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ON THE DISCONTINUOUS RIEMANN-HILBERT PROBLEM

1. INTRODUCTION

Let $\Gamma = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_m\}$ be a set of m+1 smooth Jordan-Lapunov contours lying in the complex plane.

Assume that the contour Γ_0 contain the remaining ones in this manner that the G^+ is the m+1 - connected domain bounded by $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$. By G^- we denote the complement of $G^+ + \Gamma$ to the whole plane.

We suppose that the origin of the coordinate system is placed in the domain G^+.

Moreover we assume that the orientation of contour Γ_0 (i.e. the positive direction on the Γ_0) is accordant with the orientation of the coordinate system, and the orientations of the remaining contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are opposite to the orientation of Γ_0.

2. FORMULATION OF THE PROBLEM

By the Riemann-Hilbert non-linear discontinuous problem we understand the following problem. It is required to find in the domain G^+ a function w(z) which satisfies the equation

$$\partial_{\bar{z}} w(z) + a(z)\, w(z) + b(z)\, \overline{w(z)} = f(z) \tag{1}$$

and at each point $t \in \Gamma' = \Gamma - \sum_{j=0}^{m} \sum_{i=1}^{k_j} \{c_i^{(j)}\}$ satisfies the non-linear boundary condition

$$\mathrm{re}\left[\overline{\lambda(t)}\ w\ (t)\right] = \gamma\left[t, w\ (t)\right] \tag{2}$$

where $c_i^{(j)}$ are the points of discontinuity of the function γ on the contour Γ.

3. ASSUMPTIONS

I. The angle $\Delta(t,t_1)$ that is formed by the tangents in two points $t, t_1 \in \Gamma$ satisfies the Hölder condition

$$\left|\Delta(t,t_1)\right| \leqslant k_\delta \left|t-t_1\right|^h, \quad 0 < h \leqslant 1 \tag{3}$$

where k_δ is a positive constant.

II. Complex functions $a(z)$, $b(z)$, $f(z)$ are defined in every point $z = x + iy$ of the region $G^+ + \Gamma$ and are absolutely integrable with the power $p > 2$ i.e. they belong to the class $L_p(G^+ + \Gamma)$ $p > 2$.

III. $\lambda(t) = \alpha(t) + i\,\beta(t)$, $m_\lambda < |\lambda(t)| < M_\lambda$, $|\lambda(t)| \neq 0$, where $\alpha(t)$ and $\beta(t)$ are real functions, defined and bounded for $t \in \Gamma$ and satisfying the Hölder conditions

$$\begin{aligned} \left|\alpha(t) - \alpha(t_1)\right| &\leqslant k_\alpha \left|t-t_1\right|^\nu \\ \left|\beta(t) - \beta(t_1)\right| &\leqslant k_\beta \left|t-t_1\right|^\nu, \quad 0 < \nu \leqslant h \end{aligned} \tag{4}$$

m_λ, M_λ, k_α and k_β are positive constants.

IV. The real function $\gamma(t,s)$ defined and bounded in the domain $\left[t \in \Gamma',\ |s| < \frac{\varrho_1}{\prod_{i=1}^{k_j} \left|t - c_i^{(j)}\right|^{\alpha_*}},\ j=0,1,\ldots,m\right]$ belongs to the class $h_{\alpha_*}^{\nu}$ of W. P o g o r z e l s k i [4], therefore it satisfies the following inequalities

$$|\gamma(t,s)| < \frac{M_\gamma}{\prod_{l=1}^{k_j} \left|t-c_i^{(j)}\right|^{\alpha_*}} + M'_\gamma |s|$$

$$|\gamma(t,s) - \gamma(t_1,s_1)| < \frac{k_\gamma \, |t-t_1|^{\nu}}{\left[\left|t-c_i^{(j)}\right|\left|t-c_{i+1}^{(j)}\right|\right]^{\alpha_*+\nu}} + k'_\gamma \, |s-s_1| \tag{5}$$

where M_γ , M'_γ , k_γ , k'_γ are positive constants and $c_i^{(j)}$, $i=1,2,\dots,k$, $j=0,1,\dots,m$ are the points of discontinuity of the function γ, placed on the curve Γ_j according to its positive direction; k_j denotes the number of points of discontinuity of the function γ on the curve Γ_j and $k_0+k_1+\dots+k_m = r$. The points t and t_1 are placed inside the same are $\widehat{c_i^{(j)}\, c_{i+1}^{(j)}}$ besides the point t_1 lies on the part $\widehat{t \;\; c_{i+1}^{(j)}}$ (we assume $c_{k_j+1}^{(j)} = c_1^{(j)}$).

