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INVESTIGATION OF THE NONLINEAR BOUNDARY VALUE PROBLEM IN THE THEORY OF ELASTICITY

1. INTRODUCTION

Let V be a domain in E_3 bounded by the closed Lapunov surface S. Consider the follwing boundary value problem in the theory of elasticity. Find the displacement vector-function $u(x) = [u_1(x), u_2(x), u_3(x)]$ in the domain V, such that

$$\Delta^* u(x) + \omega^2 u(x) = F(x, u(x)) \quad \text{for } x \in V \tag{1}$$

and

$$Tu(x_0) + \sigma(x_0)\, u(x_0) = G(x_0, u(x_0)) \quad \text{for } x_0 \in S \tag{2}$$

where $\Delta^* \equiv (\lambda + 2\mu)$ grad div $- \mu$rot rot, ω is a real constant, λ, μ are the Lamé's constants, $T \equiv 2\mu \frac{\partial}{\partial n} + \lambda n \operatorname{div} + \mu [n \times \operatorname{rot}]$, $n = [n_1, n_2, n_3]$ is the unit normal vector, $F(x,u) = [F_1(x,u_1,u_2,u_3), F_2(x,u_1,u_2,u_3), F_3(x,u_1,u_2,u_3)]$,

$$G(x_0,u) = [G_1(x_0,u_1,u_2,u_3),\ G_2(x_0,u_1,u_2,u_3),\ G_2(x_0,u_1,u_2,u_3)].$$

The boundary value problem of the type (1), (2) was treated by W.D.K u p r a d z e in the monograph "Potential methods in the theory of elasticity" [1]. He considered the linear problem (i.e. the vector-functions F and G weren't dependent on the displacement vector-function u).

In this paper we shall prove the existence and uniqueness of the solution of the problem (1), (2) by the potential method.

We shall admit the following assumptions

I. S is a closed Lapunov surface with the constants $C, \varkappa$; $C>0$, $0<\varkappa\leqslant 1$ such that

$$(n_{x_0}, n_{y_0}) \leqslant C\,|x_0y_0|^{\varkappa}. \tag{3}$$

II. $F_j(x,u_1,u_2,u_3)$* are real functions, defined on the set $x\in V$, $|u_s|\leqslant R$ and fulfil the Hölder-Lipschitz conditions

$$\left|F_j(x,u_1,u_2,u_3)-F_j(x',u_1',u_2',u_3')\right| \leqslant K_F\,|xx'|^{h_F}+k_F\sum_{s=1}^{3}|u_s-u_s'|, \tag{4}$$

where R, K_F, k_F are positive constants, and $0<h_F\leqslant 1$.

III. $G_j(x_0,u_1,u_2,u_3)$ are real functions, defined on the set $x_0\in S$, $|u_s|\leqslant R$ and fulfil the Hölder-Lipschitz conditions

$$\left|G_j(x_0,u_1,u_2,u_3)-G_j(x_0',u_1',u_2',u_3')\right| \leqslant k_G\Big(|x_0x_0'|^{h_G}+\sum_{s=1}^{3}|u_s-u_s'|\Big). \tag{5}$$

We shall take the additional assumption, that the functions G_j are differentiable with respect to u_k, and their derivatives fulfil the Hölder-Lipschitz conditions

$$\left|\frac{\partial}{\partial u_k}G_j(x_0,u_1,u_2,u_3)-\frac{\partial}{\partial u_k}G_j(\tilde{x}_0,\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\right| \leqslant k_G'\Big(|x_0\tilde{x}_0|^{h_G}+$$
$$+\sum_{s=1}^{3}|u_s-\tilde{u}_s|\Big), \tag{6}$$

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* In this paper the lower and upper index always takes values 1 or 2 or 3.

where k_G, k'_G are positive constants, and $0 < h_G \leq 1$.

IV. The real function $\sigma(x_0)$ is defined for $x_0 \in S$ and satisfies the Hölder condition

$$\left|\sigma(x_0) - \sigma(x'_0)\right| \leq k_\sigma \left|x_0 x'_0\right|^{h_\sigma}, \tag{7}$$

where k_σ is a positive constant, and $0 < h_\sigma \leq 1$.

2. POISSON'S EQUATION

The fundamental solution of the equation

$$\Delta^* u(x) + \omega^2 u(x) = 0$$

is the matrix $\Gamma(x,y)$ with the elements

$$\Gamma_j^{(k)}(x,y) = \frac{1}{b^2}\delta_{kj}\frac{e^{ik_2 r}}{r} - \frac{1}{\omega^2}\frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{e^{ik_1 r}}{r} - \frac{e^{ik_2 r}}{r}\right)^*$$

(here $i^2 = -1$),

where

$$\delta_{kj} = \begin{cases} 1 & k=j \\ 0 & k \neq j, \end{cases} \qquad r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2},$$

$$a^2 = \lambda + 2\mu, \quad b^2 = \mu, \quad k_1^2 = \frac{\omega^2}{a^2}, \quad k_2^2 = \frac{\omega^2}{b^2}.$$

In the monograph [1], it was proved, that the potential

$$U(x) = \frac{1}{4\pi}\int_V \varphi(y)\,\Gamma(x,y)\,dV_y \tag{8}$$

satisfies the Poisson's Equation

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* In the next we shall take the real part of $\Gamma_j^{(k)}(x,y)$.

$$\Delta^* u(x) + \omega^2 u(x) = - \psi(x), \tag{9}$$

where $\psi(x)$ is a differentiable vector-function in V.

Now, we prove that the potential (8) satisfies the equation (9), when $\psi(x)$ fulfils the Hölder condition. We shall base on the method used by W. P o g o r z e l s k i in [2].

Derivatives $\frac{\partial \Gamma_j^{(k)}(x,y)}{\partial x_s}$ have singularity like $\frac{1}{r^2(x,y)}$, and thus, the derivatives $\frac{\partial U_k(x)}{\partial x_s}$ are the absolutely convergent integrales

$$\frac{\partial U_k(x)}{\partial x_s} = \frac{1}{4\pi} \int_V \psi(y) \frac{\partial \Gamma^{(k)}(x,y)}{\partial x_s} dV_y . \tag{10}$$

2.1. INVESTIGATION OF THE SECOND DERIVATIVES OF THE POTENTIAL U(x)

Let us denote by $r(x,y,z)$ the square root

$$r(x,y,z) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 + z^2} .$$

Let $\Gamma(x,y,z)$ be the matrix of the fundamental solution, where we introduce $r(x,y,z)$ instead of $r(x,y)$. Consequently

$$W_k(x,z) = \frac{1}{4\pi} \int \psi(y) \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_s} dV_y , \quad x \in V, \quad -\infty < z < +\infty .$$

We shall prove the follwing theorem

T h e o r e m 1.

