DEMONSTRATIO MATHEMATICA
Vol. I No 4 1969

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ON SOME NON-LINEAR SYSTEM OF SINGULAR INTEGRAL EQUATIONS IN THE THEORY OF ELASTICITY

1. INTRODUCTION

Let S denote a closed Lapunov surface, and B_i - the domain bounded by S. Consider the following boundary problem in the theory of elasticity: Find the displacement vector $u(x) = u_1(x)\vec{i} + u_2(x)\vec{j} + u_3(x)\vec{k}$ in the domain B_i such that

$$\Delta^* u(x) + \omega^2 u(x) = 0 \qquad x \in B_i \tag{1}$$

(ω - constant, $\Delta^* \equiv (\lambda+2\mu)$ grad·div $- \mu\cdot$rot·rot, λ, μ-Lamé's constants) and

$$Tu(x_0) + \sigma(x_0)u(x_0) = G\big(x_0, u(x_0)\big) \qquad x_0 \in S, \tag{2}$$

where T is the well known tension operator, and $\sigma(x_0)$ is a scalar function.

By the potential methods this problem reduces to the solution of the second kind system of non-linear singular integral equations of the form

$$\varphi(x_0) - \varkappa\int_S \Big[T^{(x_0)}\Gamma(x_0,y) + \sigma(x_0)\Gamma(x_0,y)\Big]\,\varphi(y)\,dS_y = $$
$$= f\Big(x_0, \varphi(x_0)\Big) \qquad x_0 \in S, \tag{3}$$

where $\varphi(x_0) = \varphi_1(x_0)\vec{i} + \varphi_2(x_0)\vec{j} + \varphi_3(x_0)\vec{k}$ is an unknown vector, the integral is taken in the sense of the Cauchy principal value, $\Gamma(x,y)$ (the fundamental solution of the equation

(1)) is a matrix of the elements

$$\Gamma_j^{(k)}(x,y) = \frac{1}{b^2}\delta_{kj}\,\frac{e^{ik_2r}}{r} - \frac{1}{\omega^2}\cdot\frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{e^{ik_1r}}{r} - \frac{e^{ik_2r}}{r}\right)(k,j=1,2,3), \quad (4)$$

where

$$\delta_{kj} = \begin{cases} 1 & k=j \\ 0 & k\neq j, \end{cases} \qquad r = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2},$$

$$a^2 = \lambda+2\mu, \quad b^2 = \mu, \quad k_1^2 = \frac{\omega^2}{a^2}, \quad k_2^2 = \frac{\omega^2}{b^2}$$

and $T^{(x)}\Gamma(x,y)$ is a matrix of the elements

$$T_j^{(x)}\Gamma^{(k)}(x,y) = 2\mu\frac{\partial\Gamma_j^{(k)}(x,y)}{\partial n_x} + \frac{\lambda}{\lambda+2\mu}\cos(n_x,x_j)\,\frac{\partial}{\partial x_k}\,\frac{e^{ik_1r}}{r} + \quad (5)$$

$$+ \cos(n_x,x_k)\,\frac{\partial}{\partial x_j}\,\frac{e^{ik_2r}}{r} - \delta_{jk}\,\frac{\partial}{\partial n_x}\,\frac{e^{ik_2r}}{r}\ (k,j=1,2,3)$$

$f(x_0,\varphi(x_0))$ is a vector with the coordinates $f_j(x_0,\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0))$ $(j=1,2,3)$.

N-dimensional integral singular equations were treated by many authors. It was F.G.Tricomi who started investigations in this field. He considered a singular equation in the plain E_2 ([1], [2]). Tricomi's results were generalized by G. Giraud ([3]). G.Giraud considered the integral equation

$$\varphi(x) - \varkappa\int_S K(x,y)\varphi(y)dS_y = f(x), \quad (6)$$

where the kernel $K(x,y)$ is singular, $f(x)$ fulfills Hôlder's condition, and S is a closed Lapunov surface.