4. SOLUTION OF THE PROBLEM

In virtue of the results of I. N. V e k u a [5] p.229, the solution of the equation (1) is of the form

$$w(z) = \phi(z)\, e^{\omega(z)} + \tilde{w}(z) ,$$

where

$$\omega(z) = \frac{1}{\pi}\iint_{G^+}\left[a(\zeta) + b\,(\zeta)\, \frac{\overline{\phi(\zeta)}\; \overline{e^{\omega(\zeta)}}}{\phi(\zeta)\; e^{\omega(\zeta)}}\right] \frac{d\xi\, d\eta}{\zeta - z},\ \zeta = \xi + i\eta \tag{7}$$

and $e^{\omega(z)} \in C_{\frac{p-2}{p}}$ (that means that the function $e^{\omega(z)}$ is bounded and satisfies Hölder condition with the exponent $\frac{p-2}{p}$),

$\tilde{w}(z)$ is a particular solution of the equation (1) and is given by the formula

$$\tilde{w}(z) = -\frac{1}{\pi}\iint_{G^+}\left[\Omega_1(z,\zeta)\, f(\zeta) + \Omega_2(z,\zeta)\,\overline{f(\zeta)}\right] d\xi\, d\eta \qquad (8)$$

where Ω_1 and Ω_2 denote the kernels normed with respect to the domain G^+ (see [5]) and are of the form

$$\Omega_1(z,t) = X_1(z,t) + i\, X_2(z,t) = \frac{e^{\omega_1} + e^{\omega_2}}{2(t-z)}$$

$$\Omega_2(z,t) = X_1(z,t) - i\, X_2(z,t) = \frac{e^{\omega_1} - e^{\omega_2}}{2(t-z)} \qquad (9)$$

$$X_1(z,t) = \frac{e^{\omega_1(z,t)}}{2\,(t-z)}, \quad X_2(z,t) = \frac{e^{\omega_2(z,t)}}{2i(t-z)}.$$

$$\omega_j = \frac{t-z}{\pi}\iint_{G^+}\frac{a(\zeta)\, X_j\,(\zeta,t) + b(\zeta)\overline{X_j(\zeta,t)}}{(\zeta-z)\,(t-\zeta)\, X_j\,(\zeta,t)}\, d\xi\, d\eta\,, \quad (j=1,2) \qquad (10)$$

and $\tilde{w}(z) \in C_{\frac{p-2}{p}}$

The function $\phi(z)$ holomorphic in G^+ and continuous in $G^+ + \Gamma$ satisfies the non-linear boundary condition

$$\mathrm{re}\left[\overline{\lambda(t)}\; e^{\omega(t)}\, \phi(t)\right] = \gamma\left[t, w(t)\right] - \mathrm{re}\left[\lambda(t)\; \tilde{w}\,(t)\right] \qquad (11)$$

According to the results of Z. B u ć k o [1] we shall seek the solution of the problem (11) in the form

$$\phi\,(z) = \int_{\Gamma} M(z,\tau)\,\mu\,(\tau)\, d\sigma + i\, C \qquad (12)$$

where

$$M(z,\tau) = \begin{cases} \dfrac{z_k - \tau}{\tau - z} & \text{when } \tau \in \Gamma_k \quad (k=1,2,\dots,m) \\[2ex] \dfrac{\tau}{\tau - z} & \text{when } \tau \in \Gamma_0 \end{cases} \qquad (13)$$

and $\mu(t)$ is the real function, continuous on Γ', C is the real constant, σ denotes the arc-coordinate of the point $\tau \in \Gamma$.

When the point $z \in G^+$ tends to the point $t \in \Gamma$, the function $\phi(z)$ has the following boundary value

$$\lim_{z \to t} \phi(z) = \delta(t)\, \mu(t) + \int_{\Gamma} M(t,\tau)\, \mu(\tau)\, d\sigma + i\, C \quad (14)$$

where

$$\delta(t) = \begin{cases} (t-z_k)\, \bar{t}'\, \pi i\,, & t \in \Gamma_k \quad (k=1,2,\dots,m) \\ t\, \bar{t}'\, \pi i\,, & t \in \Gamma_0 \end{cases} \quad (15)$$

It is necessary to choose the unknown function $\mu(t)$ and the constant C so that the function (12) would satisfy the condition (11). According to H. Łubowicz [3] this leads to the solution of the integral equation of the form

$$A(t)\mu(t) + \int_{\Gamma} \frac{K(t,\tau)}{\tau - t}\mu(\tau)\, d\tau = \gamma\left[t, w(t)\right] - re\left[\overline{\lambda(t)}\tilde{w}(t)\right] - re\, \overline{\lambda(t)} e^{\omega} i C \quad (16)$$

where

$$A(t) = \frac{1}{2}\pi i\left[\overline{\lambda(t)} e^{\omega}\, (t-z_k)\bar{t}' - \lambda(t) e^{\bar{\omega}}\, (\bar{t}-\bar{z}_k)\, t'\right] \quad (17)$$

$$K(t,\tau) = \frac{1}{2}\left[\overline{\lambda(t)} e^{\omega(t)} (z_k - t)\bar{\tau}' + \lambda(t) e^{\omega(t)}\, (\bar{z}_k - \bar{t})\, \tau' e^{2i \arg(\tau - t)}\right] \quad (18)$$

In virtue of the assumption I and III the function $K(t,\tau)$ satisfies the Hölder condition with the exponent $\nu' = \min(\nu, \frac{p-2}{p})$

