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APPLICATION OF THE CHAPLYGIN'S METHOD IN THE THEORY OF ELASTICITY

1. INTRODUCTION

Let V be a domain in the space E_3, bounded by the closed Lapunov surface S. In this paper we shall consider the following boundary value problem in the theory of elasticity. Find the displacement vector-function $u(x) = [u_1(x), u_2(x), u_3(x)]$ in the domain V, such that

$$\Delta^* u(x) + \omega^2 u(x) = F(x,u(x)) \qquad x \in V \tag{1}$$

and

$$Tu(x_0) + \sigma(x_0)u(x_0) = 0 \qquad x_0 \in S. \tag{2}$$

Here, we admit the same notation like in the paper [4]. The problem (1),(2) is the special case of the problem which has investigated in the paper [4] by the potential method. There was given the existence and uniqueness of the solution by the Banach's Fixed Point Theorem.

The First Boundary Value Problem for the elliptic equation was treated, by the Chaplygin's method ([2]), in the paper [3], written by I.P. M y s o w s k i c h. In the paper [5], the similar problem was solved with the weak assumptions than in [3]. There, we have based on the properties of the Green's function.

In this paper we shall base on the properties of the dynamic Green's tensor of the third kind G(x,y). We shall introduce the notion of the sequence upper (lower) vector-functions, and then we are going to prove the convergence of this sequence to the solution of the problem (1),(2).

We shall admit the following assumptions

I. $F_j(x,u_1,u_2,u_3)$ are real functions, defined on the set $x\in V$, $|u_s|\leqslant R$ and fulfil the Hölder-Lipschitz condition

$$\left|F_j(x,u_1,u_2,u_3)-F_j(x',u_1',u_2',u_3')\right|\leqslant K_F\left|xx'\right|^{h_F}+k_F\sum_{s=1}^{3}\left|u_s-u_s'\right|, \tag{3}$$

where R and K_F are positive constants, $0<h_F\leqslant 1$.

The constant k_F fulfils the inequality

$$0<k_F<\frac{1}{2C}, \tag{4}$$

where

$$C_j=\sup_{x\in V+S}\left[\frac{1}{4\pi}\int_V\sum_{s=1}^{3}G_j^{(s)}(x,y)\,dV_y\right] \tag{5}$$

$$C=C_1+C_2+C_3.$$

II. The functions $F_j(x,u_1,u_2,u_3)$ satisfy the inequalities

$$F_j(x,u_1,u_2,u_3)\leqslant F_j(x',u_1',u_2',u_3'), \tag{6}$$

when $u_1\leqslant u_1'$ and $u_2\leqslant u_2'$ and $u_3\leqslant u_3'$.

III. $\sigma(x_0)$ is a real function, defined for $x_0\in S$ and fulfils the Hölder condition

$$\left|\sigma(x_0)-\sigma(x_0')\right|\leqslant k_\sigma\left|x_0x_0'\right|^{h_\sigma}, \tag{7}$$

where k_σ is a positive constant, and $0<h_\sigma\leqslant 1$.

2. GREEN'S TENSOR $G(x,y)$

We shall define the dynamic Green's tensor of the third kind similary like the tensor of the first kind, whose definition was given in the monograph [1] on p.88.

D e f i n i t i o n. The dynamic Green's tensor of the third kind is the tensor

$$G(x,y) = \begin{Vmatrix} G_1^{(1)} & G_1^{(2)} & G_1^{(3)} \\ G_2^{(1)} & G_2^{(2)} & G_2^{(3)} \\ G_3^{(1)} & G_3^{(2)} & G_3^{(3)} \end{Vmatrix} \tag{8}$$

satisfies the following conditions

a) $\Delta^* G(x,y) + \omega^2 G(x,y) = 0$ for $x,y, \in V$ and $x \neq y$,

b)

$$\lim_{x \to x_0} \left[T^{(x)} G^{(k)}(x,y) + \sigma(x) G^{(k)}(x,y)\right] = 0 \quad \text{for } x,y \in V \ (y \text{ fixed}) \text{ and } x_0 \in S$$

c) $G(x,y) = \Gamma(x,y) - g(x,y)$ for $x,y \in V$

where

$\Gamma(x,y)$ - is the matrix of the fundamental solution,

$g(x,y)$ - is the matrix od the regular solution of the equation $\Delta^* g(x,y) + \omega^2 g(x,y) = 0$