If the functions $\psi_j(x)$ for $x \in V$ are bounded and integrable, then

$$W_k(x,z) \longrightarrow \frac{\partial U_k(x)}{\partial x_s}, \quad \text{when} \quad z \longrightarrow 0.$$

P r o o f. Let $\tau(x,R_\tau)$ denote a sphere of the radius R_τ, with the centre at a point $x \in V$. We decompose $W_k(x,z)$ into the sum

$$W_k(x,z) = W_k^{\tau}(x,z) + W_k^{V-\tau}(x,z), \tag{11}$$

where τ denote the region equal to the product of $\tau(x,R_\tau)$ and V, and $V-\tau$ denote over the exterior region. In view of the weak singularity of the function $\frac{\partial \Gamma^{(k)}(x,y)}{\partial x_s}$, we have for every x and z

$$\left| W_k^{\tau}(x,z) \right| < M_\psi C_1 R_\tau , \tag{12}$$

where $M_\psi = \max_j \sup_{y\in\tau} \left|\psi_j(y)\right|$, C_1 is the positive number depending on the Lame's constants and the constant ω. Now, for an arbitrary positive ε we can choose the radius

$$R_\tau = \frac{\varepsilon}{3C_1M_\psi}, \tag{13}$$

such that

$$\left| W_k^{\tau}(x,z)\right| < \frac{\varepsilon}{3}, \tag{14}$$

for every $x \in V$ and $-\infty < z < +\infty$.

The function $W_k^{V-\tau}(x,z)$ is continuous at $z = 0$ (uniformly with respect to the point x), since x is placed outside the region of integration $V-\tau(x,R\ )$. Consequently, having fixed the sphere τ we can choose $\eta(\varepsilon)$ depending only on ε, such that

$$\left|W_k^{V-\tau}(x,z)-W_k^{V-\tau}(x,0)\right| < \frac{\varepsilon}{3}, \quad \text{when} \quad |z| < \eta(\varepsilon). \tag{15}$$

We have

$$\left|\frac{\partial U_k(x)}{\partial x_s} - W_k(x,z)\right| \leqslant \left|W_k^{\tau}(x,z)\right| + \left|W_k^{\tau}(x,0)\right| +$$

$$+ \left|W_k^{V-\tau}(x,z) - W_k^{V-\tau}(x,0)\right|. \qquad (16)$$

In view of the inequalities (14) and (15), we obtain

$$\left|\frac{\partial U_k(x)}{\partial x_s} - W_k(x,z)\right| < \varepsilon, \quad \text{when} \quad |z| < \eta(\varepsilon). \qquad (17)$$

This completes the proof.

T h e o r e m 2.

If the functions $\psi_j(x)$ for $x \in V$ satisfy the Hölder condition

$$\left|\psi_j(x) - \psi_j(y)\right| \leqslant k_\psi \left|xy\right|^{\gamma}, \qquad (18)$$

where $0 < \gamma \leqslant 1$, $k_\psi > 0$, then the functions $W_k(x,z)$ have derivatives with respect to the coordinates of the point x for $x \in V$ and $z \neq 0$, which, in a sufficiently small neighbourhood of every interior point x_0, tend uniformly to the limit

$$\frac{\partial W_k(x,z)}{\partial x_l} \rightrightarrows \frac{\psi(x)}{4\pi} \frac{\partial^2}{\partial x_s \partial x_l} \int_V \Gamma^{(k)}(x,y)\, dV_y +$$

$$+ \frac{1}{4\pi} \int_V \left[\psi(y) - \psi(x)\right] \frac{\partial^2 \Gamma^{(k)}(x,y)}{\partial x_s \partial x_l}\, dV_y, \quad \text{when} \quad z \to 0. \qquad (19)$$

P r o f. For $z \neq 0$ the function $W_k(x,z)$ has,at every interior point $x \in V$, derivatives with respect to the coordinates of x given by the regular integral

$$\frac{\partial W_k(x,z)}{\partial x_l} = \frac{1}{4\pi} \int_V \psi(y) \frac{\partial^2 \Gamma^{(k)}(x,y,z)}{\partial x_s \partial x_l}\, dV_y. \qquad (20)$$

We decompose the last integral into two terms

$$\frac{\partial W_k(x,z)}{\partial x_1} = \frac{\psi(x)}{4\pi}\frac{\partial}{\partial x_1}\int_V \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_s}\, dV_y +$$

$$+\frac{1}{4\pi}\int_V \left[\psi(y) - \psi(x)\right]\frac{\partial^2\Gamma^{(k)}(x,y,z)}{\partial x_s \partial x_1}\, dV_y . \tag{21}$$

From the Green's Theorem, we have

$$\int_V \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_s}\, dV_y = \int_{V-\tau(x_0,\varepsilon)} \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_s}\, dV_y +$$

$$- \int_{\sigma(x_0,\varepsilon)} \Gamma^{(k)}(x,y,z)\cos(n_y,x_s)dS_y \tag{22}$$

where $\sigma(x_0,\varepsilon)$ denotes the surface of the sphere $\tau(x_0,\varepsilon)$.

By virtue of the relation (22), we have

$$\frac{\partial}{\partial x_1}\int_V \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_s}\, dV_y = \int_{V-\tau(x_0,\varepsilon)} \frac{\partial^2\Gamma^{(k)}(x,y,z)}{\partial x_s\partial x_1}\, dV_y +$$

$$- \int_{\sigma(x_0,\varepsilon)} \frac{\partial \Gamma^{(k)}(x,y,z)}{\partial x_1}\cos(n_y,x_s)\, dS_y . \tag{23}$$

Since the centre x_0 of the sphere lies outside the domains of the integration, in view of Theorem 1 and formula (10), the derivative (23) tends uniformly in a certain neighbourhood of the point x_0 to the limit

$$\int_{V-\tau(x_0,\varepsilon)} \frac{\partial^2\Gamma^{(k)}(x,y,0)}{\partial x_s\partial x_1}\, dV_y - \int_{\sigma(x_0,\varepsilon)} \frac{\partial \Gamma^{(k)}(x,y,0)}{\partial x_1}\cos(n_y,x_s)\, dS_y =$$

$$= \frac{\partial^2}{\partial x_s\partial x_1}\int_V \Gamma^{(k)}(x,y)\, dV_y, \quad \text{when} \quad z \to 0. \tag{24}$$

The integrand in the second term of the sum (21) has a weak singularity. Repeating the argument of the proof of Theorem 1, we find that

$$\int_V \left[\psi(y) - \psi(x)\right] \frac{\partial^2 \Gamma^{(k)}(x,y,z)}{\partial x_s \partial x_l}\, dV_y \rightrightarrows \int_V \left[\psi(y) - \psi(x)\right] \frac{\partial^2 \Gamma^{(k)}(x,y)}{\partial x_s \partial x_l}\, dV_y , \tag{25}$$

when $z \to 0$.

This result and (24) imply the thesis of the theorem.

C o r o l l a r y .

