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Wojciech Żakowski

ON SOME PROPERTIES OF THE CAUCHY TYPE INTEGRAL IN THE EUCLIDEAN SPACE E_3

The paper considers the properties of the surface singular integral (6). This integral appears in the theory of holomorphic functions in the space E_3. It is proved that if the column $q(Q)$ satisfies the Hölder condition with the constant K and exponent $h \in (0,1)$ then the column $F(P)$ defined by the equality (21) satisfies the Hölder condition with a constant directly proportional to K and with the same exponent h as in $q(Q)$. The case when $h=1$ is also discussed, and some conclusions are drawn.

1. INTRODUCTION

Consider a bounded region D^+ in the Euclidean space E_3; the boundary S of this region is a closed Lapunov surface. Let D^- be the unbounded region, complementing the set $S+D^+$ to the space E_3.

We introduce the following notations (cf. [1]):

$$D(X,Y,Z) = \begin{Vmatrix} 0 & X & Y & Z \\ X & 0 & -Z & Y \\ Y & Z & 0 & -X \\ Z & -Y & X & 0 \end{Vmatrix}, \quad D^*(X,Y,Z) = \begin{Vmatrix} 0 & X & Y & Z \\ X & 0 & Z & -Y \\ Y & -Z & 0 & X \\ Z & Y & -X & 0 \end{Vmatrix} \tag{1}$$

and

$$M(A,Q) = -D^*\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) \frac{1}{\varrho(A,Q)} \cdot D(\cos\alpha, \cos\beta, \cos\gamma) \tag{2}$$

where $\rho(A,Q)$ denotes the dinstance of the point $A(x,y,z)$ from the point $Q(\xi,\eta,\zeta)\in S$ and $[\cos\alpha, \cos\beta, \cos\gamma]$ is the unit vector of the normal to the surface S in point Q, oriented externally with respect to the region D^+.

Let $q(Q)$ be the column

$$\begin{Vmatrix} q_1(Q) \\ q_2(Q) \\ q_3(Q) \\ q_4(Q) \end{Vmatrix} \tag{3}$$

the elements of which are defined and continuous on the surface S. The surface integral

$$\frac{1}{4\pi}\int_S M(A,Q)q(Q)ds_Q, \tag{4}$$

defined in every point A of the set $D^- + D^+$ will be called the Cauchy type integral in the space E_3. It is known, [1], that the column $p(A)$, defined in the set $D^- + D^+$ by the equality

$$p(A) = \frac{1}{4\pi}\int_S M(A,Q)\cdot q(Q)ds_Q \tag{5}$$

is holomorphic in this set, i.e. it satisfies in it the elliptic system of equations

$$D\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)p(A) = 0.$$

Furthermore, if the column (3) satisfies on the surface S the Hölder condition, then for each $P\in S$ there exists a singular integral in the sense of Cauchy principal value

$$\frac{1}{4\pi}\int_S M(P,Q)\cdot q(Q)ds_Q \tag{6}$$

and it is expressed by an absolutely convergent integral

$$\frac{1}{4\pi}\int_S M(P,Q)q(Q)ds_Q = \frac{1}{2}q(P) + \frac{1}{4\pi}\int_S M(P,Q)\left[q(Q)-q(P)\right]ds_Q. \tag{7}$$

We prove that then there exist boundary values of the column p(A) defined by the equality (5), namely

$$p^{+}(P) = \lim_{D^{+}\ni A\to P\in S} p(A), \quad \text{and} \quad p^{-}(P) = \lim_{D^{+}\ni A\to P\in S} p(A)$$

connected with the singular integral (6) in the following way.

$$p^{\pm}(P) = \pm \frac{1}{2}\, q(P) + \frac{1}{4\pi}\int_{S} M(P,Q)q(Q)ds_{Q} \, . \tag{8}$$

The formulae (8) are analogous to those of S o c h o c k i - P l e m e l j [2] which are of essential importance in the boundary problems for a complex variable function.