The existence of the tensor $G(x,y)$ follows from the existence of the solution of the boundary value problem

$$\Delta^* g(x,y) + \omega^2 g(x,y) = 0 \quad x,y \in V$$

$$\lim_{x \to x_0} \left[T^{(x)} g^{(k)}(x,y) + \sigma(x) g^{(k)}(x,y)\right] = T^{(x_0)} \Gamma^{(k)}(x_0,y) + \sigma(x_0) \Gamma^{(k)}(x_0,y) \tag{9}$$

$$x,y \in V, \quad x_0 \in S.$$

It can be proved that the functions $G_j^{(k)}(x,y)$ are positive.

3. UPPER (LOWER) VECTOR-FUNCTION

We shall prove the fundamental theorem

T h e o r e m 1. If the vector - function $v(x)$ is regular in the domain $V+S$, fulfils the inequalities

$$\Delta^*_k v(x)+\omega^2 v_k(x) \leqslant F_k(x,v_1(x),v_2(x),v_3(x)) \quad \text{for} \quad x\in V \tag{10}$$

and the boundary condition

$$Tv(x_o) + \sigma(x_o)v(x_o) = 0 \quad \text{for} \quad x_o \in S, \tag{11}$$

the functions $F_k(x,v_1(x),v_2(x),v_3(x))$ satisfy the assumption I, then

$$v_j(x) \geqslant u_j(x) , \tag{12}$$

where $u_j(x)$ are the components of the solution of the problem (1), (2),

The vector-function $v(x)$ satisfying the assumptions of this theorem will be called the upper vector-function.

P r o o f. Let be $z_j(x) = v_j(x)-u_j(x)$ for $x\in V$. Now, we shall prove that $z_j(x) \geqslant 0$. Let $\alpha(x) = [\alpha_1(x), \alpha_2(x), \alpha_3(x)]$ be the vector-function with the components satisfying the Hölder condition for $x\in V$. The exponent of this condition is equal to h_F. Moreover, we assume, that

$$\Delta^* v(x) + \omega^2 v(x) = F(x,v(x)) + \alpha(x) \quad \text{for} \quad x\in V. \tag{13}$$

Obviously, $\alpha_1(x) \leqslant 0$ for $x\in V$.

By virtue of the definition of the vector-function $z(x)$, from the equations (1) and (13), we get

$$\Delta^* z(x) + \omega^2 z(x) = F(x,v(x)) - F(x,u(x)) - \alpha(x) \quad \text{for} \quad x\in V \tag{14}$$

and the boundary condition

$$Tz(x_o) + \sigma(x_0)z(x_o) = 0 \quad \text{for} \quad x_o \in S \tag{15}$$

According to the results of the paper [4], by the definition of the Green's tensor $G(x,y)$, we can write the solution of the problem (14), (15) in the form

$$z(x) = \frac{1}{4\pi}\int_V G(x,y)\left[F(y,u(y)) - F(y,v(y))\right]dV_y - \frac{1}{4\pi}\int_V G(x,y)\,\alpha(y)\,dV_y\ , \tag{16}$$

or, using the indexes

$$z_j(x) = \frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\left[F_k(y,u_1(y),u_2(y),u_3(y)) + \right.$$

$$\left. - F_k(y,v_1(y),v_2(y),v_3(y))\right]dV_y +$$

$$-\frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\,\alpha_k(y)\,dV_y\ . \tag{17}$$

By virtue of the definition of the vector-function $\alpha(x)$ and from the assumption I, we obtain the system of inequalities

$$z_j(x) \geqslant \frac{-k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3} \left|u_s(y) - v_s(y)\right| dV_y \tag{18}$$

From here

$$z_j(x) \geqslant \frac{-k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3} \left[\left|z_s(y)\right| - z_s(y)\right]dV_y +$$

$$-\frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3} z_s(y)\,dV_y \tag{19}$$

or

$$-2z_j(x) \leq \frac{k_F}{2\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3} \left[\left|z_s(y)\right| - z_s(y)\right] dV_y + $$
$$+ \frac{k_F}{2\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3} z_s(y) dV_y . \tag{20}$$