G.Giraud found the solution of (6) in the class of functions which satisfy the Hölder condition, by the regularization method used by Tricomi. If we apply the operator

$$\psi(x) + \varkappa\int_S H(x,y;\varkappa)\psi(y)dS_y, \tag{7}$$

to the both sides of (6) then we get

$$\left[1+\varkappa^2\phi(x;\varkappa)\right]\varphi(x)+\varkappa\int_S\Big[H(x,y;\varkappa)-K(x,y)\ +$$

$$-\varkappa\int_S H(x,z;\varkappa)K(z,y)dS_z\Big]\varphi(y)dS_y = f(x)+\varkappa\int_S H(x,y;\varkappa)f(y)dS_y \tag{8}$$

$\phi(x;\varkappa)$ is the function defined by the kernels K and H.

The equation (8) is of Fredholm type if

$$1+\varkappa^2\phi(x;\varkappa)\neq 0 \tag{9}$$

and

$$H(x,y;\varkappa)-K(x,y)-\varkappa\int_S H(x,z;\varkappa)K(z,y)dS_z = O(r^{\varepsilon-2})\quad \varepsilon>0. \tag{10}$$

This equation was reduced by Giraud to a boundary problem for harmonic functions.

In this paper we shall base on a method proposed by S.G. Mikhlin ([4]) who considered the equation of the form

$$A\varphi \equiv a_0(x)\varphi(x)+\int_{E_2}\frac{f_0(x,\theta)}{r}\varphi(y)dy+\int_{E_2}F(x,y)\varphi(y)dy=p(x), \tag{11}$$

where (r,θ) are polar coordinates of the point y with respect to the pole x, and the second integral above is absolutely convergent.

Under some assumptions on the regularity of functions $a_0(x)$, $f_0(x,\theta)$ Mikhlin connected with each operator $A\varphi$ a function called by him the symbol of A (SimA).

Let

$$f(x,\theta) = \sum_{n=-\infty}^{+\infty}{}' b_n(x)e^{in\theta}, \tag{12}$$

denote Fourier series of $f(x,\theta)$. ('means that the coefficent $b_0(x)$ is equal to zero, and this condition is necessary and sufficient for the existence of the singular integral).

Denote

$$h\varphi = \frac{1}{2\pi i}\int_{E_2} \frac{e^{i\theta}}{r^2}\varphi(y)dy \tag{13}$$

then

$$A\varphi = \sum_{n=-\infty}^{+\infty} a_n(x)h^n\varphi + P\varphi, \tag{14}$$

where $a_n(x) = \frac{2\pi i^{|n|}}{|n|} b_n(x)$, and $P\varphi$ is a weakly singular operator. By the symbol of A we mean

$$\mathrm{Sim}A = \sum_{n=-\infty}^{+\infty} a_n(x)e^{in\theta} \qquad -\pi<\theta<+\pi. \tag{15}$$

For a weakly singular operator A $\mathrm{Sim}A$ is defined as zero. It may be proved that

$$\mathrm{Sim}\left(A^{(1)} + A^{(2)}\right) = \mathrm{Sim}A^{(1)} + \mathrm{Sim}A^{(2)}$$

$$\mathrm{Sim}\left(A^{(1)}A^{(2)}\right) = \mathrm{Sim}A^{(1)}\,\mathrm{Sim}A^{(2)}.$$

Now the problem of the regularization of the equation (11) is reduced to determination of the symbol $\mathrm{Sim}A^*$ from the equation

$$\mathrm{Sim}A\ \mathrm{Sim}A^* = 1. \tag{16}$$

In the paper [4] S.G. M i k h l i n introduced a notion of the symbolic matrix for a system of integral singular equations

$$\sum_{k=1}^{n} A_{jk}\varphi_k = p_j(x) \qquad j=1,2,\dots,n \tag{17}$$

(A_{jk} are the operators of the type (11)).

The condition

$$\det\left(\left\| \mathrm{Sim}A_{jk}\right\|_{(j,k=1,2,\dots,n)}\right) = 0 \tag{18}$$

is necessary and sufficient for the existence of regularizator for the system (17).