We shall write the equation (16) in the form

$$A(t)\, \mu(t) + \frac{B(t)}{\pi i}\int_{\Gamma} \frac{\mu(\tau)}{\tau - t}\, d\tau = \gamma\left[t, w(t)\right] - re\left[\overline{\lambda(t)}\ \tilde{w}(t)\right] + \\ - re\left[\overline{\lambda(t)}\ e^{\omega(t)}\ i\ C\right] - \int_{\Gamma} \tilde{K}(t,\tau)\, \mu(\tau)\, d\tau \quad (19)$$

where

$$B(t)=\pi i\ K(t,t)=\tfrac{1}{2}\pi i\left[\overline{\lambda(t)}e^{\omega}\ (t-z_k)\bar{t}'+\lambda(t)e^{\bar{\omega}}(\bar{t}-\bar{z}_k)t'\right] \quad (20)$$

$$\tilde{K}(t,\tau) = \frac{k(t,\tau)}{|\tau-t|^{1-\nu_*}} \qquad (0<\nu^*<\nu') \quad (21)$$

$$k(t,\tau) = \frac{1}{2}\Big[\overline{\lambda(t)}e^{\omega}\ (z_k-t)(\tau'-t')+ \quad (22)$$
$$+\ \lambda(t)e^{\bar{\omega}}\ (\bar{z}_k-\bar{t})(\tau' e^{2i\ \arg(\tau-t)}-t')\frac{e^{i\ \arg(t-\tau)}}{|\tau-t|^{\nu_*}}\Big]$$

and $k(t,\tau)$ satisfies the Hölder condition with the exponent $\nu'-\nu^*$.

At first we shall study the unhomogeneous characteristic equation

$$A(t)\,\mu(t) + \frac{B(t)}{\pi i}\int_{\Gamma}\frac{\mu(\tau)}{\tau-t}\,d\tau = F(t) \quad (23)$$

where we assume that $F(t)$ satisfies the Hölder condition. The functions $A(t)$, $B(t)$ defined by (17), (20) and in virtue of assumptions I,III satisfy the Hölder condition with the exponent ν'.

The index of equation (23) is given by the formula

$$\varkappa = \frac{1}{2\pi}\sum_{k=0}^{m}\left[\arg\frac{A(t)-B(t)}{A(t)+B(t)}\right]_{\Gamma_k} \quad (24)$$

By the formulae (17) and (20) we have

$$\varkappa = \frac{1}{\pi}\sum_{k=0}^{m}\left[\arg\lambda(t)\right]_{\Gamma_k}$$

The equation (23) is soluble for an arbitrary right-hand side $F(t)$ if the homogeneous equation, associated with it has only zero solution.

In this paper we shall study only the case $\varkappa > 0$. The solution of equation (23) has a form

$$\mu(t) = \frac{X^+(t) + X^-(t)}{2[A(t) + B(t)]X^+(t)} F(t) + \\ + \frac{X^+(t) - X^-(t)}{2\pi i} \int_\Gamma \frac{F(\tau)\, d\tau}{[A(\tau) + B(\tau)]X^+(\tau)\,(\tau - t)} + \\ + [X^+(t) - X^-(t)]\, P_{\varkappa-1}(t) \tag{25}$$

where $P_{\varkappa-1}(t)$ is an arbitrary polynomial of the degree not greater than $\varkappa-1$, when $\varkappa \geqslant 1$, and $P_{\varkappa-1}(t) \equiv 0$ when $\varkappa = 0$. $X^+(t)$ and $X^-(t)$ are the boundary values of the canonical solution of Hilbert problem

$$X^+(t) = \frac{A(t) - B(t)}{A(t) + B(t)} X^-(t) \tag{26}$$

Treating the right-hand side of the equation (19) as a given function and use the formula (25) we obtain an integral equation

$$\mu(t) + A^*\int_\Gamma \tilde{K}(t,\tau)\mu(\tau)d\tau - \frac{B^* Z}{\pi i}\int_\Gamma \frac{\left[\int_\Gamma \tilde{K}(\tau,\tau_1)\,\mu(\tau_1)\, d\tau_1\right]}{Z(\tau)\,(\tau-t)} d\tau = \\ = A^*\gamma[t,w(t)] - \frac{B^* Z}{\pi i}\int_\Gamma \frac{\gamma[\tau,w(\tau)]\, d\tau}{Z(\tau)\,(\tau-t)} + f^* \tag{27}$$

where

$$A^*(t) = \frac{A(t)}{A^2(t)-B^2(t)}, \quad B^*(t) = \frac{B(t)}{A^2(t)-B^2(t)}$$

$$Z(t) = [A(t) + B(t)]\, X^+(t) = [A(t) - B(t)]\, X^-(t) \tag{29}$$

$$f^* = -A^* \mathrm{re}\left[\overline{\lambda(t)}\tilde{w}(t)\right] - A^* \mathrm{re}\left[\overline{\lambda(t)} e^{\omega(t)} \, iC\right] +$$

$$+ \frac{B^* Z}{\pi i} \int_\Gamma \frac{\mathrm{re}\left[\overline{\lambda(t)}\ \tilde{w}\ (\tau)\right] d\tau}{Z(\tau)\ (\tau - t)} + \frac{B^* Z}{\pi i} \int_\Gamma \frac{\mathrm{re}\left[\overline{\lambda(t)} e^{\omega(\tau)}\ i\ C\right]}{Z(\tau)\ (\tau - t)} d\tau - 2B^* Z\ P_{\varkappa-1} \tag{30}$$

Let us examine the integral

$$J(t) = \int_\Gamma \frac{\int_\Gamma \tilde{K}(\tau, \tau_1)\ \mu(\tau_1) d\tau_1}{Z(\tau)\ (\tau - t)} d\tau \tag{31}$$

which is of the form

$$J(t) = \int_\Gamma \tilde{N}(t, \tau_1)\ \mu(\tau_1)\ d\tau_1$$

where

$$\tilde{N}(t, \tau_1) = \frac{N^*(t,\ \tau_1)}{|t - \tau_1|^{1-\beta^*}}, \qquad (0 < \beta^* < \vartheta_1) \tag{32}$$

$N^*(t,\ \tau_1)$ satisfies the Hölder condition with the exponent $\vartheta_1 - \beta^*$ [3].