Let

$$A_s = \sup_{x\in V} \left[\left|z_s(x)\right| - z_s(x)\right], \qquad A = A_1 + A_2 + A_3 . \tag{21}$$

Suppose, that $z_s(x) < 0$, then $A_s = \sup_{x\in V}\left[-2z_s(x)\right]$ and $A_s > 0$. Taking the greatest upper bound for both sides of the inequality (20), by the definition (5), we obtain the system of the inequalities

$$A_j \leq 2k_F C_j A . \tag{22}$$

Hence

$$A \leq 2k_F C A . \tag{23}$$

From the assumption (4) it follows, that $A \leq 0$. It implies $z_j \geq 0$. Similarly, we shall be able to prove

T h e o r e m 2. If the vector-function $w(x)$ is regular in the domain $V+S$, fulfils the inequalities

$$\Delta_k^* w(x) + \omega^2 w_k(x) \geq F_k(x, w_1(x), w_2(x), w_3(x)) \quad \text{for} \quad x\in V \tag{24}$$

and the boundary condition

$$Tw(x_0) + \sigma(x_0) w(x_0) = 0 \quad \text{for} \quad x_0 \in S, \tag{25}$$

the functions $F_k(x, w_1(x), w_2(x), w_3(x))$ satisfy the assumption I, then

$$w_j(x) \leq u_j(x) , \tag{26}$$

where $u_j(x)$ are the components of the solution of the problem (1),(2).

The vector-function $w(x)$ satisfying the assumptions of this theorem will be called the lower vector-function.

3.1. EXAMPLE OF THE LOWER AND UPPER VECTOR-FUNCTIONS

Let $H(x) = [H_1(x), H_2(x), H_3(x)]$ be the vector-function with the components

$$H_j(x) = -\frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) M_F^{(k)} dV_y , \qquad (27)$$

where

$$M_F^{(k)} \geq \sup_{x \in V+S} \left| F_k(x,0,0,0) \right| , \qquad M_F = \left[M_F^{(1)}, M_F^{(2)}, M_F^{(3)} \right] . \qquad (28)$$

From the definition of the Green's tensor it follows, that

$$T^{(x_o)} H_j(x_o) + \sigma(x_o) H_j(x_o) = 0 \quad \text{for} \quad x_o \in S .$$

Since $G_j^{(k)}(x,y) \geq 0$, therefore $H_j(x) \leq 0$ for $x \in V$. From the equation

$$\Delta^* H(x) + \omega^2 H(x) = M_F ,$$

we obtain

$$\Delta_k^* H(x) + \omega^2 H_k(x) - F_k(x, H_1, H_2, H_3) =$$

$$= (M_F^{(k)} - F_k(x,0,0,0)) - (F_k(x,H_1,H_2,H_3) - F_k(x,0,0,0)) . \qquad (29)$$

Hence, by the assumption II, it follows, that

$$\Delta^*_k H(x) + \omega^2 H_k(x) - F_k(x, H_1, H_2, H_3) \geq 0 .$$

From the last inequalities it follows, that the vector-function $H(x)$ is the lower vector-function.

Similarly, we shall be able to prove, that the vector-function $h(x)$ with the components

$$h_j(x) = -\frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)m_F^{(k)}dV_y \,, \tag{30}$$

where

$$m_F^{(k)} \leqslant -\sup_{x\in V+S}\left|F_k(x,0,0,0)\right|, \qquad m_F = \left[m_F^{(1)}, m_F^{(2)}, m_F^{(3)}\right], \tag{31}$$

is the upper vector-function.

4. SEQUENCE OF THE UPPER (LOWER) VECTOR-FUNCTIONS

Now, we are going to define the sequence of the upper (lower) vector-functions. Let $v^{(o)}(x)$, $w^{(o)}(x)$ denote the upper and lower vector-function, respectively.

