, t_1' \in \widehat{C_\delta C_{\delta'}} \in L \quad \text{where} \quad t_1' \in \widehat{tC_{\delta'}} \,,$$

$k_F$, $M_F$, $M_F'$ - some positive constants.

## III. THE INTEGRAL REPRESENTATION OF THE FUNCTION w(z)

According to the theory of I.N.Vekua [5] the solution of the equation (1) with condition $2^o$ is of the form of the sum $w(z) = W(z) + w_*(z)$, where $W(z)$ and $w_*(z)$ are respectively the general and particular solutions of equation (1). Let us take the function $W(z)$ in the form of the generalized Cauchy type integral with unknown density $\mu(t)$

$$W(z) = \frac{1}{2\pi i}\int_L \Omega_1(z,\tau)\,\mu(\tau)\,d\tau - \frac{1}{2\pi i}\int_L \Omega_2(z,\tau)\,\overline{\mu(\tau)}\,\overline{d\tau} \qquad (8)$$

where $\Omega_1(z,\tau)$, $\Omega_2(z,\tau)$ are basic kernels normed in respect to the domain $D^+$. The particular solution of (1) can be presented in the form

$$W_*(z) = -\frac{1}{\pi}\iint_{D^+}\Omega_1(z,\zeta)\,f(\zeta)\,d\xi\,d\eta - \frac{1}{\pi}\iint_{D^+}\Omega_2(z,\zeta)\,\overline{f(\zeta)}\,d\xi\,d\eta \qquad (8')$$

The complex function $\mu(t)$, $t \in L_*$ belongs to the class $\hbar_\alpha$ in respect to the points $C_1, C_2, \dots, C_r$. The function $W(z)$ has the limiting value

$$\lim_{\substack{z\to t\in L_*\\ z\in D^+}} W(z) = W^+(t) = \frac{1}{2}\mu(t) + W(t) \tag{9}$$

where $W(t)$ exists in the Cauchy sense. The function $w_*(z)$ according to its representation (8') is continuous in the whole plane

$$\lim_{z\to t\in L_*} w_*(z) = w_*(t) . \tag{10}$$

## IV. THE INTEGRAL REPRESENTATION $\Phi(z)$

We represent the function $\Phi(z)$ in the form proposed by Y.M.Krikunov [1], in order to solve the Hilbert type problem with the boundary condition containing derivatives:

$$\Phi(z) = \frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L \frac{\mu(\tau)}{\tau^n}\Big[(\tau - z)^{n-1}\ln\frac{z-\tau}{z} +$$

$$+ \sum_{k=0}^{n} \beta_k \tau^{n-k-1} z^k\Big] d\tau . \tag{11}$$

By $\ln\frac{z-\tau}{z}$ we understand this branch of logarithm which disappears for $z = \infty$.

It is easy to show that the same formula is valid if the unknown function $\mu(t)$ is of the class $h_\alpha$. This results from the fundamental theorem of W.Pogorzelski [3]. The n-th derivative of the function (11) is a Cauchy type integral that in the points $L_*$ possesses the limiting value defined by the formula:

$$\lim_{\substack{z\to t\in L_*\\ z\in D^-}} \frac{d^n\Phi(z)}{dz^n} = -\frac{1}{2}\mu(t)\cdot\frac{1}{t^n} + \frac{1}{2\pi i}\int_L \frac{\mu(\tau)\,d\tau}{(\tau - t)t^n} . \tag{12}$$

## V. THE SOLUTION OF THE PROBLEM

Assuming that the boundary values of the functions $w(z)$, $\Phi(z)$ and its derivatives up to the n-th order satisfy at every point $t \neq C_\nu$ $(\nu = 1,2,\dots,r)$ of the curve $L_*$ the boundary condition (2), we obtain a strongly singular integral equation with an unknown complex function $\mu(t)$ of the class $\bar{H}_\alpha$

$$a(t)\,\mu(t) + \frac{b(t)}{\pi i}\int_L \frac{\mu(\tau)d\tau}{\tau - t} + \int_L K_1(t,\tau)\,\mu(\tau)\,d\tau +$$

$$+ \int K_2(t,\tau)\,\overline{\mu(\tau)}\,\overline{d\tau} = \widetilde{g(t)} + F\left[t,U(t),U_0(t),\dots,U_n(t)\right] , \quad (13)$$

