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CONTINUITY OF TANGENTIAL DERIVATIVES OF A THERMAL POTENTIAL

In [3] we proved that if the density $\varphi(Q,\tau)$ of the thermal potential of double surface distribution is determined on a closed Lapunov surface S and satisfies Hölder's condition

$$|\varphi(Q,\tau) - \varphi(\bar{Q},\bar{\tau})| \leqslant k_\varphi(|Q\bar{Q}|^{h_\varphi} + |\tau-\bar{\tau}|^{\tilde{h}_\varphi}) \qquad (*)$$

$(k_\varphi > 0;\ h_\varphi \in (0,1];\ \tilde{h}_\varphi \in (0,1))$, and if $\{s_P\}$ is a tangent vector field given on S and fulfilling the condition[1]

$$(s_P, s_{\bar{P}}) \leqslant C_s |P\bar{P}|^{h_s} \qquad (**)$$

with $C_s > 0;\ h_s \in (0,1)$, then the values of tangential derivatives of the afore-said potential on the surface S satisfy Hölder's condition with respect to both spatial and time variables with the exponents h^* and $\tilde{h}^*$ respectively, and the coefficient k^*, subject to the following relations

$$h^* < h_\varphi;\quad \tilde{h}^* < \tilde{h}_\varphi;\quad k^* = C^* t^{-\beta} k_\varphi \qquad (***)$$

$(C^* > 0;\ \beta \in (\frac{1}{2}, 1))$.

By basing on that result, a non-linear tangential boundary problem was examined in [5], whence it turned out that relations (***) involved some additional, rather odd, conditions

[1] In (**) above P and $\bar{P}$ are arbitrary points on S, and $(s_P, s_{\bar{P}})$ denotes the angle formed by the vectors s_P and $s_{\bar{P}}$.

concerning functions given in the boundary problem (see [5], assumptions (12) and (13)), and as such should be modified, if possible.

The aim of this paper is a modification of the results of paper [3]. We shall prove that if the surface S and the vector field $\{s_P\}$ satisfy some conditions more restrictive than those in [3], then relations (***) can be transformed to the following form

$$h^* = h_\varphi; \quad \tilde{h}^* < \tilde{h}_\varphi; \quad k_* = C^* k_\varphi \tag{****}$$

which evidently enables a reduction of the afore-mentioned assumptions in [5].

The conditions concerning S and $\{s_P\}$, as well as the appropriate assumptions for $\varphi(Q,\tau)$, will be specified in the sequel.

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Let t denote a variable in a finite interval $(0,T)$, and let $A(x_1,\dots,x_n)$ be a variable point of an n-dimensional domain Ω placed in the space E_n $(n \geqslant 2)$ and bounded by a closed Lapunov surface S.

Consider the thermal potential of double surface distribution of the form

$$V(A,t) = \int_0^t\int_S (t-\tau)^{-\frac{n}{2}-1}|AQ|\cos(\overline{QA},n_Q)\exp\left[-\frac{|AQ|^2}{4(t-\tau)}\right]\varphi(Q,\tau)\,dQ\,d\tau \tag{1}$$

where n_Q denotes the inward normal to S at the point Q, and $|AQ|$ is the euclidean distance of A and Q.

The following theorem is valid.

T h e o r e m 1. Assumptions:

1°. Function $\varphi(Q,\tau)$ satisfies

$$|\varphi(Q,\tau)| \leqslant M_\varphi \tau^{-\mu_\varphi} \tag{2}$$

$$|\varphi(Q,\tau)-\varphi(\bar{Q},\bar{\tau})| \leqslant k_\varphi \tau^{-\mu_\varphi}(|Q\bar{Q}|^{h_\varphi}+|\tau-\bar{\tau}|^{\tilde{h}_\varphi}) \tag{3}$$

where

$$0<\tau \leqslant \bar{\tau} < T;\quad M_\varphi > 0;\quad k_\varphi > 0;\ \mu_\varphi \in [0,1);\ h_\varphi \in (0,1);$$

$$\tilde{h}_\varphi \in (\tfrac{1}{2},1).$$

2^o. For each point $Q \in S$ and each vector s_Q tangent to S at Q there exists a tangential derivative $\frac{d}{ds_Q}\varphi(Q,\tau)$ and

$$\left|\frac{d\varphi}{ds_Q}(Q,\tau)\right| \leqslant M'_\varphi \tau^{-\mu'_\varphi} \tag{4}$$

holds, where $M'_\varphi > 0$; $\mu'_\varphi \in [0,1)$. If, moreover, vectors s_P and s_Q belong to a continuous tangent vector field then the Hölder inequality

$$\left|\frac{d}{ds_Q}\varphi(Q,\tau) - \frac{d}{ds_{\bar{Q}}}\varphi(\bar{Q},\tau)\right| \leqslant k'_\varphi \tau^{-\mu'_\varphi}|Q\bar{Q}|^{h'_\varphi} \tag{5}$$

is valid, with $k'_\varphi > 0$ and $h'_\varphi \in (0,1]$.