W.D. K u p r a d z e in his book "Potential methods in the theory of elasticity" chapter V has proved that for the system (3) with the linear right side Fredholm's theorems hold. He made use of Giraud and Mikhlin methods. In this paper we shall prove the existence of solution of the system (3) admitting the following assumptions:

I. S is a closed Lapunov surface with the constants C, δ $C>0$, $0<\delta\leqslant 1$ such that

$$(n_x, n_y) \leqslant Cr^{\delta}(x,y). \tag{19}$$

II. The coordinates of the vector

$$f(x_0,\varphi) = f_1(x_0,\varphi_1,\varphi_2,\varphi_3)\bar{i} + f_2(x_0,\varphi_1,\varphi_2,\varphi_3)\bar{j} + f_3(x_0,\varphi_1,\varphi_2,\varphi_3)\bar{k}$$

are real functions, defined on the set $x_0 \in S, |\varphi_j|<\rho$ $(j=1,2,3)$, and fulfill Hölder-Lipschitz condition

$$\left|f_j(x_0,\varphi_1,\varphi_2,\varphi_3) - f_j(x'_0,\varphi'_1,\varphi'_2,\varphi'_3)\right| \leqslant$$
$$k_f\left[r^{h_f}(x_0,x'_0) + \sum_{s=1}^{3}|\varphi_s-\varphi'_s|\right] \quad (j=1,2,3) \tag{20}$$

where $k_f>0$, $0<h_f\leqslant 1$.

III. The real function $\sigma(x_0)$ is defined for $x_0\in S$ and satisfies the Hölder condition

$$\left|\sigma(x_0)-\sigma(x'_0)\right| \leqslant k_\sigma r^{h_\sigma}(x_0,x'_0) \tag{21}$$

where $k_\sigma \geqslant 0$, $0<h_\sigma\leqslant 1$.

2. PROOF OF EXISTENCE OF A SOLUTION FOR THE EQUATION (3)

Consider the function space Λ of all systems

$$U\left[\varphi_1,\varphi_2,\varphi_3\right] \tag{22}$$

of real functions $\varphi_1,\varphi_2,\varphi_3$ defined for $x_0\in S$ (each such system will be called a "point" of Λ). Addition of two such points and multiplication of a point by a real number are defined by the formulae

$$\left[\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0)\right] + \left[\psi_1(x_0),\psi_2(x_0),\psi_3(x_0)\right] =$$

$$= \left[\varphi_1(x_0)+\psi_1(x_0),\varphi_2(x_0)+\psi_2(x_0),\varphi_3(x_0)+\psi_3(x_0)\right] \tag{23}$$

$$m\left[\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0)\right] = \left[m\varphi_1(x_0),m\varphi_2(x_0),m\varphi_3(x_0)\right]. \tag{23'}$$

The norm $\|U\|$ of a point U is defined as

$$\|U\| = \max_j \sup_{x_0\in S} \left|\varphi_j(x_0)\right| \quad (j=1,2,3). \tag{24}$$

The distance of two points U, V in the space Λ is defined by the formulae

$$\delta(U,V) = \|U-V\|. \tag{25}$$

The space Λ (with the norm $\|\ \|$) is therefore a Banach space. In the space Λ consider the set E of all points $U\left[\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0)\right]$ for which

$$\begin{aligned} &\left|\varphi_j(x_0)\right| < \varrho && (j=1,2,3) \\ &\left|\varphi_j(x_0)-\varphi_j(x_0')\right| \leq k_\varphi r^{h_\varphi}(x_0,x_0') && (j=1,2,3), \end{aligned} \tag{26}$$

where

ϱ is the number from the assumption II,

k_φ is the positive number defined in the sequel,

h_φ is a constant with

$$0 < h_\varphi < \min(\delta, h_f, h_\sigma). \tag{27}$$

The set E is closed in the space Λ, because the limit of an uniformly convergent sequence of points $U^{(m)}\left[\varphi_1^{(m)}(x_0),\varphi_2^{(m)}(x_0),\varphi_3^{(m)}(x_0)\right]$ satisfies the conditions (26).

Moreover the set E is convex because if $0<\gamma<1$; $U,V\in E$ then $U+(1-\gamma)V\in E$. Let us transform E by the operation

$$\psi(x_o)-\varkappa\int_S\left[T^{(x_o)}\Gamma(x_o,y)+\sigma(x_o)\,\Gamma(x_o,y)\right]\psi(y)dS_y=f\big(x_o,\varphi(x_o)\big). \quad (28)$$