Let

$$\alpha_k^{(o)}(x) \equiv \Delta_k^* v^{(o)}(x)+\omega^2 v_k^{(o)}(x)-F_k(x,v_1^{(o)}(x),v_2^{(o)}(x),v_3^{(o)}(x))\leqslant 0 \tag{32}$$

$$\alpha^{(o)}(x) = \left[\alpha_1^{(o)}(x), \alpha_2^{(o)}(x), \alpha_3^{(o)}(x)\right]$$

$$\beta_k^{(o)}(x) \equiv \Delta_k^* w^{(o)}(x)+\omega^2 w_k^{(o)}(x)-F_k(x,w_1^{(o)}(x),w_2^{(o)}(x),w_3^{(o)}(x))\geqslant 0 \tag{33}$$

$$\beta^{(o)}(x) = \left[\beta_1^{(o)}(x), \beta_2^{(o)}(x), \beta_3^{(o)}(x)\right].$$

T h e o r e m 3. If the vector-function $v^{(1)}(x) = \left[v_1^{(1)}(x), v_2^{(1)}(x), v_3^{(1)}(x)\right]$ is a solution of the boundary value problem

$$\Delta^*(v^{(1)}(x)-v^{(o)}(x))+\omega^2(v^{(1)}(x)-v^{(o)}(x)) =$$

$$= k_F\,\Phi^{(1)}(x)-\alpha^{(o)}(x) \quad \text{for} \quad x\in V \tag{34}$$

$$Tv^{(1)}(x_o) + \sigma(x_o)v^{(1)}(x_o) = 0 \quad \text{for} \quad x_o \in S, \tag{35}$$

where

$$\Phi^{(1)}(x) = \Big[\sum_{k=1}^{3}(v_k^{(1)}(x) - v_k^{(o)}(x)),$$

$$\sum_{k=1}^{3}(v_k^{(1)}(x) - v_k^{(o)}(x)), \quad \sum_{k=1}^{3}(v_k^{(1)}(x) - v_k^{(o)}(x))\Big],$$

then the inequalities $v_k^{(1)}(x) < v_k^{(o)}(x)$ are fulfiled, and $v^{(1)}(x)$ is the upper vector-function.

P r o o f. We first prove that the boundary value problem (34), (35) has a unique solution. By virtue of the Poisson's Equation (27) from the paper [4] we can write

$$v^{(1)}(x)-v^{(o)}(x) = -\frac{k_F}{4\pi}\int_V G(x,y)\,\Phi^{(1)}(y)dV_y +$$

$$+\frac{1}{4\pi}\int_V G(x,y)\,\alpha^{(o)}(y)dV_y \tag{36}$$

or

$$v_j^{(1)}(x)-v_j^{(o)}(x) = -\frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\sum_{s=1}^{3}(v_s^{(1)}(y)-v_s^{(o)}(y))dVy+$$

$$+\frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\,\alpha_k^{(o)}(y)dV_y\,. \tag{37}$$

Let us

$$f_j(x) = \frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\,\alpha_k^{(o)}(y)dV_y\,,$$

$$f(x) = \big[f_1(x), f_2(x), f_3(x)\big]$$

and

$$|f_j(x)| \le N_j\,, \qquad N = N_1 + N_2 + N_3\,.$$

Consider the sequence of the functions

$$(v_j^{(1)}(x) - v_j^{(o)}(x))_o = f_j(x) \tag{38}$$

$$(v_j^{(1)}(x)-v_j^{(o)}(x))_m =$$

$$= -\frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\sum_{s=1}^{3}(v_s^{(1)}(y)-v_s^{(o)}(y))_{m-1}\,dV_y \quad \text{for} \quad m \geqslant 1.$$

Hence, by the notation (5), we obtain the inequalities

$$(v_j^{(1)}(x) - v_j^{(o)}(x))_m \leqslant N(k_F C)^m . \tag{39}$$

By virtue of the assumption (4) it follows, that the solution of the system (37) exists as the sum of the uniformly convergent series

$$v_j^{(1)}(x)-v_j^{(o)}(x) = \sum_{m=1}^{\infty}(v_j^{(1)}(x)-v_j^{(o)}(x))_m . \tag{40}$$

Now, we shall prove, that $v_j^{(1)}(x)-v_j^{(o)}(x) \leqslant 0$.