T h e s i s: For each point P on S and each vector s_P tangent to S at P, the equality

$$\lim_{A\to P} \frac{d}{ds_P} V(A,t) = (2\sqrt{\pi})^n \frac{d}{ds_P}\varphi(P,t) + \frac{d}{ds_P}V(P,t) \tag{6}$$

is satisfied, where

$$\frac{d}{ds_P}V(P,t) = \int_0^t\int_S \frac{d}{ds_P}\left\{(t-\tau)^{-\frac{n}{2}-1}|PQ|\cos(\overline{QP},n_Q)\exp\left[-\frac{|PQ|^2}{4(t-\tau)}\right]\right\}\cdot \tag{7}$$

$$\cdot[\varphi(Q,\tau)-\varphi(Q,t)]\,dQd\tau + 2^n\Gamma(\tfrac{n}{2})\int \frac{d}{ds_P}\left\{|PQ|^{1-n}\cos(\overline{QP},n_Q)\right\}[\varphi(Q,t)+$$

$$-\varphi(P,t)]dQ +$$

$$-2^n\int_S \frac{d}{ds_P}\Big\{|PQ|^{1-n}\cos(\overline{QP},n_Q)\int_0^{\frac{|PQ|^2}{4t}} q^{\frac{n}{2}-1}\ e^{-q}dq\Big\}\varphi(Q,t)dQ$$

with all integrals in (7) being absolutely convergent.

Proof is analogical to that of theorem 2 in [2], with little modifications.

Now, we shall prove the following theorem.

T h e o r e m 2. Assume that:

1°. Function $\varphi(Q,\tau)$ satisfies (2) and (3);

2°. Surface S is of Class $Ł_2(c,\varkappa)$, (see [6], p.96) and satifies the condition below:

(c) There exist on S $n-1$ fields of tangent unit vectors $\{r_P^1\},\dots,\{r_P^{n-1}\}$ such that for each pair of points P and $\bar{P}$ belonging to S, the following relations

$$\langle r_P^\alpha, r_P^\beta\rangle = \delta_\alpha^\beta \tag{8}$$

$$(r_P^\alpha, r_{\bar{P}}^\alpha) \leqslant C_r|P\bar{P}|^{h_r} \tag{9}$$

hold (where $\alpha = 1,\dots,n-1$; $\beta = 1,\dots,n-1$; $C_r > 0$, $h_r \in (0,1]$, $\langle,\rangle$ denotes the scalar product and δ_α^β is the Kronecker delta);

3°. $\{s_P\}$ is a tangent vector field given on S and fulfilling the condition (**) with

$$h_s > 1-h_\varphi. \tag{10}$$

Then, tangential derivative (7) satisfies the following Hölder condition

$$\Big|\frac{d}{ds_P}V(P,t)-\frac{d}{ds_{\bar{P}}}V(\bar{P},\bar{t})\Big| \leqslant (A_1M_\varphi+A_2k_\varphi)t^{-\mu_\varphi}(|P\bar{P}|^{h_*}+|\bar{t}-t|^{h_{**}}) \tag{11}$$

($\bar{t} \geqslant t$), where A_1 and A_2 are positive constant not depending on φ, and

$$h_* = \min(h_\varphi, \theta_1\min(h_r,h_s,\varkappa)) \tag{12}$$

$$h_{**}= \frac{\theta_2}{2} h_\varphi$$

with θ_1 and θ_2 being chosen arbitrarily in $(0,1)$.

P r o o f. We shall give here only some parts of the proof, omitting the parts which are similar to the appropriate fragments[1] of the proofs of theorems 2 and 3 in [3].

In order to prove the validity od the first part of relation (11), let us denote the successive integrals on the right-hand side of (7) by I_1, I_2 and I_3, respectively.

For the integral I_1 we keep in force the appropiate Hölder inequality obtained in [3] (formula (40)), which has the form

$$\left|I_1(P,t)-I_1(P,t)\right| \leqslant \text{const } t^{-\mu_\varphi}|P\bar{P}|^{h_s} \tag{13}$$

Passing on to the examination of the second integral, I_2, we consider a sphere K with center at P and radius $2|P\bar{P}|$, and an $(n-1)$-dimensional circular cylinder Λ with axis n_P and a constant sufficiently small radius δ. It is sufficient to consider the case when $2|P\bar{P}|<\delta$.

We have the following estimates (see [3], pp. 162-164)

$$\left|I_2^{S-S_\Lambda}(P,t)-I_2^{S-S_\Lambda}(P,t)\right| \leqslant \text{const. } k_\varphi t^{-\mu_\varphi}|P\bar{P}|^{h_0} \tag{14}$$

$$\left|I_2^{S_K}(P,t)-I_2^{S_K}(\bar{P},t)\right| \leqslant \text{const. } k_\varphi t^{-\mu_\varphi}|P\bar{P}|^{h_\varphi} \tag{15}$$

where

$$h_0 = \min(h_s, h_r, h_\varphi).$$

Further, we have

$$I_2^{S_\Lambda-S_K}(P,t)-I_2^{S_\Lambda-S_K}(\bar{P},t) = \int\limits_{S_\Lambda-S_K}\left[\frac{d}{ds_P}(|PQ|^{1-n}\cos(\overline{QP},n_Q)) + \right.$$

[1] Although in [3] the Hölder coefficient of the function φ was assumed to be limited, the results are easily extendable on the case of an unlimited Hölder coefficient (see [4], pp.144-147).

$$- \frac{d}{ds_{\bar{P}}} \left(|\bar{P}Q|^{1-n} \cos(\overline{QP}, n_Q)\right)\Big] \cdot \left[\varphi(Q,t) - \varphi(\bar{P},t)\right] dQ +$$

$$+ \left[\varphi(\bar{P},t) - \varphi(P,t)\right] \int_{S_\Lambda - S_K} \frac{d}{ds_P} \left(|PQ|^{1-n} \cos(\overline{QP}, n_Q)\right) dQ = J_1 + J_2. \tag{16}$$

Now, let us introduce two rectangular local systems of axes $Px_1, \ldots, Px_n$ and $\bar{P}\bar{x}_1, \ldots, \bar{P}\bar{x}_n$, with the axes Px_i, $\bar{P}\bar{x}_i$, Px_n and $\bar{P}\bar{x}_n$ coinciding with the vectors r_P^i, $r_{\bar{P}}^i$, n_P and $n_{\bar{P}}$, respectively[1] $(i = 1, \ldots, n-1)$.