The kernel of the equation (28)

$$T^{(x_o)}\Gamma(x_o,y)+\sigma(x_o)\,\Gamma(x_o,y)=T^{(x_o)}\overset{\circ}{\Gamma}(x_o,y)+T^{(x_o)}\Omega(x_o,y)+$$

$$+\ \sigma(x_o)\,\Gamma(x_o,y)$$

is a sum of the following

$T^{(x_o)}\overset{\circ}{\Gamma}(x_o,y)$ with the pole of order $\dfrac{1}{r^2(x_o,y)}$ and $T^{(x_o)}\Omega(x_o,y)$,

$\sigma(x_o)\,\Gamma(x_o,y)$ with the pole of order $\dfrac{1}{r(x_o,y)}$, where $\overset{\circ}{\Gamma}(x,y), T^{(x)}\overset{\circ}{\Gamma}(x,y)$ denote the matrix with the elements

$$\overset{\circ}{\Gamma}{}_j^{(k)}(x,y)=\frac{1}{2a^2b^2}\left[(a^2-b^2)\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}+(a^2+b^2)\delta_{jk}\right]\frac{1}{r(x_o,y)} \quad (29)$$

$$(k,j=1,2,3)$$

$$T_j^{(x)}\overset{\circ}{\Gamma}{}^{(k)}(x,y)=2\mu\frac{\partial\overset{\circ}{\Gamma}{}_j^{(k)}}{\partial n_x}+\frac{\lambda}{\lambda+2\mu}\cos(n_x,x_j)\,\frac{\partial}{\partial x_k}\,\frac{1}{r}+ \quad (30)$$

$$+\cos(n_x,x_k)\,\frac{\partial}{\partial x_j}\,\frac{1}{r}-\delta_{jk}\,\frac{\partial}{\partial n_x}\,\frac{1}{r} \qquad (k,j=1,2,3)$$

$\Omega(x,y)$ is equal to

$$\Omega(x,y)=\Gamma(x,y)-\overset{\circ}{\Gamma}(x,y). \quad (31)$$

By the first Fredholm Theorem [5] p.147 if the equation

$$\psi(x_o)-\varkappa\int_S\left[T^{(x_o)}\Gamma(x_o,y)+\sigma(x_o)\,\Gamma(x_o,y)\right]\psi(y)dS_y=0 \quad (32)$$

has not non-zero solution, then the only solution of (28) is given by the formulae

$$\psi(x_o) = \left[I+\varkappa^2\phi(x_o;\varkappa)\right]^{-1}\bar{f}(x_o)+\varkappa\int_S N(x_o,\xi;\varkappa)\bar{f}(\xi)dS_\xi \qquad (33)$$

where $\bar{f}(x_o) = f\big(x_o,\varphi(x_o)\big)$

$$\left[I+\varkappa^2\phi(x_o;\varkappa)\right]^{-1} =$$

$$= B(x_o;\varkappa)=\frac{1}{\Delta}\left\|\begin{matrix} 1-\Delta'(A_5^2+A_6^2) & \Delta'(A_3A_5+A_4A_6) & -\Delta'(A_1A_5+A_2A_6) \\ \Delta'(A_3A_5+A_4A_6) & 1-\Delta'(A_3^2+A_4^2) & \Delta'(A_1A_3+A_2A_4) \\ -\Delta'(A_1A_5+A_2A_6) & \Delta'(A_1A_3+A_2A_4) & 1-\Delta'(A_1^2+A_2^2) \end{matrix}\right\|$$

Δ,Δ' are the constants dependent on the Lamé constants, $A_j(x)$ $(j=1,2,\ldots,6)$ are defined in [5] p. 133, the resolvent $N(x_o,\xi;\varkappa) = B(x_o;\varkappa)H(x_o,\xi;\varkappa) + O\left[r^{h-2}(x_o,\xi)\right]$ $h>0$ where $H(x_o,\xi;\varkappa)$ is the matrix with the elements of order $\frac{1}{r^2(x_o,\xi)}$ which is the regularization matrix of (28).

