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ON THE INTEGRODIFFERENTIAL EQUATIONS OF COMPLETELY DISCONTINUOUS PROCESSES

Let $\{X_t,\ t \geqslant 0\}$ be a stochastic process which takes values belonging to a number interval I. Let us denote by F and G the conditional cumulative distribution functions of the process X_t, i.e. the probabilites

$$F(t_1,x;t_3,y) = P(X_{t_3} < y \mid X_{t_1} = x), \qquad (1)$$

$$G(t_1,x;t_2,z;t_3,y) = P(X_{t_3} < y \mid X_{t_1} = x,\ X_{t_2} = z). \qquad (2)$$

We now introduce the following functions

$$q_1(t,x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\left(X_{t+\Delta t} - X_t \neq 0 \mid X_t = x\right), \qquad (3)$$

$$q_2(t_1,x;t_2,z) = \lim_{\Delta t_2 \to 0} \frac{1}{\Delta t_2} P\left(X_{t_2+\Delta t_2} - X_{t_2} \neq 0 \mid X_{t_1} = x,\ X_{t_2} = z\right), \qquad (4)$$

$$P_1(t,x;y) = \lim_{\Delta t \to 0} P\left(X_{t+\Delta t} < y \mid X_t = x,\ X_{t+\Delta t} - X_t \neq 0\right), \qquad (5)$$

$$P_2(t_1,x;t_2,z;y) =$$

$$= \lim_{\Delta t_2 \to 0} P\left(X_{t_2+\Delta t_2} < y \mid X_{t_1} = x,\ X_{t_2} = z,\ X_{t_2+\Delta t_2} - X_{t_2} \neq 0\right). \qquad (6)$$

We say that $\{X_t,\ t \geqslant 0\}$ is a completely discontinuous Markov process [1] if for each t

$$F(t,x;t+\Delta t,y) = \left[1-q_1(t,x)\Delta t\right]E(x,y) + q_1(t,x)P_1(t,x;y)\Delta t + o(\Delta t),$$

where

$$E(x,y) = \begin{cases} 1 & \text{if } x<y, \\ 0 & \text{if } x>y. \end{cases}$$

As we know [2], [3] if $\{X_t, t\geq 0\}$ is a completely discontinuous Markov process and the functions F, q_1, P_1 have appropriate regularity properties then are satisfied the following integrodifferential equations

$$\frac{\partial F(t_1,x;t_2,y)}{\partial t_1} = $$

$$= q_1(t_1,x)\left[F(t_1,x;t_2,y) - \int_{-\infty}^{+\infty} F(t_1,z;t_2,y)\,d_z P_1(t_1,x;z)\right] \tag{7}$$

$$\frac{\partial F(t_1,x;t_2,y)}{\partial t_2} = -\int_{-\infty}^{y} q_1(t_2,z)\,d_z F(t_1,x;t_2,z) + \int_{-\infty}^{+\infty} q_1(t_2,z)P_1(t_2,z;y)\,d_z F(t_1,x;t_2,z). \tag{8}$$

The generalization of these equations for a multiple Markov process may be found in the works [4], [5].

In this paper we shall consider a completely discontinuous stochastic process, which must not be a Markov process. Namely, we shall suppose that the conditional cumulative distribution function (2) can be written as

$$G(t_1,x;t_2,z;t_3,y) = \left[1 - q_2(t_1,x;t_2,z)(t_3-t_2)\right]E(z,y) + $$
$$+ q_2(t_1,x;t_2,z)(t_3-t_2)P_2(t_1,x;t_2,z;y) + o(t_3-t_2)$$

where

$$E(z,y) = \begin{cases} 1 & \text{if } z<y \\ 0 & \text{if } z\geq y. \end{cases}$$

The aim of this paper is to give the integrodifferential equations, which are generalization of Kolmogorov's equations for Markov procesess.

We shall proof the following theorem.

Theorem. Let $\{X_t, t \geq 0\}$ be a completely discontinuous stochastic process and let (3) - (6) satisfy the conditions

1° $F(t_1,x;t_3,y)$ is a Baire function of x continuous of t_1,t_3.

2° $G(t_1,x;t_2,z;t_3,y)$ is a Baire function of x,z continuous of t_1,t_2,t_3.

3° $q_1(t,x)$, $P_1(t,x;y)$ are Baire functions of x continuous of t.

4° $q_2(t_1,x;t_2,z)$, $P_2(t_1,x;t_2,z;y)$ are Baire functions of x,z continuous of t_1,t_2.