Basing on the decomposition

$$\frac{d}{ds_P} f(P,t) - \frac{d}{ds_{\bar{P}}} f(\bar{P},t) = \sum_{\alpha=1}^{n-1} \Big\{ \left[\cos(x, s_P) - \cos(\bar{x}, s_{\bar{P}})\right] \frac{\partial}{\partial x_\alpha} f(P,t) +$$

$$+ \Big[\frac{\partial}{\partial x_\alpha} f(P,t)(1 - \cos(x_\alpha, \bar{x}_\alpha)) + \left(\frac{\partial}{\partial x_\alpha} f(P,t) - \frac{\partial}{\partial x_\alpha} f(\bar{P},t)\right) \cos(x_\alpha, \bar{x}_\alpha) +$$

$$- \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{n} \frac{\partial}{\partial x_\alpha} f(\bar{P},t) \cos(x_\beta, \bar{x}_\alpha) \Big] \cos(\bar{x}_\alpha, s_{\bar{P}}) \Big\} \tag{17}$$

and on assumptions 2° and 3°, we obtain for the integral J_1 the folowing inequality

$$|J_1| \leqslant \text{const.} k_\varphi t^{-\mu_\varphi} |P\bar{P}|^{h_0} \tag{18}$$

For the integral J_2 we have

$$J_2 = \left[\varphi(\bar{P},t) - \varphi(P,t)\right] \int_{S_\Lambda - S_K} \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left(|PQ'|^{1-n} \cos(\overline{Q'P}, n_Q)\right) (\cos(x_i, s_P) dQ +$$

$$+ \left[\varphi(\bar{P},t) - \varphi(P,t)\right] \int_{S_\Lambda - S_K} \sum_{i=1}^{n-1} \Big\{ \frac{\partial}{\partial x_i} \left(|PQ|^{1-n} \cos(\overline{QP}, n_Q)\right) +$$

$$- \frac{\partial}{\partial x_i} \left(|PQ'|^{1-n} \cos(\overline{Q'P}, n_Q)\right) \Big\} \cos(x_i, s_P) dQ = H_1 + H_2 \tag{19}$$

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[1] Here r_P^i and $r_{\bar{P}}^i$ are vectors appearing in the condition (c).

where Q' is the orthogonal projection of a point $Q \in S_\Lambda - S_K$ on the plane Π tangent to S at P.

Denoting by h_2 the integrand of the integral H_2 in (19), we can write the following equality[1]

$$h_2 = \cos(n_P, n_Q) \sum_{i=1}^{n-1} \cos(x_i, s_P) \left\{ -\frac{\partial g}{\partial \xi_i} (|PQ|^{-n} - |PQ'|^{-n}) + n\xi_i \cdot \right. \quad (20)$$

$$\left. \cdot \left(\sum_{\alpha=1}^{n-1} \frac{\partial g}{\partial \xi_\alpha} \xi_\alpha - \xi_n \right) (|PQ|^{-(n+2)} - |PQ'|^{-(n+2)}) - n\xi_i \xi_n |PQ'|^{-(n+2)} \right\} =$$

$$= k_1 + k_2 + k_3$$

in which expressions k_1 and k_2 satisfy

$$|k_1| \leqslant \text{const.} |PQ'|^{2-n} \quad (21)$$

$$|k_2| \leqslant \text{const.} |PQ'|^{2-n} \quad (22)$$

It is evident from (21) and (22) that the integrals having the integrands k_1 and k_2, respectively, satisfy an inequality analogical to (18). Hence, in order to prove our theorem, it is sufficient to examine the integral H_1 in (19) and the following integral

$$\bar{H}_2 = -n \left[\varphi(\bar{P}, t) - \varphi(P, t) \right] \int_{\Pi_\Lambda - \Pi_K} \sum_{\gamma=1}^{n-1} |PQ'|^{-(n+2)} \xi_n \xi_\gamma \, dQ' \quad (23)$$

where $\Pi_\Lambda - \Pi_K$ is the orthogonal projection of the domain $S_\Lambda - S_K$ on the plane Π.

We shall use the following relations

$$\xi_n = \frac{1}{2} \sum_{\alpha,\beta=1}^{n-1} \left(\frac{\partial^2 g(Q_*)}{\partial \xi_\alpha \partial \xi_\beta} - \frac{\partial^2 g(P)}{\partial \xi_\alpha \partial \xi_\beta} \right) \xi_\alpha \xi_\beta + \frac{1}{2} \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 g(P)}{\partial \xi_\alpha \partial \xi_\beta} \xi_\alpha \xi_\beta \quad (24)$$

$$\frac{\partial g(Q')}{\partial \xi_\alpha} = \sum_{\beta=1}^{n-1} \left(\frac{\partial^2 g(\tilde{Q})}{\partial \xi_\alpha \partial \xi_\beta} - \frac{\partial^2 g(P)}{\partial \xi_\alpha \partial \xi_\beta} \right) \xi_\beta + \sum_{\beta=1}^{n-1} \frac{\partial^2 g(P)}{\partial \xi_\alpha \partial \xi_\beta} \xi_\beta \quad (25)$$

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[1] $g(\xi_1, \ldots, \xi_{n-1})$ is a function appearing in the local representation of S in the neighbourhood of point P.

($\alpha = 1,\dots,n-1$) in which Q_* and $\tilde{Q}$ are some points inside the segment joining the points $P(0,\dots,0)$ and $Q(\xi_1,\dots,\xi_{n-1})$ of the plane Π.