The integral equation (27) has the form

$$\mu(t) + A^* \int_\Gamma \tilde{K}(t, \tau)\ \mu(\tau)\ d\tau - \frac{B^* Z}{\pi i} \int_\Gamma \tilde{N}(t, \tau)\ \mu(\tau)\ d\tau =$$

$$= A^* \gamma\left[t, w(t)\right] - \frac{B^* Z}{\pi i} \int_\Gamma \frac{\gamma\left[\tau, w\ (\tau)\right]}{Z(\tau)\ (\tau - t)} d\tau + f^* \tag{33}$$

Thus the solution of the problem (1), (2) is reduced to the solution of 4 equations in the unknown functions $w\ (z)$, $\omega(z)$, $\mu(z)$ $\phi(z)$

$$w(z) = \phi(z)\, e^{\omega(z)} + \tilde{w}(z)$$

$$\omega(z) = \frac{1}{\pi}\iint_{G^+}\left[a(\zeta) + b(\zeta)\frac{\overline{\phi(\zeta)}\; e^{\overline{\omega(\zeta)}}}{\phi(\zeta)\; e^{\omega(\zeta)}}\right]\frac{d\xi\, d\eta}{\zeta - z}$$

$$\mu(z) + A^*\int_\Gamma \tilde{K}(z,\tau)\,\mu(\tau)\,d\tau - \frac{B^* Z}{\pi i}\int_\Gamma \tilde{N}(t,\tau)\,\mu(\tau)\,d\tau = \tag{34}$$

$$= A^*\,\gamma\left[z, w(z)\right] - \frac{B^* Z}{\pi i}\int_\Gamma \gamma\frac{[\tau, w(\tau)]\,d\tau}{Z(\tau)\,(\tau - z)} + f^*$$

$$\phi(z) = \int_\Gamma M(z,\tau)\,\mu(\tau)\,d\sigma + iC$$

where $A^*, B^*, Z, f^*, \tilde{K}$ and $\tilde{N}$ depend on z, $\lambda(z), e^{\omega(z)}$.

We shall prove the existence of the solution using the topological method of J. S c h a u d e r. Let Λ be the function space consisting of all the systems of functions

$$Q(z) = \left[w(z)\;,\; \omega(z)\;,\; \mu(z)\;,\; \phi(z)\right].$$

The complex functions $w(z)$ are defined and continuous in every point $t \in G^+ + \Gamma'$ and they satisfy the inequalities

$$\max_{0<j<m}\ \sup_{z\in G^+ + \Gamma_j}\left[\prod_{i=1}^{k_j}\left|z - c_i^{(j)}\right|^{\alpha^* + \nu}|w(z)|\right] < \infty, \quad 0 < \alpha_* + \nu < 1 \tag{35}$$

The functions $\omega(z)$ are defined and continuous in every point $z \in G^+ + \Gamma$. The functions $\mu(t)$ are defined and continuous in every point $t \in \Gamma'$ and they satisfy the inequalities

$$\max_{0<j<m}\ \sup_{t\in G^+ + \Gamma_j}\left[\prod_{i=1}^{k_j}\left|t - c_i^{(j)}\right|^{\alpha_* + \nu}|\mu(t)|\right] < \infty \tag{36}$$

The functions $\phi(t)$ are defined and continuous on Γ' and satisfy the inequalities

$$\max_{0<j<m} \sup_{t\in G^{+}+\Gamma_j} \left[\prod_{i=1}^{k_j} \left| t - c_i^{(j)} \right|^{\alpha_* + \nu} \left| \phi(t) \right| \right] < \infty \tag{37}$$

The space Λ will be linear,when we define the product of point Q by the real number l and the sum of two points by the formulae

$$lQ = \left[lw,\ l\omega,\ l\mu,\ l\phi \right], \qquad Q+Q' = \left[w+w', \omega+\omega', \mu+\mu', \phi+\phi' \right] \tag{38}$$

We define the norm of point Q by the formula

$$\|Q\| = \max_{0<j<m} \sup_{z\in G^{+}+\Gamma_j} \left[\prod_{i=1}^{k_j} \left| z-c_i^{(j)} \right|^{\alpha_*+\nu} \left| w(z) \right| \right] + \sup_{z\in G^{+}+\Gamma} \left| \omega(z) \right| +$$

$$+ \max_{0<j<m} \sup_{t\in G^{+}+\Gamma_j} \left[\prod_{i=1}^{k_j} \left| t-c_i^{(j)} \right|^{\alpha_*+\nu} \left| \mu(t) \right| \right] + \tag{39}$$

$$+ \max_{0<j<m} \sup_{t\in G^{+}+\Gamma_j} \left[\prod_{i=1}^{k_j} \left| t-c_i^{(j)} \right|^{\alpha_*+\nu} \left| \phi(t) \right| \right]$$

Further, we define the distance of the two points Q and Q' by the formula

$$(Q,Q') = \| Q - Q' \| \tag{40}$$

Thus Λ is a Banach space.