From the formula (37) and (32) it implies

$$v_j^{(1)}(x)-v_j^{(o)}(x) \leqslant -\frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\sum_{s=1}^{3}\left[v_s^{(1)}(y)-v_s^{(o)}(y)\right]dV_y . \tag{41}$$

Suppose, that $v_j^{(1)}(x)-v_j^{(o)}(x) > 0$ for $x \in V$. Then the right side of the inequality (41), by the properties of the Green's tensor, is negative. This result denies our supposition. Hence $v_j^{(1)}(x)-v_j^{(o)}(x) \leqslant 0$.

Finally, we shall prove that $v^{(1)}(x)$ is the upper vector-function. From (34) and (32) we obtain the relation

$$\Delta^*_k v^{(1)}(x) + \omega^2 v_k^{(1)}(x) - F_k(x, v_1^{(1)}(x), v_2^{(1)}(x), v_3^{(1)}(x)) =$$

$$= k_F\sum_{s=1}^{3}(v_s^{(1)}(x)-v_s^{(o)}(x)) +$$

$$+ F_k(x,v_1^{(o)}(x),v_2^{(o)}(x),v_3^{(o)}(x))-F_k(x,v_1^{(1)}(x),v_2^{(1)}(x),v_3^{(1)}(x)).$$

From the assumption I and the inequalities $v_k^{(1)} \leq v_k^{(o)}$, we have

$$\Delta^*{}_k v^{(1)}(x)+\omega^2 v_k^{(1)}(x) - F_k(x,v_1^{(1)}(x),v_2^{(1)}(x),v_3^{(1)}(x)) \leq 0.$$

Hence and from the theorem 1 it implies $v_j^{(1)}(x) \geq u_j(x)$. Similarly, we shall be able to prove

T h e o r e m 4. If the vector-function $w^{(1)}(x) = \left[w_1^{(1)}(x),w_2^{(1)}(x),w_3^{(1)}(x)\right]$ is a solution of the boundary value problem

$$\Delta^*(w^{(1)}(x)-w^{(o)}(x))+\omega^2(w^{(1)}(x)-w^{(o)}(x))=k_F\Psi^{(1)}(x)-\beta^{(o)} \text{ for } x\in V \quad (42)$$

$$Tw^{(1)}(x_o)+\sigma(x_o)w^{(1)}(x_o) = 0 \quad \text{for} \quad x_o\in S, \quad (43)$$

where

$$\Psi^{(1)}(x) = \Big[\sum_{k=1}^{3}(w_k^{(1)}(x)-w_k^{(o)}(x)),\ \sum_{k=1}^{3}(w_k^{(1)}(x)-w_k^{(o)}(x)),$$
$$\sum_{k=1}^{3}(w_k^{(1)}(x)-w_k^{(o)}(x))\Big],$$

then the inequalities $w_k^{(1)}(x) \geq w_k^{(o)}(x)$ are fulfiled, and $w^{(1)}(x)$ is the lower vector-function.

Now, we shall construct two sequences of the vector-functions $\{v^{(m)}(x)\}$, $\{w^{(m)}(x)\}$ so that its components are the decreasing sequence for the functions $v_k^{(m)}(x)$ and the increasing sequence for the functions $w_k^{(m)}(x)$, respectively.

Let

$$\alpha_k^{(m-1)}(x) = \Delta^*{}_k v^{(m-1)}(x)+\omega^2 v_k^{(m-1)}(x) +$$
$$- F_k(x,v_1^{(m-1)}(x),v_2^{(m-1)}(x),v_3^{(m-1)}(x)) \leq 0 \quad (44)$$

$$\alpha^{(m-1)}(x) = \left[\alpha_1^{(m-1)}(x), \alpha_2^{(m-1)}(x), \alpha_3^{(m-1)}(x)\right]$$

$$\beta_k^{(m-1)}(x) = \Delta_k^* w^{(m-1)}(x) + \omega^2 w_k^{(m-1)}(x) +$$

$$- F_k(x, w_1^{(m-1)}(x), w_2^{(m-1)}(x), w_3^{(m-1)}(x)) \geqslant 0 \tag{45}$$

$$\beta^{(m-1)}(x) = \left[\beta_1^{(m-1)}(x), \beta_2^{(m-1)}(x), \beta_3^{(m-1)}(x)\right].$$