The functions $U_k(x)$ have the second derivatives given by the formula

$$\frac{\partial^2 U_k(x)}{\partial x_s \partial x_l} = \frac{\psi(x)}{4\pi} \frac{\partial^2}{\partial x_s \partial x_l} \int_V \Gamma^{(k)}(x,y)\, dV_y + \frac{1}{4\pi} \int_V \left[\psi(y) - \psi(x)\right] \frac{\partial^2 \Gamma^{(k)}(x,y)}{\partial x_s \partial x_l}\, dV_y . \tag{26}$$

The last relation and the formula (1.78) in the book [1] (for $\varphi \equiv 1$) imply the Poisson's Equation

$$\Delta^* U(x) + \omega^2 U(x) = -\psi(x) \quad \text{for} \quad x \in V . \tag{27}$$

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE PROBLEM (1), (2)

We seek a vector-function $u(x)$ (i.e. a solution of the problem (1), (2)) as the sum of the potential of spatial charge and the potential of surface distribution

$$u(x) = -\frac{1}{4\pi}\int_V \Gamma(x,y)F(y,u(y))\, dV_y + \int_S \Gamma(x,y)\varphi(y)\, dS_y, \quad x \in V. \tag{28}$$

We assume that the density $\varphi(y)$ satisfies the Hölder condition on the surface S. If we demand that the vector-function (28) satisfies the boundary condition (2) we obtain the nonlinear strongly singular integral equation

$$2\pi\varphi(x_o) + \int_S \left[T^{(x_o)}\Gamma(x_o,y) + \sigma(x_o)\Gamma(x_o,y)\right]\varphi(y)\, dS_y +$$

$$-\frac{1}{4\pi}\int_V \left[T^{(x_o)}\Gamma(x_o,y) + \sigma(x_o)\Gamma(x_o,y)\right]F(y,u(y))\, dV_y =$$

$$= G\Big(x_o, -\frac{1}{4\pi}\int_V \Gamma(x_o,y)F(y,u(y))\, dV_y +$$

$$+\int_S \Gamma(x_o,y)\varphi(y)dS_y\Big), \qquad x_o \in S.^{*} \tag{29}$$

To prove the existence and uniqueness of the system (28), (29) we apply the Banach's Fixed Point Theorem ([4] p. 37).

Consider the space Λ the points of which are all systems

$$U = [u_1(x),\ u_2(x),\ u_3(x),\ \varphi_1(x_o),\ \varphi_2(x_o),\ \varphi_3(x_o)],$$

of six real functions defined and continuous for $x \in V+S$, $x_o \in S$ and satisfy the inequalities

$$|u_j(x)| \leq R, \quad |\varphi_j(x_o)| \leq \varrho, \quad |\varphi_j(x_o) - \varphi_j(x_o')| \leq k_\varphi |x_o x_o'|^{h_\varphi}, \tag{30}$$

where ϱ and k_φ are arbitrarily fixed positive coefficients.

The exponent h_φ is fixed but satisfies the inequality

$$0 < h_\varphi < \min(\varkappa, h_\sigma, h_G). \tag{31}$$

The distance $\delta(U_1,U_2)$ between two points U_1 and U_2 is defined by

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* $T^{(x)}\Gamma(x,y)$ is defined in [1] p.28.

$$\delta(U_1,U_2) = \max_j \sup_{x\in V+S} \left| u_j^{(1)}(x) - u_j^{(2)}(x) \right| +$$

$$+ \max_j \sup_{x_0\in S} \left| \varphi_j^{(1)}(x_0) - \varphi_j^{(2)}(x_0) \right| +$$

$$+ \max_j H\left[\varphi_j^{(1)}(x_0) - \varphi_j^{(2)}(x_0)\right], \qquad (32)$$

where $H[\varphi_j(x_0)]$ denotes the Hölder coefficient, in an exact meaning (at the exponent h_φ) of the function $\varphi_j(x_0)$, i.e. the upper bound of the following quotient

$$H\left[\varphi_j(x_0)\right] = \sup_{x_0,x_0'\in S} \frac{\left[\varphi_j(x_0) - \varphi_j(x_0')\right]}{\left|x_0x_0'\right|^{h_\varphi}} .$$

The space Λ is metric and complete.

Consider the functional transformation in the space Λ

$$\hat{u}(x) = -\frac{1}{4\pi}\int_V \Gamma(x,y)F(y,u(y))dV_y + \int_V \Gamma(x,y)\,\hat{\varphi}(y)\,dS_y \qquad (33)$$

$$\hat{\varphi}(x_0) + \frac{1}{2\pi}\int_S \left[T^{(x_0)}\Gamma(x_0,y) + \sigma(x_0)\Gamma(x_0,y)\right]\hat{\varphi}(y)dS_y = \frac{1}{2\pi} f(x_0), \qquad (34)$$

where

$$f(x_0) = \frac{1}{4\pi}\int_V \left[T^{(x_0)}\Gamma(x_0,y) + \right.$$

$$\left. + \sigma(x_0)\Gamma(x_0,y)\right]F(y,u(y))dV_y + G(x_0,\bar{u}(x_0)) \qquad (35)$$

$$\bar{u}(x_0) = -\frac{1}{4\pi}\int_V \Gamma(x_0,y)F(y,u(y))dV_y + \int_S \Gamma(x_0,y)\,\varphi(y)dS_y\,. \qquad (36)$$

The kernel of the integral equation (34) has the strong singularity. In [1] on p.103-161 W.D. K u p r a d z e proved that for the integral equations of the type (34) with the linear right side Fredholm Theorems hold true. He based on the regularization method proposed by G. Giraud ([5]), and the definition of the symbolic matrix which is given by S.G. Mikhlin in the paper [6].

Since the homogeneous equation

$$\hat{\varphi}(x_o) + \frac{1}{2\pi}\int_S \left[T^{(x_o)}\,\Gamma(x_o,y) + \sigma(x_o)\,\Gamma(x_o,y)\right]\hat{\varphi}(y)dS_y = 0,$$

possesses only zero solution, then the equation (34) has the unique solution (we assume that the vector-function $f(x_o)$ satisfies the Hölder condition), which is of the form

$$\hat{\varphi}(x_o) = \frac{1}{2\pi}B(x_o)f(x_o) + \frac{1}{2\pi}\int_S N(x_o,\xi)f(\xi)dS_\xi\,, \tag{37}$$

where the elements of the matrix resolvent $N(x_o,\xi)$ have the singularity $\frac{1}{|x_o\xi|^2}$, the matrix $B(x_o)$ is defined in the paper [7].

In formulaes (33), (34) there are the potential of spatial charge and the potential of surface distribution. In this paper we are going to base on the two theorems, which can be proved similarly as the well known theorems relative to the Newton's Potentials.

Theorem 3.