Now, by making use of (24) and (25) we can express H_1 and $\bar{H}_2$ in the form

$$H_1 = \left[\varphi(\bar{P},t)-\varphi(P,t)\right]\Big\{ \int\limits_{S_\Lambda - S_K} \Xi_1(P,Q)\,dQ + \sum_{\alpha,\beta,\gamma=1}^{n-1} C_{\alpha\beta\gamma} \frac{\partial^2 g(P)}{\partial\xi_\alpha \partial\xi_\beta} \cdot \int\limits_{\Pi_\Lambda - \Pi_K} |PQ'|^{-(n+2)} \xi_\alpha \xi_\beta \xi_\gamma \, dQ' + \tag{26}$$

$$+ \sum_{\alpha,\gamma=1}^{n-1} \bar{C}_{\alpha\gamma} \frac{\partial^2 g(P)}{\partial\xi_\alpha \partial\xi_\gamma} \int\limits_{\Pi_\Lambda - \Pi_K} |PQ'|^{-n} \xi_\alpha \, dQ' \Big\} = Y_1 + Y_2 + Y_3;$$

$$\bar{H}_2 = \left[\varphi(\bar{P},t)-\varphi(P,t)\right]\Big\{ \int\limits_{S_\Lambda - S_K} \Xi_2(P,Q)\,dQ + $$

$$+ \sum_{\alpha,\beta,\gamma=1}^{n-1} \bar{\bar{C}}_{\alpha\beta\gamma} \frac{\partial^2 g(P)}{\partial\xi_\alpha \partial\xi_\beta \partial\xi_\gamma} \int\limits_{\Pi_\Lambda - \Pi_K} |PQ'|^{-(n+2)} \xi_\alpha \xi_\beta \xi_\gamma \, dQ' \Big\} = Z_1 + Z_2 \tag{27}$$

where $C_{\alpha\beta\gamma}$, $\bar{C}_{\alpha\gamma}$ and $\bar{\bar{C}}_{\alpha\beta\gamma}$ are constants, and $\Xi_1(P,Q)$ and $\Xi_2(P,Q)$ are some expressions satisfying

$$|\Xi_j(P,Q)| \leqslant \text{const.}\,|PQ|^{-n+1+\varkappa} \tag{28}$$

($j = 1,2$).

From (28) it follows that

$$|Y_1| \leqslant \text{const.}\,k_\varphi \; t^{-\mu_\varphi} |P\bar{P}|^{h_\varphi} \tag{29}$$

$$|Z_1| \leqslant \text{const.}\,k_\varphi \; t^{-\mu_\varphi} |P\bar{P}|^{h_\varphi}. \tag{29'}$$

We shall prove that all remaining terms on the right hand sides of (26) and (27) are equal to zero, that is, that the equalities

$$Y_i = 0 \qquad (i = 2,3) \tag{30}$$

$$Z_2 = 0 \tag{31}$$

are valid.

In fact, by introducing in the plane Π a polar coordinates system with the pole at P (see [2], p.108), each of the expressions Y_1, Y_2 and Z_2 can be written as a linear combination of products of some factors, with each product containing one of the following integrals[1]

$$\int_0^{\pi} \sin^p\omega \cos\omega \, d\omega \; ; \quad \int_0^{2\pi} \sin^q\omega \cos\omega \, d\omega \; ; \quad \int_0^{2\pi} \sin^{2r+1}\omega \, d\omega \tag{32}$$

where p, q and r are non-negative integers.

All integrals above are equal to zero, whence (30) and (31) hold.

On joining the results (21), (29), (29'), (30) and (31) we get for the integral J_2 in (16) the following estimate

$$|J_2| \leqslant \text{const.}k_\varphi \, t^{-\mu_\varphi} |P\bar{P}|^{h_\varphi} \tag{33}$$

and by virtue of (14), (15), (18) and (33) we obtain

$$|I_2(\bar{P},t) - I_2(\bar{P},t)| \leqslant \text{const.}k_\varphi t^{-\mu_\varphi} |P\bar{P}|^{h_o} \tag{34}$$

$(h_o = \min(h_r, h_s, h_\varphi))$.

Now, we pass on to the examination of the integral $I_3(P,t)$ in (7).

We can write

$$I_3(P,t) = -2^n \Big\{ \int_S f(P,Q,t)\left[\varphi(Q,t) - \varphi(P,t)\right] dQ + \tag{35}$$

$$+ \varphi(P,t) \int_S f(P,Q,t)\,dQ \Big\} = \bar{I}_3(P,t) + \tilde{I}_3(P,t)$$

$$J_3(\bar{P},t) = -2^n \Big\{ \int_S f(\bar{P},Q,t)\left[\varphi(Q,t) - \varphi(\bar{P},t)\right] dQ + \tag{35'}$$

$$+ \varphi(\bar{P},t) \int_S f(\bar{P},Q,t)\,dQ \Big\} = \bar{I}_3(\bar{P},t) + \tilde{I}_3(\bar{P},t)$$

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[1] The form of the integral contained by a product depends on the values of the indexes α, β and γ appearing in this product.

where

$$f(X,Q,t) = \frac{d}{ds_X}\left[|XQ|^{1-n}\cos(\overline{QX},n_Q)\int_0^{\frac{|XQ|^2}{4t}} q^{\frac{n}{2}-1}\, e^q dq\right] \qquad (36)$$

(with $X = P$ or $X = \bar{P}$).