Denote by

$$M_B = \max_{k,j}\sup_{x_o\in S}\left|B_j^{(k)}(x_o;\varkappa)\right| \qquad (k,j=1,2,3)$$

$$M_f = \max_{j}\sup_{x_o\in S}\left|f_j\big(x_o,\varphi_1(x_o),\varphi_2(x_o),\varphi_3(x_o)\big)\right| \qquad (j=1,2,3)$$

$$M_\sigma = \sup_{x_o\in S}\left|\sigma(x_o)\right|$$

$$M_N = \max_{k,j}\sup_{x_o\in S}\left|\int_S N_j^{(k)}(x_o,\xi;\varkappa)dS_\xi\right| \qquad (k,j=1,2,3).$$

From the decomposition

$$\psi_j(x_o) = \sum_{k=1}^{3} B_j^{(k)}(x_o;\varkappa)f_k\big(x_o,\varphi_1(x_o),\varphi_2(x_o),\varphi_3(x_o)\big) +$$

$$+\varkappa\sum_{k=1}^{3}\int_S N_j^{(k)}(x_0,\xi;\varkappa)\Big[f_k\big(\xi,\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\big)+$$

$$-f_k\big(x_0,\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0)\big)\Big]dS_\xi + \tag{34}$$

$$+\varkappa\sum_{k=1}^{3}f_k\big(x_0,\varphi_1(x_0),\varphi_2(x_0),\varphi_3(x_0)\big)\int_S N_j^{(k)}(x_0,\xi;\varkappa)dS_\xi \quad (j=1,2,3)$$

the assumption II and (26), (27) we get the estimate

$$\left|\psi_j(x_0)\right| \leqslant 3\left[M_B M_f + |\varkappa| M_N(k_f + k_f k_\varphi + M_f)\right]. \tag{35}$$

To prove that ψ_j satisfy the Hölder condition we must obtain some estimates

$$B(x_0;\varkappa)-B(x_0';\varkappa) = \frac{\Delta'}{\Delta}\begin{Vmatrix} b_{11}(x_0,x_0') & b_{12}(x_0,x_0') & b_{13}(x_0,x_0') \\ b_{21}(x_0,x_0') & b_{22}(x_0,x_0') & b_{23}(x_0,x_0') \\ b_{31}(x_0,x_0') & b_{32}(x_0,x_0') & b_{33}(x_0,x_0') \end{Vmatrix}$$

where

$$b_{11}(x_0,x_0') = \left[A_5(x_0')-A_5(x_0)\right]\left[A_5(x_0')+A_5(x_0)\right]+$$
$$+\left[A_6(x_0')-A_6(x_0)\right]\left[A_6(x_0')+A_6(x_0)\right]$$

$$b_{12}(x_0,x_0') = \left[A_3(x_0)-A_3(x_0')\right]A_5(x_0)+\left[A_5(x_0)-A_5(x_0')\right]A_3(x_0')+$$
$$+\left[A_4(x_0)-A_4(x_0')\right]A_6(x_0)+\left[A_6(x_0)-A_6(x_0')\right]A_4(x_0')$$

$$b_{13}(x_0,x_0') = \left[A_1(x_0')-A_1(x_0)\right]A_5(x_0')+\left[A_5(x_0')-A_5(x_0)\right]A_1(x_0)+$$
$$+\left[A_2(x_0')-A_2(x_0)\right]A_6(x_0')+\left[A_6(x_0')-A_6(x_0)\right]A_2(x_0)$$

$$b_{21}(x_0,x_0') = b_{12}(x_0,x_0')$$

$$b_{22}(x_0,x_0') = \left[A_3(x_0')-A_3(x_0)\right]\left[A_3(x_0')+A_3(x_0)\right]+$$
$$+\left[A_4(x_0')-A_4(x_0)\right]\left[A_4(x_0')+A_4(x_0)\right]$$

$$b_{23}(x_0,x_0') = \left[A_1(x_0)-A_1(x_0')\right]A_3(x_0)+\left[A_3(x_0)-A_3(x_0')\right]A_1(x_0')+$$
$$+\left[A_2(x_0)-A_2(x_0')\right]A_4(x_0)+\left[A_4(x_0)-A_4(x_0')\right]A_2(x_0')$$

$$b_{31}(x_0,x_0') = b_{13}(x_0,x_0')$$

$$b_{32}(x_0,x_0') = b_{23}(x_0,x_0')$$

$$b_{33}(x_0,x_0') = \left[A_1(x_0')-A_1(x_0)\right]\left[A_1(x_0')+A_1(x_0)\right] +$$
$$+\left[A_2(x_0')-A_2(x_0)\right]\left[A_2(x_0')+A_2(x_0)\right]$$