Let us examine in the space Λ a set E of all the points Q which satisfy the conditions

$$\left| w(z) \right| \leqslant \frac{\rho_1}{\prod_{i=1}^{k_j} \left| z-c_i^{(j)} \right|^{\alpha_*}}, \quad \left| w(z)-w(z') \right| \leqslant \frac{\varkappa_1 \left| z-z' \right|^{\theta}}{\left[\left| z-c_i^{(j)} \right| \left| z'-c_{i+1}^{(j)} \right| \right]^{\alpha_*+\theta}}$$

$$\left| \omega(z) \right| \leqslant \rho_2, \quad \left| \omega(z) - \omega(z') < \varkappa_2 \left| z - z' \right|^{\theta} \tag{41}$$

$$|\mu(z)| \leqslant \frac{\varrho_3}{\prod_{i=1}^{k_j} |z-c_i^{(j)}|^{\alpha_*}}, \quad |\mu(z)-\mu(z')| \leqslant \frac{\varkappa_3 |z-z'|^{\theta}}{\left[|z-c_i^{(j)}|\,|z'-c_{i+1}^{(j)}|\right]^{\alpha_*+\theta}}$$

$$|\phi(z)| < \frac{\varrho_4}{\prod_{i=1}^{k_j} |z-c_i^{(j)}|^{\alpha_*}}, \quad |\phi(z)-\phi(z')| \leqslant \frac{\varkappa_4 |z-z'|^{\theta}}{\left[|z-c_i^{(j)}|\,|z'-c_{i+1}^{(j)}|\right]^{\alpha_*+\theta}}$$

where $z \in \widehat{c_i^{(j)}\, c_{i+1}^{(j)}}$, $z' \in \widehat{z\, c_{i+1}^{(j)}}$, $|z-c_i^{(j)}| < |z'-c_i^{(j)}|$, $j=0,\dots,m$, $\theta \leqslant \frac{1}{4} \min (\nu, \frac{p-2}{p})$, ϱ_2, ϱ_3, ϱ_4, $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, $\varkappa_4$ are arbitrary positive constants and ϱ_1 is a positive constant from the assumption V.

The set E is closed and convex.

Without loss of generality we may admit $|z - z'| \leqslant 1$.

Let us transform the set E into the set E' by means of the relations

$$\hat{w}(z) = \phi(z)\, e^{\omega(z)} + \widetilde{w}(z)$$

$$\hat{\omega}(z) = \frac{1}{\pi} \iint_{G^+} \left[a(\zeta) + b(\zeta) \frac{\overline{\phi(\zeta)}\ e^{\overline{\omega(\zeta)}}}{\phi(\zeta)\ e^{\omega(\zeta)}} \right] \frac{d\xi\, d\eta}{\zeta - z} \tag{42}$$

$$\hat{\mu}(z) + \frac{1}{\pi i} \int_{\Gamma} M^*(z,\tau)\, \hat{\mu}(\tau)\, d\tau = A^* \gamma[z, w(z)] - \frac{B^* Z}{\pi i} \int_{\Gamma} \frac{\gamma[\tau, w(\tau)]\, d\tau}{Z(\tau)\ (\tau - z)} + f^*$$

$$\hat{\phi}(z) = \int_{\Gamma} M(z,\tau)\, \mu(\tau)\, d\sigma + iC$$

where

$$M^*(t,\tau) = \pi i\, A^* \widetilde{K}(t,\tau) - B^* Z\, \widetilde{N}(t,\tau) \tag{43}$$

The third equation of the system is a Fredholm equation with a weak singularity and an unknown function $\hat{\mu}(t)$ in the form

$$\hat{\mu}(z) + \frac{1}{\pi i}\int_{\Gamma} M^*(z,\tau)\,\hat{\mu}(\tau)\,d\tau = \Omega^*(t,\mu,\omega) \tag{44}$$

where $\Omega^*(t,\mu,\omega)$ denotes the right-hand side of the third equation of the system.

Let us admit that the homogeneous equation

$$\hat{\mu}(t) + \frac{1}{\pi i}\int_{\Gamma} M^*(t,\tau)\,\hat{\mu}(\tau)\,d\tau = 0 \tag{45}$$

has only the zero solution. The unique solution of the equation (44) is given by the formula

$$\hat{\mu}(t) = \Omega^*(t,\mu,\omega) + \int_{\Gamma} \mathfrak{M}(t,\tau)\,\Omega^*\left[\tau,\mu(\tau),\,\omega(\tau)\right]d\tau \tag{46}$$

where $\mathfrak{M}(t,\tau)$ is the resolvent kernel with a weak singularity, depending only on the function $M^*(t,\tau)$. The function $\mathfrak{M}(t,\tau)$ has the same singularity.