where

$$a(t) = \frac{1}{2}\left[a_0(t) + \frac{b_n(t)}{t^n}\right] \quad b(t) = \frac{1}{2}\left[a_0(t) - \frac{b_n(t)}{t^n}\right] , \quad (14)$$

$$U(t) = \frac{1}{2\pi i}\int_L \Omega_1(t,\tau)\,\mu(\tau)d\tau - \frac{1}{2\pi i}\int_L \Omega_2(t,\tau)\,\overline{\mu(\tau)}\,\overline{d\tau} \quad (15)$$

$$U_0(t) = \frac{(-1)^n}{(n-1)!}\,\frac{1}{2\pi i}\int_L \frac{\mu(\tau)}{\tau^n}\left[(\tau - t)^{n-1}\ln\frac{t-\tau}{t} +\right.$$

$$\left. + \sum_{k=0}^{n} \beta_k\,\tau^{n-k-1}\,t^k\right]d\tau \quad (16)$$

$$U_l(t) = \frac{(-1)^{n+1}}{(n-l-1)!}\,\frac{1}{2\pi i}\int_L \frac{\mu(\tau)}{\tau^n}\left\{(\tau - t)^{n-l-1}\left[\ln\frac{\tau - t}{t} +\right.\right.$$

$$+ \left(\frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{n-l}\right)\frac{\tau}{t} +$$

$$+ \frac{(\tau - t)^{n-l}}{n-l}\left(\frac{l-1}{n-1} + \frac{l-2}{n-2} + \dots + \frac{1}{n-l-1}\right)\frac{\tau}{t^2} +$$

$$+\frac{(\tau-t)^{n-l-1}}{(n-l-1)(n-l-2)}\left(\frac{l-1}{n-1}+\frac{l-2}{n-2}+\dots+\frac{1}{n-l-2}\right)\frac{(l-2)!}{t^{l-2}}+\dots+ \tag{17}$$

$$+\frac{(\tau-t)^{n-2}}{(n-1)(n-2)\dots(n-l)}\frac{(l-1)!}{t^{l}}\tau+$$

$$+\sum_{k=1}^{n-2}k(k-1)\dots(k-l+1)\beta_k\tau^{n-k-1}t^{k-1}\Big]d\tau\ , \qquad (l=1,2,\dots,n-1)$$

$$U_n(t)=\frac{t^{-n}}{2\Pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t} \tag{18}$$

$$\widetilde{g(t)}=g(t)+\frac{a_o(t)}{\Pi}\Big[\iint_{D^+}\Omega_1(t,\zeta)\ f(\zeta)\ d\xi\ d\eta+$$

$$+\iint_{D^+}\Omega_2(t,\zeta)\ \overline{f(\zeta)}d\xi\ d\eta\Big]-\frac{1}{\Pi}\int_L A_o(t,\tau)\Big[\iint_{D^+}\Omega_1(\tau,\zeta)f(\zeta)\ d\xi\ d\eta+$$

$$+\iint_{D^+}\Omega_2(\tau,\zeta)\overline{f(\zeta)}\ d\xi\ d\eta\Big]d\tau\ . \tag{19}$$

The kernels $K_1(t,\tau)$ and $K_2(t,\tau)$ are completely defined by the given function and are of the form

$$K_j(t,\tau)=\frac{K_j^o(t,\tau)}{|t-\tau|^{\lambda_1}}\qquad (j=1,2)\qquad \lambda_1=\max\left(\lambda,\frac{2}{p}\right) \tag{20}$$

where the functions $K_j^o(t,\tau)$ satisfy, with regard to the both variables, the Hölder condition with the exponent $\delta^*=\min\left(\beta,\lambda,\frac{2}{p}\right)$.