5° There exists the limit

$$\lim_{\Delta t_1 \to 0} \int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;z).$$

Then the following integrodifferential equations are satisfied

$$\left.\frac{\partial G(t_1,x;\tau,x;t_3,y)}{\partial \tau}\right|_{\tau=t_1} = q_1(t_1,x)G(t_1,x;t_3,y) + \tag{9}$$
$$- \lim_{\Delta t_1 \to 0} q_1(t_1,x)\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;y)$$

$$\frac{\partial F(t_1,x;t_3,y)}{\partial t_3} = -\int_{-\infty}^{y} q_2(t_1,x;t_3,z)d_zF(t_1,x;t_3,z) + \tag{10}$$
$$+\int_{-\infty}^{+\infty} q_2(t_1,x;t_3,z)P_2(t_1,x;t_3,z;y)d_zF(t_1,x;t_3,z).$$

Proof. The Chapman-Kolmogorov equation for a non-Marcovian stochastic process is given by

$$F(t_1,x;t_3,y) = \int_{-\infty}^{+\infty} G(t_1,x;t_2,z;t_3,y)d_zF(t_1,x;t_2,z). \qquad (11)$$

It follows from the formula (11) that

$$F(t_1,x;t_3+\Delta t_3,y) = \int_{-\infty}^{+\infty} G(t_1,x;t_3,z;t_3+\Delta t_3,y)d_zF(t_1,x;t_3,z). \quad (12)$$

Since the process is a completely discontinuous process, then for the function G we have

$$\begin{aligned} G(t_1,x;t_3,z;t_3+\Delta t_3,y) &= \left[1 - q_2(t_1,x;t_3,z)\Delta t_3\right]E(z,y) + \\ &+ q_2(t_1,x;t_3,z)\Delta t_3 \; P_2(t_1,x;t_3,z;y) + o(\Delta t_3) \end{aligned} \qquad (13)$$

where

$$E(z,y) = \begin{cases} 1 & \text{if} \quad z < y \\ 0 & \text{if} \quad z \geq y. \end{cases}$$

By use of (13) the expression (12) becomes

$$\begin{aligned} F(t_1,x;t_3+\Delta t_3,y) &= \int_{-\infty}^{+\infty} E(z,y)d_zF(t_1,x;t_3,z) + \\ &- \Delta t_3 \int_{-\infty}^{+\infty} q_2(t_1,x;t_3,z)E(z,y)d_zF(t_1,x;t_3,z) + \\ &+ \Delta t_3 \int_{-\infty}^{+\infty} q_2(t_1,x;t_3,z)P_2(t_1,x;t_3,z;y)d_zF(t_1,x;t_3,z). \end{aligned} \qquad (14)$$

The existence of the integrals on the right hand side of (14) follows from the assumptions relative to the functions q_2, P_2, F.

By using the properties of the function $E(z,y)$ and dividing both sides (12) by Δt_3 we obtain

$$\frac{F(t_1,x;t_3+\Delta t_3,y) - F(t_1,x;t_3,y)}{\Delta t_3} =$$

$$= -\int_{-\infty}^{y} q_2(t_1,x;t_3,z)d_zF(t_1,x;t_3,z) +$$

$$+\int_{-\infty}^{+\infty} q_2(t_1,x;t_3,z)P_2(t_1,x;t_3,z;y)d_zF(t_1,x;t_3,z) + o(1).$$

Hence, in the limit, as $\Delta t_3 \to 0$, we obtain the equation (10). We shall now derive the equation (9).

It follows from the formula (11) that

$$F(t_1,x;t_3,y) = \int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zF(t_1,x;t_1+\Delta t_1,z). \quad (15)$$

But

$$F(t_1,x;t_1+\Delta t_1,z) = \left[1 - q_1(t_1,x)\,\Delta t_1\right]E(x,z) +$$

$$+ q_1(t_1,x)\,\Delta t_1 P_1(t_1,x;z) + o(\Delta t_1).$$

Equation (15) can now be written as

$$F(t_1,x;t_3,y) = \left[1 - q_1(t_1,x)\Delta t_1\right]\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zE(x,z)+$$

$$+ q_1(t_1,x)\,\Delta t_1\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;z) + o(\Delta t_1). \quad (16)$$

The existence of the integrals on the right hand side of (16) follows from the assumptions relative to the functions q_1, P_1, G.

By using the properties of the function $E(z,y)$ from (16) we obtain

$$F(t_1,x;t_3,y) = G(t_1,x;t_1+\Delta t_1,x;t_3,y) +$$

$$- q_1(t_1,x)\Delta t_1 G(t_1,x;t_1+\Delta t_1,x;t_3,y) +$$

$$+ q_1(t_1,x)\Delta t_1\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;z)+o(\Delta t_1). \quad (17)$$

Because $F(t_1,x;t_3,y) = G(t_1,x;t_1,x;t_3,y)$ therefore dividing both side (17) by Δt_1 we have

$$\frac{G(t_1,x;t_1+\Delta t_1,x;t_3,y) - G(t_1,x;t_1,x;t_3,y)}{\Delta t_1} =$$

$$= q_1(t_1,x)G(t_1,x;t_1+\Delta t_1,x;t_3,y) +$$

$$- q_1(t_1,x)\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;z) + o(1).$$

Hence, in the limit, as $\Delta t_1 \to 0$ we obtain

$$\left.\frac{\partial G(t_1,x;\tau,x;t,y)}{\partial \tau}\right|_{\tau = t_1} = q_1(t_1,x)G(t_1,x;t_1,x;t_3,y)+$$

$$- \lim_{\Delta t_1 \to 0} q_1(t_1,x)\int_{-\infty}^{+\infty} G(t_1,x;t_1+\Delta t_1,z;t_3,y)d_zP_1(t_1,x;z) .$$

This completes the proof of the theorem.

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