If the functions $F_k(x,u_1(x),u_2(x),u_3(x))$ for $x\in V$ are bounded and integrable then the functions

$$U_j(x) = \frac{1}{4\pi}\sum_{k=1}^{3}\int_V \Gamma_j^{(k)}(x,y)F_k(y,u_1(y),u_2(y),u_3(y))dV_y \tag{38}$$

$$T_jU(x) = \frac{1}{4\pi}\sum_{k=1}^{3}\int_V T_j^{(x)}\,\Gamma^{(k)}(x,y)F_k(y,u_1(y),u_2(y),u_3(y))dV_y\,, \tag{39}$$

satisfy the inequalities

$$\left|U_j(x)\right| \leqslant p_1M_F \tag{40}$$

$$\left|T_jU(x)\right| \leqslant p_2M_F \tag{41}$$

and the Hölder conditions

$$\left| U_j(x) - U_j(x') \right| \leq q_1 M_F \left| xx' \right| \qquad (42)$$

$$\left| T_j U(x) - T_j U(x') \right| \leq q_2 M_f \left| xx' \right| \theta , \qquad (43)$$

where $M_F = \max_k \sup_{x \in V+S} \left| F_k(x, u_1(x), u_2(x), u_3(x)) \right|$, p_1, p_2, q_1, q_2 are positive numbers which aren't depend from the function F_k, θ is a positive number and satisfies the inequality $0 < \theta < 1$.

T h e o r e m 4.

If the functions $\varphi_k(x_o)$ for $x_o \in S$ are bounded and integrable then the functions

$$V_j(x_o) = \sum_{k=1}^{3} \int_S \Gamma_j^{(k)}(x_o, y)\, \varphi_k(y) dS_y , \qquad (44)$$

satisfy the inequalities

$$\left| V_j(x_o) \right| \leq p_3\, \varrho \qquad (45)$$

and the Hölder conditions

$$\left| V_j(x_o) - V_j(x_o') \right| \leq q_3 \varrho \left| x_o x_o' \right|^{\theta} , \qquad (46)$$

where $0 < \theta < 1$, p_3, q_3 are positive numbers which aren't depending on the function $\varphi_k(x_o)$.

Now, we present some definitions

$$M_B = \max_{j,k} \sup_{x_o \in S} \left| B_j^{(k)}(x_o) \right| , \quad M_N = \max_{j,k} \sup_{x_o \in S} \left| \int_S N_j^{(k)}(x_o, \xi) dS_\xi \right|$$

(the integrals $\int_S N_j^{(k)}(x_o, \xi) dS_\xi$ are taken in the sense of the Cauchy principal value), $M_G = \max_k \sup_{x_o \in S} \left| G_k(x_o, u_1, u_2, u_3) \right|$, $M_\sigma = \sup_{x_o \in S} \left| \sigma(x_o) \right|$.

We shall prove the following lemma

Lemma 1.

If the numbers M_F, M_G, k_G are sufficiently small, the number ϱ is sufficiently large, then the transformation (33), (34) associates with every point of the space Λ a point of the same space.

Proof. We first prove that the functions $f_k(x_o)$ satisfy the Hölder condition. In view of the formula (35) we can write

$$f_k(\xi) - f_k(x_o) = \frac{1}{4\pi}\sum_{s=1}^{3}\int_V\Big[T_k^{(\xi)}\Gamma^{(s)}(\xi,y) + - T_k^{(x_o)}\Gamma^{(s)}(x_o,y)\Big]F_s(y,u_1(y),u_2(y),u_2(y),u_3(y))dV_y +$$

$$+\frac{1}{4\pi}\Big[\sigma(\xi)-\sigma(x_o)\Big]\sum_{s=1}^{3}\int_V\Gamma_k^{(s)}(\xi,y)F_s(y,u_1(y),u_2(y),u_3(y))\,dV_y +$$

$$+\frac{1}{4\pi}\sigma(x_o)\sum_{s=1}^{3}\int_V\Big[\Gamma_k^{(s)}(\xi,y)-\Gamma_k^{(s)}(x_o,y)\Big]F_s(y,u_1(y),u_2(y),u_3(y))dVy+$$

$$+ G_k(\xi,\bar u_1(\xi),\bar u_2(\xi),\bar u_3(\xi)) - G_k(x_o,\bar u_1(x_o),\bar u_2(x_o),\bar u_3(x_o)). \quad (47)$$

To estimate the righ side we prove two inequalities. By virtue of (36), we have

$$\bar u_k(\xi) - \bar u_k(x_o) = -\frac{1}{4\pi}\sum_{s=1}^{3}\int_V\Big[\Gamma_k^{(s)}(\xi,y) + - \Gamma_k^{(s)}(x_o,y)\Big]F_s(y,u_1(y),u_2(y),u_3(y))\,dV_y +$$

$$+\sum_{s=1}^{3}\int_S\Big[\Gamma_k^{(s)}(\xi,y) - \Gamma_k^{(s)}(x_o,y)\Big]\,\varphi_s(y)\,dS_y\,. \quad (48)$$

On the basis of the inequalities (42),(46), this implies that

$$\left|\bar{u}_k(\xi)-\bar{u}_k(x_o)\right| \leqslant (q_1M_F + q_3\varrho)\ \left|x_o\xi\right|^{\theta}. \tag{49}$$

This inequality and the assumption III imply the second inequality

$$\left|G_k(\xi,\bar{u}_1(\xi),\bar{u}_2(\xi),\bar{u}_3(\xi)) - G_k(x_o,\bar{u}_1(x_o),\bar{u}_2(x_o),\bar{u}_3(x_o))\right| \leqslant$$

$$\leqslant k_G(1+3q_1M_F+3q_3\varrho)\ \left|x_o\xi\right|^{h_G}. \tag{50}$$

Hence and from the inequalities (31), (40), (42),(43),assumption IV, finally, we have the conditions

$$\left|f_k(\xi)-f_k(x_o)\right| \leqslant \left[q_2M_F+p_1M_Fk_\sigma+q_1M_\sigma M_F+k_G(1+3q_1M_F+3q_3\varrho)\right]\left|x_o\xi\right|^{h_\varphi}. \tag{51}$$

Now, in view of the formula (37) we can write the inequalities

$$\left|\hat{\varphi}_j(x_o)\right| \leqslant \frac{1}{2\pi}\sum_{k=1}^{3}\left|B_j^{(k)}(x_o)f_k(x_o)\right| +$$

$$+\frac{1}{2\pi}\sum_{k=1}^{3}\left|\int_S N_j^{(k)}(x_o,\xi)\left[f_k(\xi) - f_k(x_o)\right]\,dS_y\right| +$$

$$+\frac{1}{2\pi}\sum_{k=1}^{3}\left|f_k(x_o)\right|\ \left|\int_S N_j^{(k)}(x_o,\xi)dS_\xi\right|. \tag{52}$$

From here and from the inequalities (40), (41), (51) we obtain the inequalities

$$\left|\hat{\varphi}_j(x_o)\right| \leqslant \frac{3}{2\pi}\Big\{C_2\left[q_3M_F+p_1M_Fk_\sigma+q_1M_\sigma M_F+k_G(1+3q_1M_F+3q_3\varrho)\right] +$$

$$+ (M_B+M_N)(p_2M_F+p_1M_\sigma M_F+M_G)\Big\}. \tag{53}$$

To prove, that the functions $\hat{\varphi}_j(x_o)$ satisfy the Hölder condition we estimate the following expressions

$$\hat{\varphi}_j(x_o) - \hat{\varphi}_j(x_o') = \frac{1}{2\pi}\sum_{k=1}^{3}\left[B_j^{(k)}(x_o) - B_j^{(k)}(x_o')\right] f_k(x_o) +$$

$$+ \frac{1}{2\pi}\sum_{k=1}^{3} B_j^{(k)}(x_o')\left[f_k(x_o) - f_k(x_o')\right] + \frac{1}{2\pi}\left[D_j(x_o) - D_j(x_o')\right], \quad (54)$$

where

$$D_j(x_o) = \sum_{k=1}^{3}\int_S N_j^{(k)}(x_o,\xi) f_k(\xi)dS_\xi \ . \quad (55)$$