On considering the sphere K and the cylinder Λ introduced above in the examination of I_2; on decomposing the surface S into three parts: $S-S_\Lambda$, $S_\Lambda-S_K$ and S_K, and on making use of (24) and (25) we can show that

$$\left|\bar{I}_3(P,t)-\bar{I}_3(\bar{P},t)\right| \leqslant \text{const.}k_\varphi t^{-\mu_\varphi}|P\bar{P}|^{h_0} \qquad (37)$$

In order to examine the difference of integrals $\tilde{I}_3(P,t)$ and $\tilde{I}_3(\bar{P},t)$, let us introduce two $(n-1)$-dimensional circular cylinders W and W_1, with the common radius δ' and the axes of revolution coinciding with n_P and $n_{\bar{P}}$ respectively. Assume that δ' is so small that $(S_W \cup S_{W_1}) \subset S_\Lambda$.

The following decomposition

$$\tilde{I}_3(P,t)-\tilde{I}_3(\bar{P},t) = \left[\tilde{I}_3^{S-S_\Lambda}(P,t)-\tilde{I}_3^{S-S_\Lambda}(\bar{P},t)\right] +$$

$$+\left[\tilde{I}_3^{S_\Lambda-S_W}(P,t)-\tilde{I}_3^{S_\Lambda-S_{W_1}}(\bar{P},t)\right] + \left[\tilde{I}_3^{S_W}(P,t)-\tilde{I}_3^{S_{W_1}}(\bar{P},t)\right] =$$

$$= U_1 + U_2 + U_3 \qquad (38)$$

is valid.

For the first two terms on the right-hand side of (38), the estimate

$$|U_i| \leqslant (\bar{A}_1 M_\varphi + \bar{A}_2 k_\varphi)t^{-\mu_\varphi}|P\bar{P}|^{h_0} \qquad (39)$$

(where $i = 1$ or 2, and $\bar{A}_1$ and $\bar{A}_2$ are positive constants) is easily obtained.

In order to examine the third member, U_3, let us note that by virtue of (24), (25), (30) and (31) we have

$$\tilde{I}_3^{S_W}(P,t) = \varphi(P,t)\int_{S_W} \overset{*}{\omega}(P,t;Q)dQ \qquad (40)$$

$$\tilde{I}_3^{S_{W_1}}(\bar{P},t) = \varphi(\bar{P},t) \int_{S_{W_1}} \overset{*}{\omega}_1(\bar{P},t;Q)dQ \tag{41}$$

where $\overset{*}{\omega}(P,t;Q)$ and $\overset{*}{\omega}_1(\bar{P},t;Q)$ are some expressions satisfying the inequalities

$$|\overset{*}{\omega}(P,t;Q)| \leqslant \text{const.}|PQ'|^{-n+1+\varkappa} \tag{42}$$

$$|\overset{*}{\omega}_1(\bar{P},t;Q)| \leqslant \text{const.}|\bar{P}Q''|^{-n+1+\varkappa} \tag{42'}$$

in which Q' is understood as above and Q'' denotes the orthogonal projection of the point $Q \in S_{W_1}$ on the plane $\bar{P}\bar{x}_1 \ldots \bar{x}_{n-1}$.

We can write

$$|U_3| \leqslant \left|\tilde{I}_3^{S_K}(P,t)\right| + \left|\tilde{I}_3^{S_K}(\bar{P},t)\right| + \\ + \left|\tilde{I}_3^{S_W-S_K}(P,t) - \tilde{I}_3^{S_W-S_K}(\bar{P},t)\right| + \overset{*}{I}'_3(\bar{P},t) \tag{43}$$

where

$$\overset{*}{I}_3(\bar{P},t) = \varphi(\bar{P},t) \int_{S_{WW_1}} |\overset{*}{\omega}_1(\bar{P},t;Q)|\, dQ \tag{43'}$$

with $S_{WW_1} = (S_W \cup S_{W_1}) \setminus (S_W \cap S_{W_1})$.

It follows from (42) and (42') that the following inequalities

$$\left|\tilde{I}_3^{S_K}(X,t)\right| \leqslant \text{const.}M_\varphi\, t^{-\mu_\varphi}|P\bar{P}|^\varkappa \tag{44}$$

$$\overset{*}{I}_3(\bar{P},t) \leqslant \text{const.}M_\varphi\, t^{-\mu_\varphi}|P\bar{P}| \tag{45}$$

hold, for X equal to either P or $\bar{P}$.

Now, it can easily be observed that in order to estimate the difference of integrals $\tilde{I}_3^{S_W-S_K}(P,t)$ and $\tilde{I}_3^{S_W-S_K}(\bar{P},t)$ (see (40) and (41)), the following expressions

$$e_1 = \sum_{\alpha,\beta=1}^{n-1} \left[\frac{\partial^2 g(\tilde{Q})}{\partial\xi_\alpha\partial\xi_\beta} - \frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}\right]\xi_\beta - \sum_{\alpha,\beta=1}^{n-1}\left[\frac{\partial^2 g_1(\tilde{\bar{Q}})}{\partial\bar{\xi}_\alpha\partial\bar{\xi}_\beta} - \frac{\partial^2 g_1(\bar{P})}{\partial\bar{\xi}_\alpha\partial\bar{\xi}_\beta}\right]\bar{\xi}_\beta \tag{46}$$

$$e_2 = \sum_{\alpha=1}^{n-1} \frac{\partial g(Q')}{\partial \xi_\alpha} \left(|PQ|^{-n} - |PQ'|^{-n}\right) - \sum_{\alpha=1}^{n-1} \frac{\partial g_1(Q'')}{\partial \bar{\xi}_\alpha} \left(|\bar{P}Q|^{-n} - |\bar{P}Q''|^{-n}\right) \qquad (47)$$

$$e_3 = \sum_{\alpha,\beta,\gamma=1}^{n-1} \left[\frac{\partial^2 g(Q_*)}{\partial \xi_\alpha \partial \xi_\beta} - \frac{\partial^2 g(P)}{\partial \xi_\alpha \partial \xi_\beta} \right] \xi_\alpha \xi_\beta \xi_\gamma +$$