T h e o r e m 5. If the vector-function $v^{(m)}(x) = \left[v_1^{(m)}(x), v_2^{(m)}(x), v_3^{(m)}(x)\right]$ is a solution of the boundary value problem

$$\Delta^*(v^{(m)}(x) - v^{(m-1)}(x)) + \omega^2(v^{(m)}(x) - v^{(m-1)}(x)) =$$

$$= k_F \Phi^{(m)}(x) - \alpha^{(m-1)}(x) \quad \text{for} \quad x \in V \tag{46}$$

$$Tv^{(m)}(x_o) + \sigma(x_o)v^{(m)}(x_o) = 0 \quad \text{for} \quad x_o \in S, \tag{47}$$

where

$$\Phi^{(m)}(x) = \Big[\sum_{k=1}^{3}(v_k^{(m)}(x) - v_k^{(m-1)}(x)),$$

$$\sum_{k=1}^{3}(v_k^{(m)}(x) - v_k^{(m-1)}(x)), \quad \sum_{k=1}^{3}(v_k^{(m)}(x) - v_k^{(m-1)}(x))\Big]$$

and the vector-function $w^{(m)}(x) = \left[w_1^{(m)}(x), w_2^{(m)}(x), w_3^{(m)}(x)\right]$ is a solution of the boundary value problem

$$\Delta^*(w^{(m)}(x) - w^{(m-1)}(x)) + \omega^2(w^{(m)}(x) - w^{(m-1)}(x)) =$$

$$= k_F \Psi^{(m)}(x) - \beta^{(m-1)}(x) \quad \text{for} \quad x \in V \tag{48}$$

$$Tw^{(m)}(x_o) + \sigma(x_o)w^{(m)}(x_o) = 0 \quad \text{for} \quad x_o \in S, \tag{49}$$

where

$$\Psi^{(m)}(x) = \Big[\sum_{k=1}^{3} (w_k^{(m)}(x) - w_k^{(m-1)}(x)),$$

$$\sum_{k=1}^{3} (w_k^{(m)}(x) - w_k^{(m-1)}(x)), \quad \sum_{k=1}^{3} (w_k^{(m)}(x) - w_k^{(m-1)}(x)) \Big],$$

then the following inequalities hold true

$$v_k^{(m-1)}(x) \geqslant v_k^{(m)}(x) \geqslant u_k(x), \quad w_k^{(m-1)}(x) \leqslant w_k^{(m)}(x) \leqslant u_k(x).$$

This theorem can be proved similarly, like the theorem 3.

From the theorem 5, by the induction, it follows

$$v_k^{(o)}(x) \geqslant v_k^{(1)}(x) \geqslant \dots \geqslant v_k^{(m)}(x) \geqslant \dots \geqslant u_k(x) \quad \text{for} \quad x \in V$$

$$w_k^{(o)}(x) < w_k^{(1)}(x) \leqslant \dots \leqslant w_k^{(m)}(x) \leqslant \dots \leqslant u_k(x) \quad \text{for} \quad x \in V.$$

5. CONVERGENCE OF THE SEQUENCE OF THE UPPER (LOWER) VECTOR-FUNCTION

The sequences $\{v_k^{(m)}(x)\}$, $\{w_k^{(m)}(x)\}$ are convergent for $m \to \infty$ because they are monotonic and bounded, for exapmle, the sequence $\{v_k^{(m)}(x)\}$ is bounded by the function $H_k(x)$. We shall be able to prove.

T h e o r e m 6. The sequences $\{v_k^{(m)}(x)\}$, $\{w_k^{(m)}(x)\}$ are uniformly convergent for $m \to \infty$ to the solution of the problem (1), (2).