$$A_1(x_0)-A_1(x_0') = a_{11}(x^0)\left[\cos(n_{x_0},x_2) - \cos(n_{x_0'},x_2)\right] +$$
$$- a_{12}(x^0)\left[\cos(n_{x_0},x_1) - \cos(n_{x_0'},x_1)\right]$$

$$A_2(x_0)-A_2(x_0') = a_{21}(x^0)\left[\cos(n_{x_0},x_2) - \cos(n_{x_0'},x_2)\right] +$$
$$- a_{22}(x^0)\left[\cos(n_{x_0},x_1) - \cos(n_{x_0'},x_1)\right]$$

$$A_3(x_0)-A_3(x_0') = a_{11}(x^0)\left[\cos(n_{x_0},x_3) - \cos(n_{x_0'},x_3)\right] +$$
$$- a_{13}(x^0)\left[\cos(n_{x_0},x_1) - \cos(n_{x_0'},x_1)\right]$$

$$A_4(x_0)-A_4(x_0') = a_{21}(x^0)\left[\cos(n_{x_0'},x_3) - \cos(n_{x_0},x_3)\right] +$$
$$- a_{23}(x^0)\left[\cos(n_{x_0'},x_1) - \cos(n_{x_0},x_1)\right]$$

$$A_5(x_0)-A_5(x_0') = a_{12}(x^0)\left[\cos(n_{x_0'},x_3) - \cos(n_{x_0},x_3)\right] +$$
$$- a_{13}(x^0)\left[\cos(n_{x_0},x_2) - \cos(n_{x_0'},x_2)\right]$$

$$A_6(x_0)-A_6(x_0') = a_{22}(x^0)\left[\cos(n_{x_0}',x_3) - \cos(n_{x_0'},x_3)\right] +$$
$$- a_{23}(x^0)\left[\cos(n_{x_0},x_2) - \cos(n_{x_0'},x_2)\right]$$

But

$$\left|\cos(n_{x_0},x_j) - \cos(n_{x_0'},x_j)\right| \leq (n_{x_0},n_{x_0'}) \leq Cr^{\delta}(x_0,x_0') \quad (j=1,2,3).$$

Hence

$$\left| \overset{(k)}{B_j}(x_0;\varkappa) - \overset{(k)}{B_j}(x'_0;\varkappa) \right| \leqslant C_1 \cdot C r^{\delta}(x_0,x'_0) \qquad (k,j=1,2,3) \quad (36)$$

where C_1 is the constant dependent on Lamé's constants.

By [6] and theorem 4 in [5], p.111, the functions

$$D_j(x_0;\varkappa) = \sum_{k=1}^{3} \int_S \overset{(k)}{N_j}(x_0,\xi;\varkappa)\bar{f}_k(\xi)dS_\xi \qquad (j=1,2,3) \qquad (37)$$

satisfy the condition

$$\left| D_j(x_0;\varkappa) - D_j(x'_0;\varkappa) \right| \leqslant k_N k_f (1+k_\varphi) r^{h_\varphi}(x_0,x'_0) \quad (j=1,2,3) \ (38)$$

i.e. the Hölder condition with the exponent h_φ and the coefficient being proportional to that of the functions $\bar{f}_k(\xi)$.

From the decomposition

$$\psi_j(x_0)-\psi_j(x'_0) = \sum_{k=1}^{3} \left[\overset{(k)}{B_j}(x_0;\varkappa) - \overset{(k)}{B_j}(x'_0;\varkappa) \right] \bar{f}_k(x_0) + \qquad (39)$$

$$+ \sum_{k=1}^{3} \overset{(k)}{B_j}(x'_0)\left[\bar{f}_k(x_0)-\bar{f}_k(x'_0)\right] + \varkappa\left[D_j(x_0;\varkappa)-D_j(x_0;\varkappa)\right] \qquad (j=1,2,3),$$

the assumption II and the inequalities (36), (38) we have

$$\left|\psi_j(x_0)-\psi_j(x'_0)\right| < 3\left[C_1 C M_f + k_f(M_B+|\varkappa|\, k_N)(1+k_\varphi)\right] r^{h_\varphi}(x_0,x'_0) \qquad (40)$$

$$(j=1,2,3).$$

A point $V\left[\psi_1(x_0),\psi_2(x_0),\psi_3(x_0)\right]$ belongs to E if the following inequalities are satisfied

$$3\left[M_B M_f + |\varkappa|\, M_N(k_f+k_f k_\varphi+M_f)\right] \leqslant \varrho$$
$$3\left[C_1 C M_f + k_f(M_B+|\varkappa|\, k_N)(1+k_\varphi)\right] \leqslant k_\varphi \qquad (41)$$

But these inequalities are fulfilled if $k\varphi, \varrho$ are sufficiently large and k_f such that

$$k_f < \frac{M_B+|\varkappa|\, k_N}{3}. \qquad (42)$$

L e m m a 1.