By virtue of the inequality (51) from this paper and (26), (28) from the paper [7], we have

$$\left|\hat{\varphi}_j(x_o) - \hat{\varphi}_j(x_o')\right| \leq \frac{3}{2\pi}\Big\{C_3 CM_F(p_2+p_1M_\sigma)+C_3CM_G +$$

$$+ (M_B+k_N)\left[q_2M_F+p_1M_Fk_\sigma+q_1M_\sigma M_F+k_G(1+3q_1M_F+3q_3\varrho)\right]\Big\}\left|x_ox_o'\right|^{h\varphi}, \quad (56)$$

where the positive numbers C_3, k_N depend on the Lame's constants and the constant ω .

From the formula (33) and the inequalities (40),(45),(53) it follows

$$\left|\hat{u}_j(x)\right| \leq p_1M_F + \frac{3}{2\pi}\, p_3\Big\{C_2\left[q_2M_F+p_1M_Fk_\sigma+q_1M_GM_F+k_G(1+3q_1M_F+3q_3\varrho)\right]+$$

$$+(M_B+M_N)(p_2M_F+p_1M_\sigma M_F+M_G)\Big\} \ . \quad (57)$$

On the basis of the inequalities (53), (56), (57) it implies, that the transformation (33), (34) associates with every point of the space Λ a point of the same space and this is the sufficient condition, if the following system of the inequalities is fulfiled

$$M_F(a_1+a_2k_\sigma+a_3M_\sigma+a_4k_G\varrho)+a_6k_G+a_7M_G \leq R \ ,$$

$$M_F(b_1+b_2k_\delta+b_3M_\delta+b_4k_G)+k_G(b_5+b_6\varrho)+b_7M_G \leq \varrho, \qquad (58)$$

$$M_F(c_1+c_2k_\delta+c_3M_\delta+c_4k_G)+k_G(c_5+c_6\varrho)+c_7M_G \leq k_\varphi,$$

where the positive numbers $a_1,\dots,a_7$, $b_1,\dots,b_7$, $c_1,\dots,c_7$ depend on the Lame's constants and the constant ω.

We take the number ϱ equal to ϱ_0 so that for the sufficiently small numbers M_F, M_G, and $k_G < \frac{2\pi}{9C_2q_3}$ the second inequality (58) is fulfiled. The remaining inequalities are fulfiled on the basis of the assumptions for numbers M_F, M_G, k_G.

Thus, lemma 1 is proved.

L e m m a 2 (Hadamard's Lemma)

If the functions $G_k(x,u_1,u_2,u_3)$ satisfy the assumption III, then the differences $\Delta G_k = G_k(x,\bar{u}_1,\bar{u}_2,\bar{u}_3)-G_k(x,u_1,u_2,u_3)$ can be written as the sum the products

$$\Delta G_k = g_k^{(1)}(x,u_1,\bar{u}_1,\bar{u}_2,\bar{u}_3)(\bar{u}_1-u_1) + g_k^{(1)}(x,u_1,u_2,\bar{u}_2,\bar{u}_3)(\bar{u}_2-u_2)+$$

$$+ g_k^{(3)}(x,u_1,u_2,u_3,\bar{u}_3)(\bar{u}_3-u_3), \qquad (59)$$

where all of the functions $g_k^{(s)}(x,t_1,t_2,t_3,t_4)$ satisfy the Hölder-Lipschitz condition

$$\left|g_k^{(s)}(x,t_1,t_2,t_3,t_4)-g_k^{(s)}(x',t'_1,t'_2,t'_3,t'_4)\right| \leq k'_G\left(|xx'|^{h_G}+\sum_{j=1}^{4}|t_j-t'_j|\right). \qquad (60)$$

The proof of this lemma is similar as the proof of the lemma 1 in the paper [8] on p.106.

Let us $U_1 = [u_1^{(1)}(x), u_2^{(1)}(x), u_3^{(1)}(x), \varphi_1^{(1)}(x_0), \varphi_2^{(1)}(x_0), \varphi_3^{(1)}(x_0)]$, $U_2 = [u_1^{(2)}(x), u_2^{(2)}(x), u_3^{(2)}(x), \varphi_1^{(2)}(x_0), \varphi_2^{(2)}(x_0), \varphi_3^{(2)}(x_0)]$ be points of the space $\mathbf{\Lambda}$. Let $\hat{U}_1, \hat{U}_2$ be the images of the points U_1, U_2 after the transformation (33),(34).

L e m m a 3.

Let α be the number

$$\alpha = \max \left\{ k_F \left[A_1+A_2k_G+A_3k_\sigma+A_4M_\sigma+k'_G(A_5+A_6M_F+A_7\varrho_0)\right], \right.$$
$$\left. A_8k_G+k'_G(A_9+A_{10}M_F+A_{11}\varrho_0)\right\}, \quad (61)$$

where the positive constants $A_1,\ldots,A_{11}$ depend on the Lame's constant ω. If the numbers k_F,k_G,k'_G are sufficiently small so that the inequality

$$\alpha < 1 \quad (62)$$

and the inequalities (58) are fulfiled, then

$$\delta\,(\hat{U}_1,\hat{U}_2) \leqslant \alpha\,\delta\,(U_1,U_2). \quad (63)$$

P r o o f. We first prove some inequalities. From the formula (36) we have

$$\bar{u}_j^{(1)}(x)-\bar{u}_j^{(2)}(x) = -\frac{1}{4\pi}\sum_{k=1}^{3}\int_V \Gamma_j^{(k)}(x,y)\left[F_k(y,u_1^{(1)}(y),u_2^{(1)}(y),u_3^{(1)}(y)) + \right.$$
$$\left. - F_k(y,u_1^{(2)}(y),u_2^{(2)}(y),u_3^{(2)}(y))\right]dV_y +$$
$$+\sum_{k=1}^{3}\int_S \Gamma_j^{(k)}(x,y)\left[\varphi_k^{(1)}(y)-\varphi_k^{(2)}(y)\right]dS_y. \quad (64)$$

By virtue of the assumption II and the inequalities (40), (45), we obtain

$$\left|\bar{u}_j^{(1)}(x)-\bar{u}_j^{(2)}(x)\right| \leqslant 3p_1k_F \max_k \sup\left|u_k^{(1)}-u_k^{(2)}\right| + 3p_3 \max_k \sup\left|\varphi_k^{(1)}-\varphi_k^{(2)}\right|. \quad (65)$$