According to the assumption $3^o$, the equation (13) is of normal type. Let us consider the characteristic equation of (13) with the right-hand side as follows:

$$a(t)\ \mu(t)+\frac{b(t)}{\Pi i}\int_L\frac{\mu(\tau)\ d\tau}{\tau-t}=\psi(t)\ ,\qquad \psi(t)\in H_\alpha \tag{21}$$

The index of the Hilbert problem corresponding to the equation (21) has the form

$$\varkappa = \text{ind } G(t) = \text{ind } \frac{a(t) - b(t)}{a(t) + b(t)} = \text{ind } t^{-n} \frac{b_n(t)}{a_o(t)} = \varkappa_1 - n \quad (22)$$

$$\varkappa_1 = \text{ind } b_n(t) - \text{ind } a_o(t) \quad (23)$$

and is called the index of the mixed problem. The solution of equation (21) in the case $\varkappa \geqslant 0$ is given by the formula

$$\mu(t) = a^*(t)\,\psi(t) - \frac{b^*(t)Z(t)}{\Pi i} \int_L \frac{\psi(\tau)\,d\tau}{Z(\tau)(\tau - t)} + b^*(t)Z(t)P_{\varkappa-1}(t) \quad (24)$$

$$a^*(t) = \frac{a(t)}{a^2(t) - b^2(t)} \quad (25)$$

$$b^*(t) = \frac{b(t)}{a^2(t) - b^2(t)} \quad (26)$$

$$Z(t) = \left[a(t) + b(t)\right] X^+(t) = \left[a(t) - b(t)\right] X^-(t) \quad (27)$$

and $P_{\varkappa-1}(t)$ - an arbitrary polynomial of the degree $\varkappa-1$ with complex coefficients and $P_{\varkappa-1} = 0$ for $\varkappa = 0$.

In view of equation (24) we conclude that the function $\mu(t) \in \tilde{H}_\alpha$ satisfying equation (13) satisfy the equation

$$\hat{N}\mu = \mu(t) + \int N_1(t,\tau)\,\mu(\tau)\,d\tau + \int_L N_2(t,\tau)\,\overline{\mu(\tau)}\,\overline{d\tau} =$$

$$= F^*\left[t, U(t), U_o(t), \ldots, U_n(t)\right] \quad (28)$$

where

$$N_1(t,\tau) = a^*(t)K_2(t,\tau) - \frac{b^*(t)Z(t)}{\Pi i} \int_L \frac{K_1(t_1,\tau)}{Z(t_1)(t_1 - t)}\,dt_1 \quad (29)$$

$$N_2(t,\tau) = a^*(t)K_2(t,\tau) + \frac{b^*(t)Z(t)}{\Pi . i} \int_L \frac{K_2(t_1,\tau)}{Z(t_1)(t_i-t)} dt_1 \quad (30)$$

$$F^*[t,U(t),U_o(t),\dots,U_n(t)] = K^*\left[\widetilde{g(t)}+F[t,U(t),U_o(t),\dots,U_n(t)]\right] +$$

$$+ b^*(t)\ Z(t)\ P_{\varkappa-1}(t) \quad (31)$$

$$K^*[s(t)] = a^*(t)s(t) - \frac{b^*(t)Z(t)}{\Pi i} \int_L \frac{s(\tau)d\tau}{Z(\tau)(\tau - t)} \quad (32)$$

The kernels $N_j(t,\tau)$ $(j = 1,2)$ are of the form

$$N_j(t,\tau) = \frac{N_j^o(t,\tau)}{|t - \tau|^{1-\sigma}} \qquad (j = 1,2) \quad \sigma = \min(\sigma^*,\ 1 - \sigma^*) \quad (33)$$

where the functions $N_j^o(t,\tau)$ satisfy, with respect to both variables, the Hölder condition with the exponent

$$\sigma(1 - \theta)\ , \qquad 0 < \theta < 1 \quad (34)$$

The non-linear integral equation (28) will be solved by the Schauder fixed point method. Let us consider a functional space $\Lambda$, whose elements are the complex functions $\mu(t)$ defined and continuous in the domain $L_* = L - \sum_{k=1}^{r} C_k$ and satisfying the condition:

$$\sup_{t\in L_*} \prod_{\nu=1}^{r} |t - C_\nu|^{\alpha+h} |\mu(t)| < \infty \quad (35)$$

$\alpha$, h - being the fixed parameters of the class $\hbar$.