In virtue of the assumption IV

$$\gamma\left[t,w(t)\right] \leqslant \frac{M_\gamma + M'_\gamma}{\prod_{i=1}^{k_j}\left|t-c_i^{(j)}\right|^{\alpha_*}} \tag{47}$$

$$\left|\gamma\left[t,w(t)\right]-\gamma\left[\tau,w(\tau)\right]\right| \leqslant \frac{\left[k_\gamma + k'_\gamma\,\varkappa_1\right]\left|t-\tau\right|^{\theta}}{\left[\left|t-c_i^{(j)}\right|\left|\tau-c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}} \qquad j = 0,1,\dots,m$$

Now we shall examine the integral

$$J(t) = \int_{\Gamma} \frac{\gamma\left[\tau,w(\tau)\right]d\tau}{Z(\tau)\,(\tau-t)} \tag{48}$$

From (47) it follows that $\gamma \in \tilde{h}^{\theta}_{\alpha_*}$. In virtue of the results of [4]

$$|J(t)| < \frac{C\, M^*_\gamma + C' k^*_\gamma}{\prod_{i=1}^{k_j} |t-c_i^{(j)}|^{\alpha_*}} \tag{49}$$

$$|J(t) - J(\tau)| < \frac{[C_1 M^*_\gamma + C'_1\; k^*_\gamma]\,|t-\tau|^{\theta}}{[|t-c_i^{(j)}|\,|\tau - c_{i+1}^{(j)}|]^{\alpha_*+\theta}}$$

where $\theta = \min(\nu, \frac{p-2}{p})$, the positive constants C, C', C_1, C'_1 depend on G^+ and Γ, are independent of the function γ and Z while

$$M^*_\gamma = (M_\gamma + M'_\gamma \rho_1)\, M'_Z$$

$$k^*_\gamma = M'_Z\left[k_\gamma + k'_\gamma \alpha_1 + M'_Z\, k_Z\, C'_\Gamma\, (M_\gamma + M'_\gamma \rho_1)\right] \tag{50}$$

$$M'_Z = \sup_{t\in\Gamma}\left|\frac{1}{Z}\right|, \quad C'_\Gamma = \max_{0<j<m}\sup_{t,\tau\in\Gamma} \frac{\left[|t-c_i^{(j)}|\,|\tau-c_{i+1}^{(j)}|\right]^{\alpha_*+\theta}}{\prod_{i=1}^{k_j}|t-c_i^{(j)}|^{\alpha_*}}$$

and k_Z is the Hölder coefficient of the function Z.

The function f^* given by the formula (30) has, according to the assumptions and inequalities (41) the supremum M_f, and satisfies the Hölder condition with the exponent $\frac{1}{2}\nu'$ and the coefficient k_{f^*}.

So

$$\sup_{t\in\Gamma}|\Omega^*| \leqslant \frac{M_{\Omega^*}}{\prod_{i=1}^{k_j}|t-c_i^{(j)}|^{\alpha_*}} \qquad j=0,1,\dots,m \tag{51}$$

where

$$M_{\Omega^*} = M_A\,(M_\gamma + M'_\gamma\rho_1) + \pi^{-1} M_B M_Z\,(C\, M^*_\gamma + C' k^*_\gamma) + M_{f^*} C_\Gamma$$

$$C_\Gamma = \sup_{t\in\Gamma} \prod_{i=1}^{k_j}|t-c_i^{(j)}|^{\alpha_*}$$

$$M_Z = \sup_{t\in\Gamma}|Z(t)|, \quad M_A = \sup_{t\in\Gamma}|A^*|, \quad M_B = \sup_{t\in\Gamma}|B^*|$$

and the constants (52) depend on m_λ, G^+, Γ and p.

Further

$$\left|\Omega^*\left[t,\mu(t),\omega(t)\right] - \Omega^*\left[t_1, \mu(t_1), \omega(t_1)\right]\right| \leq$$

$$\frac{K_{\Omega^*}|t - t_1|^\theta}{\left[\left|t-c_i^{(j)}\right| \left|t_1-c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}} \tag{53}$$

where

$$K_{\Omega^*} = k_A C'_\Gamma (M_\gamma + M'_\gamma \varrho_1) + (k_\gamma + k'_\gamma \varkappa_1)M_A + \pi^{-1}k_{BZ} C'_\Gamma(CM^*_\gamma + C' k^*_\gamma) +$$
$$+ \pi^{-1}M_{BZ} (C_1 M^*_\gamma + C'_1 k^*_\gamma) \tag{54}$$

k_A and k_{BZ} denote Hölder coefficients for the functions A^* and B^*Z, respectivetly and according to (28), (29) these coefficients depend on Γ, function ω and λ, $M_{BZ} = \sup_\Gamma |B^* Z|$. Thus the function Ω^* belongs to the class $h^\theta_{\alpha_*}$ $(\theta \leq \frac{1}{4} \min(\nu, \frac{p-2}{p})$.

L e m m a 1. The set E' is a subset of the set E.

P r o o f. From (46) we have

$$|\hat{\mu}(t)| \leq \frac{M_{\Omega^*}(1+M_{\mathfrak{M}}) + K_{\Omega^*}M_\Gamma}{\prod_{i=1}^{k_j} \left|t - c_i^{(j)}\right|^{\alpha_*}} \tag{55}$$

where

$$M_{\mathfrak{M}} = \sup_{t\in\Gamma}\left|\int_\Gamma \mathfrak{M}(t,\tau)\, d\tau\right|$$

$$M_\Gamma = \max_{0<j\leq m} \sup_{t\in\Gamma} \prod_{i=1}^{k_j} \left|t-c_i^{(j)}\right|^{\alpha_*} \int_\Gamma \frac{|\mathfrak{M}(t,\tau)|\, |t-\tau|^\theta}{\left[\left|t-c_i^{(j)}\right|\left|\tau-c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}}\, d\tau$$

In view of (41), (42)

$$|\hat{w}(z)| < \frac{\rho_4}{\prod_{i=1}^{k_j} |z-c_i^{(j)}|^{\alpha_*}} e^{\rho^2} + M_{\tilde{w}} \tag{56}$$

where $M_{\tilde{w}} = \sup_{z \in G^+ + \Gamma} |\tilde{w}(z)|$.