From the formula (34) we have

$$\hat{\varphi}^{(1)}(x_o)-\hat{\varphi}^{(2)}(x_o)+\frac{1}{2\pi}\int_S\left[T^{(x_o)}\Gamma(x_o,y)+6(x_o)\Gamma(x_o,y)\right]\left[\hat{\varphi}^{(1)}(y)-\hat{\varphi}^{(2)}(y)\right]dS_y =$$
$$=\frac{1}{2\pi}\bar{f}(x_o), \quad (66)$$

where

$$\bar{f}(x_o) = \frac{1}{4\pi}\int_V \left[T^{(x_o)}\Gamma(x_o,y)+\sigma(x_o)\Gamma(x_o,y)\right]\left[F(y,u^{(1)}(y))-F(y,u^{(2)}(y))\right]dV_y +$$

$$+ G(x_o,\bar{u}^{(1)}(x_o)) - G(x_o,\bar{u}^{(2)}(x_o)), \tag{67}$$

We shall prove that the functions $f_k(x_o)$ satisfy the Hölder condition. Now, we are going to investigate the differences

$$\bar{f}_k(\xi) - \bar{f}_k(x_o) = J_k^{(1)} + J_k^{(2)} + J_k^{(3)} + J_k^{(4)}, \tag{68}$$

where

$$J_k^{(1)} = \frac{1}{4\pi}\sum_{s=1}^{3}\int_V\left[T_k^{(\xi)}\Gamma^{(s)}(\xi,y)-T_k^{(x_o)}\Gamma^{(s)}(x_o,y)\right]\left[F_s(y,u_1^{(1)}(y),u_2^{(1)}(y),\right.$$

$$\left. u_3^{(1)}(y)) - F_s(y,u_1^{(2)}(y),u_2^{(2)}(y),u_3^{(2)}(y))\right]dV_y$$

$$J_k^{(2)} = \frac{1}{4\pi}\left[\sigma(\xi)-\sigma(x_o)\right]\sum_{s=1}^{3}\int_V \Gamma_k^{(s)}(\xi,y)\left[F_s(y,u_1^{(1)}(y),u_2^{(1)}(y),u_3^{(1)}(y)) +\right.$$

$$\left. - F_s(y,u_1^{(2)}(y),u_2^{(2)}(y),u_3^{(2)}(y))\right]dV_y$$

$$J_k^{(3)} = \frac{1}{4\pi}\sigma(x_o)\sum_{s=1}^{3}\int_V\left[\Gamma_k^{(s)}(\xi,y)-\Gamma_k^{(s)}(x_o,y)\right]\left[F_s(y,u_1^{(1)}(y),u_2^{(1)}(y),\right.$$

$$\left. u_3^{(1)}(y)) - F_s(y,u_1^{(2)}(y),u_2^{(2)}(y),u_3^{(2)}(y))\right]dV_y$$

$$J_k^{(4)} = \left[G_k(\xi,\bar{u}_1^{(1)}(\xi),\bar{u}_2^{(1)}(\xi),\bar{u}_3^{(1)}(\xi)) - G_k(\xi,\bar{u}_1^{(2)}(\xi),\bar{u}_2^{(2)}(\xi),\bar{u}_3^{(2)}(\xi))\right]+$$

$$-\left[G_k(x_o,\bar{u}_1^{(1)}(x_o),\bar{u}_2^{(1)}(x_o),\bar{u}_3^{(1)}(x_o))-G_k(x_o,\bar{u}_1^{(2)}(x_o),\bar{u}_2^{(2)}(x_o),\bar{u}_3^{(2)}(x_o))\right].$$

In view of the assumptions II, IV and the inequalities (40), (42), (43) we get

$$\left|J_k^{(1)}\right| \leq 3q_2k_F \max_j \sup \left|u_j^{(1)} - u_j^{(2)}\right| \left|x_o\xi\right|^{\theta} \tag{69}$$

$$\left|J_k^{(2)}\right| \leq 3p_1k_\delta k_F \max_j \sup \left|u_j^{(1)} - u_j^{(2)}\right| \left|x_o\xi\right|^{h_G} \tag{70}$$

$$\left|J_k^{(3)}\right| \leq 3q_1M_\delta k_F \max_j \sup \left|u_j^{(1)} - u_j^{(2)}\right| \left|x_o\xi\right| . \tag{71}$$

To estimate $J_k^{(4)}$ we apply the lemma 2. From the formula (59) it follows

$$J_k^{(4)} = \sum_{s=1}^{3} \left[g_k^{(s)}(\xi)-g_k^{(s)}(x_o)\right]\left[\bar{u}_s^{(1)}(\xi)-\bar{u}_s^{(2)}(\xi)\right]+$$

$$+\sum_{s=1}^{3} g_k^{(s)}(x_o)\left[(\bar{u}_s^{(1)}(\xi)-\bar{u}_s^{(2)}(\xi))-(\bar{u}_s^{(1)}(x_o)-\bar{u}_s^{(2)}(x_o))\right], \tag{72}$$

where

$$g_k^{(1)}(x) = g_k^{(1)}(x,\bar{u}_1^{(1)}(x),\bar{u}_1^{(2)}(x),\bar{u}_2^{(2)}(x),\bar{u}_3^{(2)}(x))$$

$$g_k^{(2)}(x) = g_k^{(2)}(x,\bar{u}_1^{(1)}(x),\bar{u}_2^{(1)}(x),\bar{u}_2^{(2)}(x),\bar{u}_3^{(2)}(x))$$

$$g_k^{(3)}(x) = g_k^{(3)}(x,\bar{u}_1^{(1)}(x),\bar{u}_2^{(1)}(x),\bar{u}_3^{(1)}(x),\bar{u}_3^{(2)}(x)) .$$

The inequalities (31),(49),(60) imply

$$\left| g_k^{(s)}(\xi)-g_k^{(s)}(x_o)\right| \leq k_G'(e_1+e_2M_F+e_3\varrho_o) \left|x_o\xi\right|^{h\varphi}, \tag{73}$$

where the positive numbers e_1, e_2, e_3 depend on the Lamé's constants and the constant ω . The functions $g_k^{(s)}$ are bounded

$$\left| g_k^{(s)}(x_o)\right| < k_G . \tag{74}$$

In view of the assumption II, the formula (36) and the inequalities (42),(46) we get

$$\left|(\bar{u}_s^{(1)}(\xi)-\bar{u}_s^{(2)}(\xi))-(\bar{u}_s^{(1)}(x_o)-\bar{u}_s^{(2)}(x_o))\right| \leqslant$$

$$\leqslant 3q_1k_F \max_j \sup\left|u_j^{(1)}-u_j^{(2)}\right|\left|x\xi\right|+3q_3 \max_j \sup\left|\varphi_j^{(1)}-\varphi_j^{(2)}\right|\left|x_o\xi\right|^{\theta}. \quad (75)$$

By virtue of the formula (72) and from the inequalities (65),(73),(74),(75) we have

$$\left|J_k^{(4)}\right| \leqslant \Big\{ 9k_F\left[p_1k_G'(e_1+e_2M_F+e_3\varrho_o)+q_1k_G\right] \max_j \sup\left|u_j^{(1)}-u_j^{(2)}\right| +$$

$$+9\left[p_3k_G'(e_1+e_2M_F+e_3\varrho_o)+q_3k_G\right] \max_j \sup\left|\varphi_j^{(1)}-\varphi_j^{(2)}\right|\Big\} \left|x_o\xi\right|^{h\varphi}. \quad (76)$$