The sum of two points of the space and the product of a point by a number is defined in a known way. The distance of two points of $\Lambda$ is defined as the norm of their difference

$$\tilde{\varrho}(\mu_1,\ \mu_2) = \|\mu_1 - \mu_2\|\ , \quad (36)$$

where by the norm of the point $\mu(t)$ we understand

$$\|\mu\| = \sup_{t\in L_*}\Big[\prod_{\nu=1}^{r}|t - c_\nu|^{\alpha+h}\,|\mu(t)|\Big]\,. \tag{37}$$

The so defined space is a Banach space. Let us consider in $\Lambda$ a set E of all points $\mu$ that satisfy the inequalities:

$$\prod_{\nu=1}^{r}|t - c_\nu|^{\alpha+h}|\mu(t)| \leqslant \varrho \tag{38}$$

$$\Big[|t - c_\nu|\,|t_1 - c_{\nu+1}|\Big]^{\alpha+h}|\mu(t) - \mu(t_1)| \leqslant \omega |t-t_1|^h. \tag{38'}$$

The constant $\varrho$ and $\omega$ are arbitraryly positive reals; but $\alpha$ and h satisfy the relations:

$$h = \min\left(\beta,\ \theta\beta,\ (1-\theta)\delta,\ \frac{p-2}{p}\right), \qquad 0 < \alpha+h < 1$$

$t_1$, $t_2$ being arbitrary points placed on the same arc $\widehat{c_r c_{r+1}}$ between the successive points of discontinuity such that $t_1 \in \widehat{tc_{r+1}}$. The set E is closed and convex (see Żakowski [7]). Let us transform each element of the set E by means of the transformation:

$$\widehat{\mu(t)} + \int_L N_1(t,\tau)\,\widehat{\mu(\tau)}\,d\tau + \int_L N_2(t,\tau)\,\overline{\widehat{\mu(\tau)}}\,\overline{d\tau} = F^*\big[t, U(t), \dots, U_n(t)\big]\,, \tag{39}$$

according to which to every element of corresponds the exactly defined element of the set E'. Now, we are going to find the conditions for E' to be a subset of E.

L e m m a 1. The function $F\big[t, U(t), U_0(t) \dots, U_n(t)\big]$ satisfies the inequalities

$$\left|F\left[t,U(t),U_0(t),\ldots,U_n(t)\right]\right| \quad \frac{M_F + M_F'(A_1\varrho + A_2\omega)}{\prod_{\nu=1}^{r}\left|t - C_\nu\right|^\alpha} \tag{40}$$

$$\left|F\left[t,U(t),\ldots,U_n(t)\right] - F\left[t_1,U(t_1),\ldots,U_n(t_1)\right]\right| < \frac{K_F(B_1\varrho + B_2\omega + B_3)}{W(t,t_1)} \tag{40'}$$

where the constants $A_1, B_1$ depend on the curve L, on the norms of the functions $A(z)$, $B(z)$ and on the number $n$; the constants $A_2, B_2, B_3$ depend only on the curve and the exponent $h$. The form of the inequality (40) results from the assumption $6^o$ and the properties of the Cauchy type integrals $U(t)$, $U_n(t)$ (formulae (15), (18)).

L e m m a 2. The function $\widetilde{g(t)}$ is of the class $\hbar_\alpha$ i.e. it satisfies the inequalities

$$\left|\widetilde{g(t)}\right| < \frac{M_{\tilde g}}{\prod_{\nu=1}^{r}\left|t - C_\nu\right|^\alpha}$$

$$\left|\widetilde{g(t)} - \widetilde{g(t_1)}\right| < \frac{k_{\tilde g}\left|t - t_1\right|^h}{W(t_1,t_2)} \tag{41}$$

This follows from the assumption $5^o$ and the representation (19).

L e m m a 3. The function $F^*\left[t,U(t),U_0(t),\ldots,U_n(t)\right]$ defined by formulae (32) is of the class $\hbar_\alpha$.

P r o o f. According to the form (32) of the function $F^*$ and in view of the inequalites (40), (40') we apply the principal theorem of the function of the class $\hbar_\alpha$, and obtain the following evaluation for $F^*$

$$\left|F^*\left[t,U(t),U_0(t),\ldots,U_n(t)\right]\right| < \frac{M_{F^*}}{\prod_{\nu=1}^{r}\left|t - C_\nu\right|^\alpha} \tag{42}$$

$$\left|F\left[t,U(t),\dots,U_n(t)\right]-F^*\left[t_1,U(t_1),\dots,U_n(t_1)\right]\right|<\frac{k_{F^*}\left|t-t_1\right|^h}{W(t,t_1)}, \tag{42}$$

where

$$\begin{aligned} M_{F^*} &= p_1 M_F + (p_2\varrho + p_3\omega)\, M_F' + (p_4\varrho + p_5\omega)k_F + p_6 \\ k_{F^*} &= p_1' M_F + (p_2'\varrho + p_3'\omega) M_F' + (p_4'\varrho + p_5'\omega)k_F + p_6' \end{aligned} \tag{42'}$$

and where the constants $p_i$, $p_i'$ $(i=1,\dots,6)$ depend on the given functions and on the curve L, but they are independent on $\mu(t)$, q.e.d.