According to (42) and monography of Vekua [5] p. 178 we have

$$|\hat{\omega}(z)| \leqslant M_p \left[L_p(a) + L_p(b)\right] \tag{57}$$

where the positive constant M_p depends on p and the region G^+

$$|\hat{\phi}(z)| < \frac{M_{\rho_3} + |C| \, C_\Gamma + M_1 \varkappa_3}{\prod_{i=1}^{k_j} |t - c_i^{(j)}|^{\alpha_*}} \tag{58}$$

where $M = \sup_{t \in \Gamma} \left| \int_\Gamma M(t,\tau) \right| d\tau$

$$M_1 = \max_{0<j<m} \sup_{t \in \Gamma} \left(\prod_{i=1}^{k_j} |t-c^{(j)}|^{\alpha_*} \int_\Gamma \frac{|M(t,\tau)| \, |t-\tau|^\theta}{\left[|t-c_i^{(j)}| \, |\tau-c_{i+1}^{(j)}|\right]^{\alpha_*+\theta}} d\tau \right)$$

Then

$$|\hat{\mu}(t) - \hat{\mu}(t_1)| < \frac{k_\mu |t, t_1|^\theta}{\left[|t-c_i^{(j)}| \, |t_1-c_{i+1}^{(j)}|\right]^{\alpha_*+\theta}} \tag{59}$$

where

$$k_\mu = \pi^{-1}\left[M_{\Omega^*}(1+M_{\mathfrak{M}}) + K_{\Omega^*} M_\Gamma\right] K_{M^*} C_\Gamma^* + k_A \, (M_\gamma + M_\gamma' \rho_1) \, C_\Gamma' + k_\gamma +$$

$$k_\gamma' \varkappa_1 + \pi^{-1} M_B \, M_z \left[C_1 \, M_\gamma^* + C_1 k_\gamma^* + \pi^{-1} k_{BZ} \, (C M_\gamma^* + C' k_\gamma^*) \, C_\Gamma' + k_{f^*} C_\Gamma''\right.$$

$$K_{M^*} = \max_{0<j<m} \sup_{t \in \Gamma} \int_\Gamma \frac{|M(t,\tau) - M^*(t_1, \tau)|}{|t-t_1|^\theta \prod_{i=1}^{k_j} |\tau - c_i^{(j)}|^{\alpha_*}} d\tau$$

$$C''_\Gamma = \max_{0<j\leq m} \sup_{t,t_1\in\Gamma} \left[\left|t-c_i^{(j)}\right|\left|t_1 - c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}$$

and

$$\left|\hat{w}(t) - \hat{w}(t_1)\right| \leq \frac{\left[\varkappa_4\, e^{\varrho_2} + \varkappa_2\, e^{\varrho_2}\, \varrho_4\, C'_\Gamma + C''_\Gamma\, k_{\tilde{w}}\right]\left|t - t_1\right|^\theta}{\left[\left|t - c_i^{(j)}\right|\left|t_1 - c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}} \quad (60)$$

where $k_{\tilde{w}}$ is the Hölder coefficient of the function $\tilde{w}(t)$.

In virtue of W. Pogorzelski [4] we have

$$\left|\hat{\phi}(t) - \hat{\phi}(t_1)\right| \leq \frac{\left[\tilde{C}_1\, \varrho_3 + \tilde{C}_2\, \varkappa_3\right]\left|t-t_1\right|^\theta}{\left[\left|t-c_i^{(j)}\right|\left|t_1-c_{i+1}^{(j)}\right|\right]^{\alpha_*+\theta}} \quad (61)$$

where the positive constants $\tilde{C}_1$ and $\tilde{C}_2$ are independent of the function $\mu(t)$, and

$$\left|\hat{\omega}(t) - \omega(t_1)\right| \leq M'_p \left[L_p(a) + L_p(b)\right] \left|t - t_1\right|^\theta \quad (62)$$

where M'_p is a positive constant depending on p and G^+.

The estimates (59), (60), (61) and (62) holds for $\theta < \frac{1}{4}\min\left(\nu, \frac{p-2}{p}\right)$.

The transformed point $\hat{Q}$ will belong to the set E, if the following inequalities are satisfied