$$- \sum_{\alpha,\beta,\gamma=1}^{n-1} \left[\frac{\partial^2 g_1(Q_{**})}{\partial \bar{\xi}_\alpha \partial \bar{\xi}_\beta} - \frac{\partial^2 g_1(\bar{P})}{\partial \bar{\xi}_\alpha \partial \bar{\xi}_\beta} \right] \bar{\xi}_\alpha \bar{\xi}_\beta \bar{\xi}_\gamma \qquad (48)$$

should be examined. Here Q_* and $\tilde{Q}$ are understood as in (24) and (25), Q_{**} and $\tilde{\tilde{Q}}$ denote some points inside the segment joining $\bar{P}$ and Q'' (see (42')), while $\xi_n = g(\xi_1, \ldots, \xi_{n-1})$ and $\bar{\xi}_n = g_1(\bar{\xi}_1, \ldots, \bar{\xi}_{n-1})$ are local representations of S in the neighbourhoods of points P and $\bar{P}$ respectively, with $(\xi_1, \ldots, \xi_n)$ denoting the coordinates of a point $Q \in S_W - S_K$ in the local system $Px_1 \ldots x_n$, and $(\bar{\xi}_1, \ldots, \bar{\xi}_n)$ - the coordinates of that point in the local system $\bar{P}\bar{x}_1 \ldots \bar{x}_n$.

Note first of all that the following relations

$$\bar{\xi}_n = g_1(\bar{\xi}_1, \ldots, \bar{\xi}_{n-1}) = \sum_{\nu=1}^{n-1} a_{\nu n}(\xi_\nu - \tilde{x}_\nu) + a_{nn}\left[g(\xi_1, \ldots, \xi_{n-1}) - \tilde{x}_n\right] \equiv$$

$$\equiv \tilde{g}(\xi_1, \ldots, \xi_{n-1});$$

$$\bar{\xi}_\alpha = \sum_{\nu=1}^{n} a_{\nu\alpha}(\xi_\nu - \tilde{x}_\nu) \qquad (\alpha = 1, \ldots, n-1) \qquad (49')$$

$$\xi_\beta = \sum_{\mu=1}^{n} b_{\mu\beta} \bar{\xi}_\mu + \tilde{x}_\beta \qquad (\beta = 1, \ldots, n) \qquad (50)$$

are valid, where $a_{ij} = b_{ji} = \cos(\bar{\xi}_j, \xi_i)$, and $\tilde{x}_1, \ldots, \tilde{x}_n$ denote the coordinates of point $\bar{P}$ with respect to the system of axes $Px_1 \ldots x_n$.

It follows from (49) - (50) that

$$\frac{\partial^2 g_1(Q'')}{\partial \bar{\xi}_\alpha \partial \bar{\xi}_\beta} = \sum_{i,j=1}^{n-1} a_{nn} b_{\alpha i} b_{\beta j} \frac{\partial^2 g(Q')}{\partial \xi_i \partial \xi_j} \qquad (51)$$

whence

$$\left|\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g_1(\bar P)}{\partial\bar\xi_\alpha\partial\bar\xi_\beta}\right|\leqslant|1-a_{nn}|\sum_{i,j=1}^{n-1}\left|\frac{\partial^2 g(\bar P')}{\partial\xi_i\partial\xi_j}\right||b_{\alpha i}||b_{\beta j}| + \tag{52}$$
$$+\left|\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g(\bar P')}{\partial\xi_\alpha\partial\xi_\beta}\right|+|1-b_{\beta\beta}|\left|\frac{\partial^2 g(\bar P')}{\partial\xi_\alpha\partial\xi_\beta}\right|+$$
$$+|1-b_{\alpha\alpha}||b_{\beta\beta}|\left|\frac{\partial^2 g(\bar P')}{\partial\xi_\alpha\partial\xi_\beta}\right|+\sum_{i,j=1}^{n-1}{}'\left|\frac{\partial^2 g(P)}{\partial\xi_i\partial\xi_j}\right||b_{\alpha i}||b_{\beta j}|$$

where $\bar P'$ denotes the point of Π with coordinates $(\tilde x_1,\dots,\tilde x_{n-1};0)$, and $\sum_{i,j=1}^{n-1}{}'$ is understood as $\sum_{i,j=1}^{n-1}$ with $i\neq\alpha$ and $j\neq\beta$.

However, due to the assumption (c) and the special choice of the systems $Px_1\dots x_n$ and $\bar P\bar x_1\dots\bar x_n$, we have

$$\begin{aligned}|a_{\nu\alpha}|&\leqslant \text{const.}\,|P\bar P|^{h_r}\quad(\nu\neq\alpha)\\ |1-a_{\alpha\alpha}|&\leqslant\text{const.}\,|P\bar P|^{2h_r}\end{aligned}\tag{53}$$

where $\alpha=1,\dots,n$. Hence, and by making use of the inequality

$$\left|\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g(\bar P)}{\partial\xi_\alpha\partial\xi_\beta}\right|\leqslant\text{const.}\,|P\bar P|^{\varkappa}\tag{54}$$

(which results from the relation $S\in L_2(C,\varkappa)$), we obtain for the difference on the left-hand side of (52) the following estimate

$$\left|\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g_1(\bar P)}{\partial\bar\xi_\alpha\partial\bar\xi_\beta}\right|\leqslant\text{const.}\,|P\bar P|^{\hat h}\tag{55}$$

where $\hat h=\min(h_r,\varkappa)$.