L e m m a 4. The set $E'$ containing the points transformed $\hat{\mu}$ by means of the transformation (39) is a subset of the set E if the constants of the problem satisfy the system of inequalities:

$$(1 + M_{\gamma_1} + M_{\gamma_2})\, M_{F^*} \leqslant \varrho \qquad k_{F^*} + M_{F^*}\,(k_{\gamma_1} + k_{\gamma_2}) \leqslant \omega \tag{43}$$

where $M_{\gamma_1} = \int_L \left|\gamma_1(t,\tau)\right| d\tau$, $\quad M_\gamma = \int_L \left|\gamma_2(t,\tau)\right| d\tau$,

$$k_{\gamma_i} = \sup_{t,t_1\in L_*} \frac{\int_L \left|\gamma_i(t,\tau) - \gamma_i(t_1,\tau)\right| d\tau}{\left|t-t_1\right|^h}, \qquad (i = 1,2)$$

P r o o f. In view of the lemma 2 and the results of G.F.Mandżawidze [2], the solution of equation (39) can be given in the form

$$\mu(t) = F^*\left[t,U(t),\dots,U_n(t)\right] + \int_L \gamma_1(t,\tau)\, F^*\left[\tau,U(\tau),\dots,U_n(\tau)\right] d\tau + \int_L \gamma_2(t,\tau)\, \overline{F^*\left[\tau,U(\tau),\dots,U_n(\tau)\right]}\, \overline{d\tau} \tag{44}$$

where $F^*$ is the function of the class $\mathfrak{h}_\alpha$ and the homogeneous equation $\hat{N}\mu = 0$ has a zero solution only. On the basis of the inequalities (38), (42), we obtain the inequalities (43) and substituting formulae (42') into (43) we obtain the condition

$$M'_F + k_F < \frac{1}{P(1+M_{\tau_1}+M_{\tau_2}+k_{\tau_1}+k_{\tau_2})}, \quad P=\max(p_i+p'_i), \quad (i=1,\dots,5). \quad (45)$$

Since the choice of the constants $\varrho$, $\omega$ is arbitrary, it is evident that conditions (43) are always satisfied, provided that the constants $M'_F$, $k_F$ are sufficiently small and satisfy the inequality (45).

L e m m a 5. The transformation (39) is continuous in the space $\Lambda$.

P r o o f. The proof of this lemma follows from the property (42) and from the properties of the singular integrals appearing in equation (44), considered by W. Pogorzelski [4].

Thus all the conditions of the theorem of Schauder are satisfied and we can infer that in the set E there is at least one point $\mu^*$ fixed with respect to the transformation (39), provided that inequality (45) is satisfied. This implies that the function $\mu^*(t) \in \mathfrak{h}_\alpha$ satisfies equation (15) and can be chosen as an density in the integral representations (8), (13). Hence we conclude that there exists at least one solution of the boundary problem (1), (2).

T h e o r e m. Assume that the given functions $A(z)$, $B(z)$, $a_0(t)$, $A_0(t,\tau)$, $b_k(t)$, $B_k(t,\tau)$ $(k = 1,\dots,n)$, $g(t)$, $F[t,U(t),\dots,U_n(t)]$ satisfy conditions $2^o$ – $6^o$. Assume further that the contour L satisfies supposition $1^o$, the constants $M'_F$, $k'_F$ of the problem (1), (2) satisfy inequality (45), and the index of equation (25) is non-negative. Let the homogeneous equation $\hat{N}\mu = 0$ have only zero solution. Then there exist a function $\Phi(z)$ holomorphic in the domain $D^-$ and zero at infinity, and the function $w(z)$ satisfying in $D^+$ equation

(1), such that the boundary values of $\Phi(z)$ and $w(z)$ satisfy on $L_*$ the condition (2). All such functions are defined by formulae (8), (13), where $\mu(t)$ is a solution of integral equation (15) and belongs to the class $\mathfrak{h}_\alpha$.

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