Finally, from the definition of the exponent h_φ and from the inequalities (69),(70),(71),(76) we have

$$\left|\bar{f}_k(\xi)-\bar{f}_k(x_o)\right| \leqslant$$

$$\leqslant \Big\{3k_F\left[q_2+p_1k_\sigma+q_1M_\sigma+3q_1k_G+3p_1k_G'(e_1+e_2M_F+e_3\varrho_o)\right] \max_j \sup\left|u_j^{(1)}-u_j^{(2)}\right| +$$

$$+9\left[p_3k_G'(e_1+e_2M_F+e_3\varrho_o)+q_3k_G'\right] \max_j \sup\left|\varphi_j^{(1)}-\varphi_j^{(2)}\right|\Big\} \left|x_o\xi\right|^{h\varphi}. \quad (77)$$

Hence, since the functions $\bar{f}_k(x_o)$ satisfy the Hölder condition, the integral equation (66) has the unique solution

$$\hat{\varphi}^{(1)}(x_o)-\hat{\varphi}^{(2)}(x_o) = \frac{1}{2\pi} B(x_o)\bar{f}(x_o)+\frac{1}{2\pi}\int_S N(x_o,\xi)\bar{f}(\xi)dS_\xi . \quad (78)$$

According to the assumption III, the inequalities (40) and the formula (67) we get the estimate

$$\left|\bar{f}_k(x_o)\right| \leqslant 3p_1k_F(1+3k_G) \max_j \sup\left|u_j^{(1)}-u_j^{(2)}\right|+9p_3k_G \max_j \sup\left|\varphi_j^{(1)}-\varphi_j^{(2)}\right| \quad (79)$$

From the formula (78) we have

$$\left|\hat{\varphi}_j^{(1)}(x_o)-\hat{\varphi}_j^{(2)}(x_o)\right| \leq \frac{1}{2\pi}\sum_{k=1}^{3}\left|B_j^{(k)}(x_o)\right|\left|\bar{f}_k(x_o)\right| +$$

$$+\frac{1}{2\pi}\sum_{k=1}^{3}\left|\int_S N_j^{(k)}(x_o,\xi)\left[\bar{f}_k(\xi)-\bar{f}_k(x_o)\right]dS_\xi\right|+\frac{1}{2\pi}\sum_{k=1}^{3}\left|\bar{f}_k(x_o)\right|\left|\int_S N_j^{(k)}(x_o,\xi)\,dS_\xi\right|. \tag{80}$$

From the inequalities (77),(79),(80) it follows

$$\left|\hat{\varphi}_j^{(1)}(x_o)-\hat{\varphi}_j^{(2)}(x_o)\right| \leq \frac{9}{2\pi}k_F\Big\{p_1(M_B+M_N)(1+3k_G)+C_2\big[q_2+p_1k_\sigma+q_1M_\sigma+3q_1k_G +$$

$$+3p_1k_G'(e_1+e_2M_F+e_3\rho_o)\big]\Big\}\max_k \sup\left|u_k^{(1)}-u_k^{(2)}\right| +$$

$$+\frac{27}{2\pi}\Big\{p_3k_G(M_B+M_N)+C_2\big[p_3k_G'(e_1+e_2M_F+e_3\rho_o)+q_3k_G\big]\Big\}\max_k \sup\left|\varphi_k^{(1)}-\varphi_k^{(2)}\right|. \tag{81}$$

From the equation (33) we have

$$\hat{u}_j^{(1)}(x)-\hat{u}_j^{(2)}(x) = -\frac{1}{4\pi}\sum_{k=1}^{3}\int_V \Gamma_j^{(k)}(x,y)\Big[F_k(y,u_1^{(1)}(y),u_2^{(1)}(y),u_3^{(1)}(y)) +$$

$$-F_k(y,u_1^{(2)}(y),u_2^{(2)}(y),u_3^{(2)}(y))\Big]dV_y +$$

$$+\sum_{k=1}^{3}\int_S \Gamma_j^{(k)}(x,y)\left[\hat{\varphi}_k^{(1)}(y)-\hat{\varphi}_k^{(2)}(y)\right]dS_y . \tag{82}$$

According to the assumption II, the inequalities (40), (45),(81) we obtain

$$\left|\hat{u}_j^{(1)}(x)-\hat{u}_j^{(2)}(x)\right| \leq \frac{9}{2\pi}k_F\Big\{\frac{2\pi}{3}p_1+p_1(M_B+M_N)(1+3k_G)+C_2\big[q_2+p_1k_\sigma+q_1M_\sigma+3q_1k_G +$$

$$+3p_1k_G'(e_1+e_2M_F+e_3\rho_o)\big]\Big\}\max_k \sup\ u_k^{(1)}-u_k^{(2)}\Big| +$$

$$+\frac{81}{2\pi}p_3\Big\{p_3k_G(M_B+M_N)+C_2\big[p_3k_G'(e_1+e_2M_F+e_3\rho_o)+q_3k_G\big]\Big\}\max_k \sup\left|\varphi_k^{(1)}-\varphi_k^{(2)}\right|. \tag{83}$$

Now, we estimate the Hölder coefficient $H\left[\hat{\varphi}_j^{(1)}(x_o) + -\hat{\varphi}_j^{(2)}(x_o)\right]$. From the formula (78) we have

$$\left[\hat{\varphi}_j^{(1)}(x_o)-\hat{\varphi}_j^{(2)}(x_o)\right]-\left[\hat{\varphi}_j^{(1)}(x_o')-\hat{\varphi}_j^{(2)}(x_o')\right]=\frac{1}{2\pi}\sum_{k=1}^{3}\left[B_j^{(k)}(x_o)-B_j^{(k)}(x_o')\right]\bar{f}_k(x_o)+$$

$$+\frac{1}{2\pi}\sum_{k=1}^{3}B_j^{(k)}(x_o')\left[\bar{f}_k(x_o)-\bar{f}_k(x_o')\right]+\frac{1}{2\pi}\left[\bar{D}_j(x_o)-\bar{D}_j(x_o')\right], \tag{84}$$

where

$$\bar{D}_j(x_o) = \sum_{k=1}^{3}\int_S N_j^{(k)}(x_o,\xi)\bar{f}_k(\xi)dS_\xi. \tag{85}$$

Consequently, we obtain the inequalities

$$H\left[\hat{\varphi}_j^{(1)}(x_o)-\hat{\varphi}_j^{(2)}(x_o)\right]\leqslant\frac{1}{2\pi}\sum_{k=1}^{3}H\left[B_j^{(k)}(x_o)\right]\left|\bar{f}_k(x_o)\right| +$$

$$+\frac{1}{2\pi}\sum_{k=1}^{3}\left|B_j^{(k)}(x_o')\right| H\left[\bar{f}_k(x_o)\right]+\frac{1}{2\pi}H\left[\bar{D}_j(x_o)\right]. \tag{86}$$