Now, let us observe that for the expression (46) the following equality

$$e_1=\sum_{\alpha,\beta=1}^{n-1}\left\{\left[\frac{\partial^2 g(\tilde Q)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g_1(\tilde{\bar Q})}{\partial\bar\xi_\alpha\partial\bar\xi_\beta}\right]\xi_\beta+\left[\frac{\partial^2 g_1(\bar P)}{\partial\bar\xi_\alpha\partial\bar\xi_\beta}-\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}\right]\bar\xi_\beta+\right.$$
$$\left.+(\xi_\beta-\bar\xi_\beta)\left[\frac{\partial^2 g(\tilde Q)}{\partial\xi_\alpha\partial\xi_\beta}-\frac{\partial^2 g(P)}{\partial\xi_\alpha\partial\xi_\beta}\right]\right\}$$

holds.

In order to estimate the difference $\left[\frac{\partial^2 g(\tilde{Q})}{\partial\xi_\alpha\partial\xi_\beta} - \frac{\partial^2 g_1(\tilde{\tilde{Q}})}{\partial\bar{\xi}_\alpha\partial\bar{\xi}_\beta}\right]$, we make use of the equalities

$$\sum_{\beta=1}^{n-1}\frac{\partial^2 g_1(\tilde{\tilde{Q}})}{\partial\bar{\xi}_\alpha\,\partial\bar{\xi}_\beta}\bar{\xi}_\beta = \frac{\partial g_1(Q'')}{\partial\bar{\xi}_\alpha} - \frac{\partial g_1(\bar{P})}{\partial\bar{\xi}_\alpha} =$$

$$= a_{nn}b_{\alpha\alpha}\left[\frac{\partial g(Q')}{\partial\xi_\alpha} - \frac{\partial g(\bar{P}')}{\partial\xi_\alpha}\right] + {\sum_{i=1}^{n}}' a_{nn}b_{\alpha i}\left[\frac{\partial g(Q')}{\partial\xi_i} - \frac{\partial g(\bar{P}')}{\partial\xi_i}\right]; \tag{56}$$

$$\sum_{\beta=1}^{n-1}\frac{\partial^2 g(\tilde{Q})}{\partial\xi_\alpha\partial\xi_\beta}\xi_\beta = \frac{\partial g(Q')}{\partial\xi_\alpha} - \frac{\partial g(P)}{\partial\xi_\alpha}$$

where the symbol $\sum'$ is understood as above.

On substracting the equalities in (56) from each other, and on making use of (49) - (50) and of

$$\frac{\partial g(P)}{\partial\xi_\alpha} - \frac{\partial g(\bar{P}')}{\partial\xi_\alpha} = \sum_{\beta=1}^{n-1}\frac{\partial^2 g(\hat{P})}{\partial\xi_\alpha\partial\xi_\beta}(-\tilde{x}_\beta) \tag{57}$$

$$|\bar{P}'\ Q'| \leqslant \text{const.}|PQ'| \tag{58}$$

($Q \in S_W - S_K$, and $\hat{P} \in \overline{P\bar{P}'}$), we obtain

$$\left|\sum_{\beta=1}^{n-1}\left(\frac{\partial^2 g_1(\tilde{\tilde{Q}})}{\partial\bar{\xi}_\alpha\,\partial\bar{\xi}_\beta} - \frac{\partial^2 g(Q)}{\partial\xi_\alpha\partial\xi_\beta}\right)\bar{\xi}_\beta\right| \leqslant \text{const}\ |PQ'||P\bar{P}|^{\hat{h}} \tag{59}$$

Finally, let us note that the following sequence of relations

$$|\xi_\beta - \bar{\xi}_\beta| = \left|\xi_\beta - \left\{\sum_{\nu=1}^{n-1}a_{\nu\beta}(\xi_\nu - \tilde{x}_\nu) + a_{n\beta}\left[g(\xi_1,\dots,\xi_{n-1}) - \tilde{x}_n\right]\right\}\right| \leqslant$$

$$\leqslant |\xi_\beta||1 - a_{\beta\beta}| + |\tilde{x}_\beta||a_{\beta\beta}| + {\sum_{\nu=1}^{n-1}}'|a_{\nu\beta}||\xi_\nu - \tilde{x}_\nu| +$$

$$+|a_{n\beta}||g(\xi_1,\dots,\xi_{n-1}) - \tilde{x}_n| \leqslant$$

$$\leqslant \text{const.}|P\bar{P}|^{h_r}|PQ'|^{1-h_r} \tag{60}$$

is valid.

By (55), (59) and (60) we get for the expression e_1 in (46) the estimate of the form

$$|e_1| < \text{const.}\cdot(|P\bar{P}|^{\hat{h}}|PQ'|+|P\bar{P}|^{h_r}|PQ'|^{1-h_r+\varkappa}) \tag{61}$$

The expression e_2 (see (47)) can be examined by basing on the equality

$$\frac{\partial g(Q')}{\partial\xi_\alpha}(|PQ|^{-n}-|PQ'|^{-n}) - \frac{\partial g_1(Q'')}{\partial\bar{\xi}_\alpha}(|\bar{P}Q|^{-n}-|\bar{P}Q''|^{-n}) =$$

$$= \left[\frac{\partial g(Q')}{\partial\xi_\alpha} - \frac{\partial g_1(Q'')}{\partial\bar{\xi}_\alpha}\right](|PQ|^{-n}-|PQ'|^{-n}) + \tag{62}$$

$$+\frac{\partial g_1(Q'')}{\partial\bar{\xi}_\alpha}\left[(|PQ|^{-n}-|PQ'|^{-n}) +\right.$$

$$\left.-(|\bar{P}Q|^{-n}-|\bar{P}Q'|^{-n})+(|\bar{P}Q''|^{-n}-|\bar{P}Q'|^{-n})\right]$$