The transformation (28) of the set E is continuous in the space Λ.

Proof. Let $U^{(m)}\left[\varphi_1^{(m)}(x_o),\ \varphi_2^{(m)}(x_o),\ \varphi_3^{(m)}(x_o)\right]$ be a convergent sequence of points in E to a point $U\left[\varphi_1(x_o),\varphi_2(x_o),\varphi_3(x_o)\right]$, that is

$$\lim_{m\to\infty}\delta\left(U^{(m)},U\right)=0 \tag{43}$$

and let $V^{(m)}\left[\psi_1^{(m)}(x_o),\psi_2^{(m)}(x_o),\psi_3^{(m)}(x_o)\right]$, and $V\left[\psi_1(x_o),\psi_2(x_o),\psi_3(x_o)\right]$ denote the image of $U^{(m)}$ and U by the transformation (28).

We have to prove that

$$\lim_{m\to\infty}\delta\left(V^{(m)},V\right)=0. \tag{44}$$

By the definition of the transformation (28) and by (33) we get

$$\begin{aligned}\psi_j^{(m)}(x_o)-\psi_j(x_o)=\sum_{k=1}^{3}B_j^{(k)}(x_o;\varkappa)\Big[f_k\left(x_o,\varphi_1^{(m)}(x_o),\varphi_2^{(m)}(x_o),\varphi_3^{(m)}(x_o)\right)\ +\\ -\ f_k\left(x_o,\varphi_1(x_o),\varphi_2(x_o),\varphi_3(x_o)\right)\Big]\ +\\ +\varkappa\sum_{k=1}^{3}\int_S N_j^{(k)}(x_o,\xi;\varkappa)\Big[f_k\left(\xi,\varphi_1^{(m)}(\xi),\varphi_2^{(m)}(\xi),\varphi_3^{(m)}(\xi)\right)\ +\\ -\ f_k\left(\xi,\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\right)\Big]\cdot dS_\xi \quad (j=1,2,3).\end{aligned} \tag{45}$$

From the assumption II and (26),(27) we have

$$\begin{aligned}\left|f_k\left(\xi,\varphi_1^{(m)}(\xi),\varphi_2^{(m)}(\xi),\varphi_3^{(m)}(\xi)\right)-f_k\left(x_o,\varphi_1^{(m)}(x_o),\varphi_2^{(m)}(x_o),\varphi_3^{(m)}(x_o)\right)\right|\leqslant\\ \leqslant \text{const.}\,r^{h\varphi}(x_o,\xi) \quad (k=1,2,3)\end{aligned} \tag{46}$$

$$\begin{aligned}\left|f_k\left(x_o,\varphi_1(x_o),\varphi_2(x_o),\varphi_3(x_o)\right)-f_k\left(\xi,\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\right)\right|\\ \leqslant \text{const.}\ r^{h\varphi}(x_o,\xi) \quad (k=1,2,3)\end{aligned} \tag{47}$$

$$\left|f_k\, x_0, \overset{(m)}{\varphi_1}(x_0), \overset{(m)}{\varphi_2}(x_0), \overset{(m)}{\varphi_3}(x_0)) - f_k\left(x_0, \varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0)\right)\right| <$$

$$\max_j \sup_{x_0 \in S} \left|\overset{(m)}{\varphi_j}(x_0) - \varphi_j(x_0)\right| \tag{48}$$

$$(k, j = 1, 2, 3).$$

Denote by $\sigma(x^o, \varepsilon)$ the subset of S contained in the ball $\tau(x^o, \varepsilon)$ with the center in x^o and with the radius ε. Let $x \in \sigma(x^o, \varepsilon)$. We can write (45) in the following way