By virtue of the inequalities (26), (28) from the paper [7] and the inequality (79), it follows

$$H\left[B_j^{(k)}(x_o)\right]\leqslant C_3C \tag{87}$$

$$H\left[\bar{D}_j(x_o)\right] \leqslant 3p_1k_Nk_F(1+3k_G)\max_k\sup\left|u_k^{(1)}-u_k^{(2)}\right| +$$

$$+3p_3k_Nk_G\max_k\sup\left|\varphi_k^{(1)}-\varphi_k^{(2)}\right|. \tag{88}$$

Finally, from the inequalities (77),(79),(86), (87), (88) we have

$$H\left[\hat{\varphi}_j^{(1)}(x_o)-\hat{\varphi}_j^{(2)}(x_o)\right]\leqslant\frac{27}{2\pi}\Big\{C_3Cp_1k_F(1+3k_G)+M_Bk_F\left[q_2+p_1k_\delta+q_1M_\delta+\right.$$

$$+3q_1k'_G+3p_1k'_G(e_1+e_2M_F+e_3\varrho_o)+\frac{1}{3}p_1k_Nk_F(1+3k_G)\Big\}\max_k\sup\left|u_k^{(1)}-u_k^{(2)}\right|+$$

$$+\frac{27}{2\pi}\Big\{C_3Cp_3k_G+3M_B\left[p_3k'_G(e_1+e_2M_F+e_3\varrho_o)+q_3k_G\right]+$$

$$+\frac{1}{3}p_3k_Nk_G\Big\}\max_k\sup\left|\varphi_k^{(1)}-\varphi_k^{(2)}\right|. \qquad (89)$$

Let us denote

$$A_1=3p_1+\frac{9}{2\pi}\left[p_1(2M_B+2M_N+3C_3C+k_N)+q_2(2C_2+3M_B)\right],$$

$$A_2=\frac{27}{2\pi}\left[p_1(2M_B+2M_N+3C_3C+k_N)+q_1(2C_2+3M_B)\right],\ A_3=\frac{9}{2\pi}p_1(2C_2+3M_B),$$

$$A_4=\frac{9}{2\pi}q_1(2C_2+3M_B),\ A_5=\frac{27}{2\pi}p_1e_1(2+3M_B),\ A_6=\frac{27}{2\pi}p_1e_2(2+3M_B),$$

$$A_7=\frac{27}{2\pi}p_1e_3(2+3M_B),$$

$$A_8=\frac{27}{2\pi}\left\{(3p_3+1)\left[p_3(M_B+M_N)+C_3q_3\right]+p_3(C_3C+\frac{1}{3}k_N)+3M_Bq_3\right\},$$

$$A_9=\frac{27}{2\pi}p_3e_1\left[C_2(3p_3+1)+3M_B\right],\quad A_{10}=\frac{27}{2\pi}p_3e_2\left[C_2(3p_3+1)+3M_B\right],$$

$$A_{11}=\frac{27}{2\pi}p_3e_3\left[C_2(3p_3+1)+3M_B\right].$$

Then, in view of the inequalities (81), (83), (89), the defition (32) and the assumption respectively the numbers k_F, k_G, k'_G it follows the lemma is true.

We conclude from the lemma 1 and 3, by the Banach's Fixed Point Theorem, that the system of the integral equations (28), (29) has the unique solution $u^*(x)$, $\varphi^*(x_0)$ in the space $\boldsymbol{\Lambda}$.

Remark. The solution $u^*(x)$, $\varphi^*(x_0)$ can be appoint by the iterative method in the space $\boldsymbol{\Lambda}$.

From the integral equations (28),(29) it follows that the vector-function $u^*(x)$ satisfies the boundary condition (2) in all points of the surface S. Owing to the continuity of the vector-function $F(y,u^*(y))$ the integral $\int_V \Gamma(x,y)F(y,u^*(y))dV_y$ satisfies the Hölder condition.

Hence, the vector-function $F(y,u^*(y))$ satisfies the Hölder condition for $y \in V$. According to the corollary from the theorem 2, the second derivatives of the vector-function $u^*(x)$ exist and $u^*(x)$ satisfies the equation (1) for $x \in V$.

Finally, we can formulate the theorem

T h e o r e m 5.

If the assumptions I-IV are fulfiled and the numbers M_F, M_G, k_F, k_G, k'_G are sufficiently small, then there exists only one regular vector-function in the form of the sum (28) which satisfies the equation (1) and the boundary condition (2). This solution $u(x)$ can be appoint by the iterative of the equations

$$u^{(m+1)}(x) = -\frac{1}{4\pi}\int_V \Gamma(x,y)F(y,u^{(m)}(y))dV_y + \int_S \Gamma(x,y)\,\varphi^{(m+1)}(y)dS_y$$

$$2\pi\varphi^{(m+1)}(x_o) + \int_S \left[T^{(x_o)}\Gamma(x_o,y) + \sigma(x_o)\Gamma(x_o,y)\right]\varphi^{(m+1)}(y)dS_y =$$

$$= \frac{1}{4\pi}\int_V \left[T^{(x_o)}\Gamma(x_o,y) + \sigma(x_o)\Gamma(x_o,y)\right]F(y,u^{(m)}(y))dV_y + G(x_o,\bar{u}^{(m)}(x_o)),$$

where

$$\bar{u}^{(m)}(x_o) = -\frac{1}{4\pi}\int_V \Gamma(x_o,y)F(y,u^{(m)}(y))dV_y + \int_S \Gamma(x_o,y)\varphi^{(m)}(y)dS_y.$$

As $u^{(o)}(x)$, $\varphi^{(o)}(x_o)$ we take an arbitrary point from the space Λ.

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BIBLIOGRAPHY

[1] W.D. K u p r a d z e (В.Д.Купрадзе): Potential methods in the theory of elasticity (in Russian), Moscow 1963.

[2] W. P o g o r z e l s k i : Étude de la solution fondamentale de l'équation elliptique, Ann.Polon.Math. 4 (1957), 247-284.

[3] W. P o g o r z e l s k i : Integral equations and their application, t.II, (in Polish), Warsaw 1959.

[4] L.A. L u s t e r n i k, W.L. S o b o l e w : Elements of the functional analysis (in Polish), Warsaw 1959.

[5] G. G i r a u d : Equations á intégrales principales, Ann. Scient. Ecole norm. supér. 51, fasc. 3 et 4(1934), 251-372.

[6] S.G. M i k h l i n (С.Г.Михлин): Singular integral equations with two independent variables (in Russian), Mat. Sbor. 43,4(1936), 535-550.

[7] J. C h m a j : On some non-linear system of singular integral equations in the theory of elasticity, Demonstr. Math. 1 (1969), 193--207.

[8] W. P o g o r z e l s k i : Sur la resolution du probléme aux dérivées tangentielles pout l'équation parabolique par la méthode des approximations successives, Ann. Polon. Math. 7(1959), 93-112.

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