$$
\begin{aligned}
&\varrho_4\, e^{\varrho_2} + M_{\tilde{w}}\, C_\Gamma \leq \varrho_1 \\
&M_p\left[L_p(a) + L_p(b)\right] \leq \varrho_2 \\
&M_\gamma D_1 + M'_\gamma D_2 \varrho_1 + M_\gamma M'_\gamma D_3 \varrho_1 + k_\gamma D_4 + k'_\gamma D_5 \varkappa_1 + M_\gamma D_6 k_\gamma + M_\gamma k'_\gamma D_7 \varkappa_1 + D_8 \leq \varrho_3 \\
&\varrho_3\, M + \tilde{C}\, C_\Gamma + \varkappa_3 M_1 \leq \varrho_4 \qquad (63) \\
&e^{\varrho_2}\, \varkappa_4 + C'_\Gamma\, e^{\varrho_2}\, \varkappa_2\, \varrho_4 + C''_\Gamma k_{\tilde{w}} \leq \varkappa_1 \\
&M'_p\left[L_p(a) + L_p(b)\right] \leq \varkappa_2 \\
&M_\gamma D_9 + M'_\gamma\, D_{10}\, \varrho_1 + k_\gamma\, D_{11} + k'_\gamma\, D_{12}\, \varkappa_1 + D_{13} \leq \varkappa_3 \\
&\tilde{C}_1\, \varrho_3 + \tilde{C}_2\, \varkappa_3 \leq \varkappa_4
\end{aligned}
$$

where D_i $(i=1,2,\dots,13)$ are the positive constants, $(D_1, D_2, \dots, D_7$ are independent of the function $f(z)$ in the equation (1) while D_8 is independent of the function γ of the boundary condition (2)).

The choice of the constants ϱ_2, ϱ_3, ϱ_4, $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, $\varkappa_4$ being arbitrary so it is possible to choose them in such a way that the inequalities (63) will be satisfied provided

$$M'_\gamma + k'_\gamma < \frac{1}{C^*} \qquad (64)$$

where $C^* = \max(C_1^*, C_2^*)$

$$C_1^* = e^{M_p\left[L_p(a)+L_p(b)\right]}\Big\{\left[M(D_2+D_3M_\gamma)+M_1D_{10}\right]\left[1+C'_r\left(L_p(a)+L_p(b)\right)M'_p\right] +$$

$$+ \tilde{C}_1\ (D_2 + D_3\ M_\gamma) + \tilde{C}_2\ D_{10}\Big\}$$

$$C_2^* = e^{M_p\left[L_p(a)+L_p(b)\right]}\Big\{\left[M(D_5+D_7M_\gamma)+M_1D_{12}\right]\left[1+C'_rM'_p(L_p(a)+L_p(b)\right] +$$

$$+ \tilde{C}_1\ (D_5 + D_7\ M_\gamma) + \tilde{C}_2\ D_{12}\Big\}.$$

L e m m a 2. The transformation (42) is continuous in the space Λ.

P r o o f. Let $Q^{(n)}\left[w^{(n)}, \omega^{(n)}, \mu^{(n)}, \phi^{(n)}\right]$ $n=1,2,\dots$ be an arbitrary sequence of points of the set E convergent in the sense of the norm (39) to the point $Q\left[w,\omega,\mu,\phi\right]$ of this set, i.e.

$$\delta(Q^{(n)}, Q) \longrightarrow 0, \quad \text{when } n \longrightarrow \infty. \qquad (65)$$

Thus the set of the transformed points $\hat{Q}^{(n)}\left[\hat{w}^{(n)}, \hat{\omega}^{(n)}, \hat{\mu}^{u)}, \hat{\phi}^{(n)}\right]$ is convergent in the sense of the norm (39) to the point $\hat{Q}\left[\hat{w}, \hat{\omega}, \hat{\mu}, \hat{\phi}\right]$ of the transformation (42). This property is evident for all of the components of the system (34), except the integral (48). The integrals in this form were examined by W. Pogorzelski [4] p.198, W. Żakowski [2] and others.

Applying the method used in the quoted above papers and assumption IV concerning the function γ, analogicaly we can prove that $\hat{Q}^{(n)} \to \hat{Q}$ in the norm (39) when $n \to \infty$. This completes the proff of the continuity of transformation (42).

Lemma 3. The set E' is compact in the space Λ.

Proof. In order to show that the set E' is compact we have to prove that we can choose from every sequence $\hat{Q}^{(n)}$ such a subsequence $\hat{Q}^{(n_k)}$ that is convergent in norm (39).

It is easy to show similarly as in the paper W.Żakowski [8], that there exists such a sub-sequence $\psi^{(n_k)}$ which is convergent in the set $G^+ + \Gamma$ and which satisfies the Cauchy condition.

Thus all the assumptions of the Schauder theorem are satisfied, hence there exists in the set E at least one fixed point $Q^*[w^*, \omega^*, \mu^*, \phi^*]$ of the transformation (42), i.e. a point satisfying the system (34).

The functions w^*, ω^*, μ^*, ϕ^* are solutions of the system (34). So the function w^* is a solution of the discontinuous generalized Riemann-Hilbert problem in the multiconnected domain.

The function w^* belongs to $h^{\theta}_{\alpha_*}$ where $\theta \leq \frac{1}{4}\min\left(\nu, \frac{p-2}{2}\right)$ and it satisfies the equation (1) so it possesses the generalized derivate $w^*_{\bar{z}}$ of the $L_p(G^+ + \Gamma)$ class i.e. $w^* \in D_{1,p}(G^+ + \Gamma)$.

Theorem. If: a) the assumptions I-IV are satisfied, b) the homogeneous equation (45) has only zero solution, c) the index $\varkappa$ of the problem is non negative and d) the constants of the problem M'_γ and k'_γ are sufficiently small, so that the inequality (64) holds, then there exists in the multi-connected domain G^+ at least one generalized solution $w(z) \in D_{1,p}$ $p > 2$ of the equation (1), which satisfies the boundary condition (2) in every point $t \in \Gamma'$.

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