and by using the relations[1]

$$|QQ'| < \text{const.}\cdot|PQ'|^2 \tag{63}$$

$$|Q'Q''| \leqslant \text{const.}\cdot|P\bar{P}|\,|PQ'| \tag{63'}$$

$$\left|\frac{\partial g(Q')}{\partial\xi_\alpha} - \frac{\partial g_1(Q'')}{\partial\bar{\xi}_\alpha}\right| = \frac{\partial g(Q')}{\partial\xi_\alpha}\left[1-b_{\alpha\alpha}+(1-a_{nn})b_{\alpha\alpha}\right]+$$

$$-a_{\alpha n}b_{\alpha\alpha} - \sum_{\substack{\nu=1\\ \nu\neq\alpha}}^{n-1}\left[a_{\nu n}+a_{nn}\frac{\partial g(Q')}{\partial\xi_\nu}\right]b_{\alpha\nu}$$

which are valid when $Q \in S_W - S_K$.

As a result we obtain

$$|e_2| \leqslant \text{const.}\cdot(|P\bar{P}|\,|PQ'|^{-(n-1)}+|P\bar{P}|^{h_r}|PQ'|^{-(n-1)}) \tag{65}$$

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[1] Inequality (63') is taken from [1], p.207.

The expression e_3, given by (48), is studied similarly as e_1 and the result is as follows

$$|e_3| \leqslant \text{const.}(|P\bar{P}|^{\hat{h}}|PQ'|^3 + |P\bar{P}|^{h_r}|PQ'|^{3-h_r+\varkappa}) \qquad (66)$$

Basing on (61), (65) and (66) we get the inequality

$$|\tilde{I}_3^{S_W-S_K}(P,t) - \tilde{I}_3^{S_W-S_K}(\bar{P},t)| \leqslant (C_1 M_\varphi + C_2 k_\varphi) t^{-\mu_\varphi} |P\bar{P}|^{\tilde{h}} \qquad (67)$$

where C_1 and C_2 are positive constants, while $\tilde{h} = \min[h_\varphi, \theta \min(h_r, h_g, \varkappa)]$, with θ being an arbitrary number in (0,1).

On joining (37), (39), (44), (45) and (67) we obtain

$$|I_3(P,t) - I_3(\bar{P},t)| \leqslant (\tilde{C}_1 M_\varphi + \tilde{C}_2 k_\varphi) t^{-\mu_\varphi} |P\bar{P}|^{\tilde{h}} \qquad (68)$$

and the validity of the first part of the required Hölder condition (11) follows immediately from (13), (34) and (68).

Now, we shall prove the second part of condition (11).

Let us consider again the expression (7).

For the integrals I_1 and I_2 in this expression we keep in force the conditions[1] obtained in [3], p.175-179. Hence

$$|I_1(P,t) - I_1(P,\bar{t})| \leqslant \text{const.} k_\varphi t^{-\mu_\varphi} |t-\bar{t}|^{\theta \tilde{h}_\varphi} \qquad (69)$$

$$|I_2(P,t) - I_2(P,\bar{t})| \leqslant \text{const.} k_\varphi t^{-\mu_\varphi} |t-\bar{t}|^{h_*} \qquad (70)$$

$(\bar{t} \geqslant t;\ h_* = \theta \frac{h_\varphi}{2};\ \theta \in (0,1))$.

The integral $I_3(P,t)$ in (7) can be examined by way of expressing it according to the scheme (35), and by subsequent consideration of the sphere K, with the center at P and the radius $(\bar{t}-t)^{1/2}$, and of the cylinder W introduced above. The

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[1] Let us note that in virtue of the relation $S \in \mathcal{L}_2(c,\varkappa)$, we can put in these conditions $\varkappa = 1$.

reasoning is similar to that in the proof of (68), but is much simpler. As a result we get

$$\left|I_3(P,\bar{t})-I_3(P,t)\right| \leqslant (\overset{*}{C}_1 M_\varphi+\overset{*}{C}_2 k_\varphi)t^{-\mu_\varphi}\left|\bar{t}-t\right|^{h_{**}} \qquad (71)$$

On joining (69), (70) and (71) we obtain the second part of Hölder condition (11), q.e.d.

BIBLIOGRAPHY

[1] A.B o r z y m o w s k i; Własności pochodnych stycznościowych pewnego rozwiązania równania parabolicznego w obszarze niewalcowym i ich zastosowanie, Prace matem. 8 (1964) 193-215.

[2] A.B o r z y m o w s k i; Własność brzegowa pochodnej stycznościowej potencjału warstwy podwójnej względem równania przewodnictwa, Zesz.Nauk.Polit.Warsz. Matematyka 2 (1964) 99-112.

[3] A.B o r z y m o w s k i; Regularna ciągłość wartości brzegowych pochodnych stycznościowych potencjału warstwy podwójnej względem równania przewodnictwa, Zesz.Nauk.Polit. Warsz., Matematyka 3 (1964), 159-182.

[4] A.B o r z y m o w s k i; O pewnych własnościach pochodnych potencjału warstwy podwójnej w teorii równania przewodnictwa,Zesz.Nauk. Polit. Warsz., Matematyka 6 (1965) 135-148.

[5] A.B o r z y m o w s k i; Pewne zagadnienie graniczne z pochodnymi stycznościowymi dla równania przewodnictwa cieplnego, Zesz. Nauk. Polit. Warsz., Matematyka 8 (1966) 25-40.

[6] N.M.G ü n t h e r; Die Potentialtheorie und ihre Anwendung auf Grundaufgaben der Mathematischen Physic.Leipzig 1957.

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