P r o o f. From (46), (47) and (48), (49) after substitution (44) and (45), respectively, we have

$$\Delta^*(v^{(m)}(x)-w^{(m)}(x)) + \omega^2(v^{(m)}(x)-w^{(m)}(x)) =$$

$$= k_F(\Phi^{(m)}(x) - \Psi^{(m)}(x)) + F(x,v^{(m-1)}(x)) - F(x,w^{(m-1)}(x))$$

$$\text{for} \quad x \in V \tag{50}$$

$$T(v^{(m)}(x)-w^{(m)}(x))+\sigma(x_o)(v^{(m)}(x_o)-w^{(m)}(x_o)) = 0 \quad \text{for} \quad x_o \in S. \tag{51}$$

Similarly like in the proof of the theorem 3, we shall write the solution of the problem (50),(51) in the following form

$$v^{(m)}(x)-w^{(m)}(x) = -\frac{k_F}{4\pi}\int_V G(x,y)(\Phi^{(m)}(y)- \Psi^{(m)}(y))\, dV_y + \\ - \frac{1}{4\pi}\int_V G(x,y)\, F(y,v^{(m-1)}(y))-F(y,w^{(m-1)}(y))\, dV_y \tag{52}$$

or

$$v_j^{(m)}(x)-w_j^{(m)}(x) = -\frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\sum_{s=1}^{3}(v_s^{(m)}(y)-w_s^{(m)}(y))dV_y + \\ + \frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y) \sum_{s=1}^{3}(v_s^{(m-1)}(y)-w_s^{(m-1)}(y))\, dV_y + \\ - \frac{1}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)(F_k(y,v_1^{(m-1)}(y),v_2^{(m-1)}(y),v_3^{(m-1)}(y)) + \\ - F_k(y,w_1^{(m-1)}(y),w_2^{(m-1)}(y),w_3^{(m-1)}(y))\quad dV_y . \tag{53}$$

Hence, by the assumption II, we obtain the inequalities

$$v_j^{(m)}(x)-w_j^{(m)}(x) \leq \frac{k_F}{4\pi}\int_V \sum_{k=1}^{3} G_j^{(k)}(x,y)\sum_{s=1}^{3}(v_s^{(m-1)}(y)-w_s^{(m-1)}(y))\, dV_y . \tag{54}$$

Taking the greatest upper hound for the both sides and using the definition (5), we get

$$\sup_{V+S}(v_j^{(m)}-w_j^{(m)}) \leq k_F C_j \sum_{s=1}^{3} \sup (v_s^{(m-1)}-w_s^{(m-1)}). \tag{55}$$

If we introduce the notation $\sum_{s=1}^{3} \sup_{V+S} (v_s^{(o)}-w_s^{(o)}) = L$, we obtain

$$\sup_{V+S} (v_j^{(m)}-w_j^{(m)}) \leq (k_F C)^m L \, . \tag{56}$$

From the inequality (4) it follows that $v_j^{(m)}(x)-v_j^{(m)}(x) \rightrightarrows 0$ for $m \to \infty$. Hence, $v^{(m)}(x) \rightrightarrows u(x)$ and $w^{(m)}(x) \rightrightarrows u(x)$ for $m \to \infty$ and $x \in V+S$.

From (44), (46), (47) we have

$$v^{(m)}(x) = -\frac{k_F}{4\pi}\int_V G(x,y)\, \Phi^{(m)}(y)dV_y - \frac{1}{4\pi}\int_V G(x,y)F(y,v^{(m-1)}(y))dV_y . \tag{57}$$

Now, if we take the limit for the both sides of this formula we get the solution of the problem (1),(2)

$$u(x) = -\frac{1}{4\pi}\int_V G(x,y)F(y,u(y))dV_y \, .$$

This solution is unique.

Suppose, there exist two vector-functions $u'(x)$, $u''(x)$ as the limit of two different sequences of the upper vector-functions. For the difference $u'(x)-u''(x)$, we get

$$u'(x)-u''(x) = -\frac{1}{4\pi}\int_V G(x,y) \; F(y,u'(y))-F(y,u''(y)) \; dV_y \, . \tag{58}$$

Hence for the greatest upper bound of the components we have

$$\sup_{V+S} \left| u'_j-u''_j \right| \leq k_F C_j \sup_{V+S} \sum_{s=1}^{3} \left| u'_s-u''_s \right| . \tag{59}$$

Finally, by virtue of the inequality (4) it follows $u'_j(x) \equiv u''_j(x)$ for $x \in V+S$.

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