$$\overset{(m)}{\psi_j}(x_0) - \psi_j(x_0) = \sum_{k=1}^{3} \overset{(k)}{B_j}(x_0; \varkappa)\left[f_k\left(x_0, \overset{(m)}{\varphi_1}(x_0), \overset{(m)}{\varphi_2}(x_0), \overset{(m)}{\varphi_3}(x_0)\right) + \right.$$

$$\left. - f_k\left(x_0, \varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0)\right)\right] + \tag{49}$$

$$+ \varkappa \sum_{k=1}^{3} \int_{\sigma(x^o;\varepsilon)} \overset{(k)}{N_j}(x_0, \xi; \varkappa)\left[f_k(\xi, \overset{(m)}{\varphi_1}(\xi), \overset{(m)}{\varphi_2}(\xi), \overset{(m)}{\varphi_3}(\xi)) + \right.$$

$$\left. - f_k\left(x_0, \overset{(m)}{\varphi_1}(x_0), \overset{(m)}{\varphi_2}(x_0), \overset{(m)}{\varphi_3}(x_0)\right)\right] dS_\xi +$$

$$+ \varkappa \sum_{k=1}^{3} \int_{\sigma(x^o;\varepsilon)} \overset{(k)}{N_j}(x_0, \xi; \varkappa)\left[f_k\left(x_0, \varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0)\right) + \right.$$

$$\left. - f_k(\xi, \varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\right] dS_\xi +$$

$$+ \varkappa \sum_{k=1}^{3} \left[f_k\left(x_0, \overset{(m)}{\varphi_1}(x_0), \overset{(m)}{\varphi_2}(x_0), \overset{(m)}{\varphi_3}(x_0)\right) + \right.$$

$$\left. - f_k\left(x_0, \varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0)\right)\right] \int_{\sigma(x^o;\varepsilon)} N_f^{(k)}(x_0, \xi; \varkappa) dS_\xi +$$

$$+ \varkappa \sum_{k=1}^{3} \int_{S-\sigma(x^o;\varepsilon)} \overset{(k)}{N_j}(x_0, \xi; \varkappa)\left[f_k(\xi, \overset{(m)}{\varphi_1}(\xi), \overset{(m)}{\varphi_2}(\xi), \overset{(m)}{\varphi_3}(\xi)) + \right.$$

$$\left. - f_k(\xi, \varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\right] dS_\xi \qquad (j = 1, 2, 3).$$

Since $\overset{(m)}{\varphi_j}(x_0)$ tends uniformly to $\varphi_j(x)$ for $x_0 \in S$ and (46), (47), (48) by (49) for $\varepsilon \to 0$ it is evident that $\psi_j^{(m)}(x_0)$

tends uniformly to $\psi_j(x_0)$ for $x_0 \in S$ and that completes the proof of the continuity of the transformation (28).

L e m m a 2.

The set $\{\psi_j(x_0)\}$ (j=1,2,3) is compact.

Proof. The functions $\psi_j(x_0)$ (j = 1,2,3) are uniformly bounded and uniformly continuous for $x_0 \in S$ because they fulfill the Hölder condition with the same exponent and constant. So the set $\{\psi_j(x_0)\}$ is compact by virtue of the theorem of Arzela.

So, all the assumptions of the Schauder Theorem [7] are satisfied. Thus this theorem implies that there is at least one point $U^*[\varphi_1^*(x_0),\varphi_2^*(x_0),\varphi_3^*(x_0)]$ invariant for the transformation (28). Another words, there exists a function $\varphi^*(x_0) = \varphi_1^*(x_0)\vec{i} + \varphi_2^*(x_0)\vec{j} + \varphi_3^*(x_0)\vec{k}$ which is a solution of the system (3). Thus we have proved the following theorem.

T h e o r e m

If I,II,III hold and the constant k_f is sufficiently small and satisfies the inequality (42), then the system (3) has at least one solution $\varphi(x_0) = \varphi_1(x_0)\vec{i} + \varphi_2(x_0)\vec{j} + \varphi_3(x_0)\vec{k}$ which satisfies the Hölder condition with the exponent h_φ from (27).

R e m a r k

It could be proved that each solution of the system (3) is also a solution of the problem (1),(2).

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Received, November 9th 1968.

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