2. PROPERTIES OF THE MATRIX M(P,Q)

The kernel M(P,Q) of the integral (6), defined by the equalities (1) and (2), may, after easy calculations, be expressed in the form

$$M(P,Q) = \frac{1}{\varrho^{2}(P,Q)}\, M^{o}(P,Q) \tag{9}$$

here

$$M^{o}(P,Q) = \begin{Vmatrix} \vec{s}\cdot\vec{n} & W_x & W_y & W_z \\ -W_x & \vec{s}\cdot\vec{n} & W_z & -W_y \\ -W_y & -W_z & \vec{s}\cdot\vec{n} & W_x \\ -W_z & W_y & -W_x & \vec{s}\cdot\vec{n} \end{Vmatrix} \tag{10}$$

where $\vec{n}$ is the unit vector $[\cos\alpha, \cos\beta, \cos\gamma]$ introduced above, $\vec{s}$ - the unit vector of the vector $\vec{PQ}$, $\vec{s}\cdot\vec{n}$ - the scalar product, and $[W_x, W_y, W_z]$ is the vector product $\vec{s}\times\vec{n}$.

Since $\det M^{o}(P,Q) = 1$, then in view of the equality (9) we have

$$\det M(P,Q) = \frac{1}{\varrho^{8}(P,Q)} \, . \tag{11}$$

We shall now consider some properties of the matrices (9) and (10). Substituting into (7) the column $q(Q)$ with elements $q_1 = 1$ and $q_2 = q_3 = q_4 = 0$ we obtain

$$\frac{1}{4\pi}\int_S M(P,Q)\cdot \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix} ds_Q = \begin{Vmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{Vmatrix}$$

i.e.

$$\frac{1}{4\pi}\int_S \frac{M^o_{ik}(P,Q)}{\varrho^2(P,Q)}\, ds_Q = \frac{1}{2}\,\delta_{ik}, \tag{12}$$

where $M^o_{ik}(P,Q)$ is the element of the i-th line and the k-th column of the matrix (10), and δ_{ik} - the Kronecker delta.

The elements $M^o_{ik}(P,Q)$ obviously satisfy the condition

$$\left|M^o_{ik}(P,Q)\right| \leqslant 1. \tag{13}$$

Next, we shall show that for every three points P_1, P_2 and Q, $(Q \neq P_2)$ of the surface S the following inequality is fulfilled

$$\left|M^o_{ik}(P_1,Q) - M^o_{ik}(P_2,Q)\right| \leqslant 2\pi\,\frac{\varrho(P_1,P_2)}{\varrho(Q,P_2)} \qquad (i,k = 1,2,3,4). \tag{14}$$

Assume that points P_1, P_2 and Q do not lie on the same straight line and consider two cases: $i=k$ and $i\neq k$.

If $i=k$ then $M^o_{ik}(P,Q) = \cos(\vec{PQ},\vec{n})$ thus, denoting $\theta = \sphericalangle(\overrightarrow{P_1Q},\overrightarrow{P_2Q})$, we get

$$\left|M^o_{ik}(P_1,Q) - M^o_{ik}(P_2,Q)\right| \leqslant \theta \leqslant \pi \sin\frac{\theta}{2}. \tag{15}$$

Since

$$\frac{\rho(P_0,P_2)}{\sin\frac{\theta}{2}} = \frac{\rho(Q,P_2)}{\sin\delta},$$

where $\delta = \sphericalangle(\overrightarrow{P_0Q},\overrightarrow{P_0P_2})$ and P_0 denotes a point at which the bisetrix of the angle θ in the triangle QP_1P_2 intersects the side P_1P_2, so, on account of the inequality $\rho(P_1,P_2) > \rho(P_0,P_2)$ we have

$$\rho(P_1,P_2)\sin\delta > \rho(Q,P_2)\sin\frac{\theta}{2},$$

i.e.

$$\sin\frac{\theta}{2} < \frac{\rho(P_1,P_2)}{\rho(Q,P_2)}. \tag{16}$$

Now, taking into account the estimates (15) and (16) we get the inequality (14).

If $i \neq k$, then $M^o_{ik}(P,Q)$ is equal (with an accuracy to one sign) to one of the coordinates of the vector product $\vec{s}\times\vec{n}$. Let $\vec{s}_1 = [\cos\alpha_1,\cos\beta_1,\cos\gamma_1]$ and $\vec{s}_2 = [\cos\alpha_2,\cos\beta_2,\cos\gamma_2]$ be the unit vectors of the vector $\overrightarrow{P_1Q}$ and $\overrightarrow{P_2Q}$, correspondingly. Consider the case when $M^o_{ik}(P,Q) = W_x$; the remaining cases are considered analogously. Since

$$W_x(P_1,Q) = \cos\beta_1\cos\gamma - \cos\gamma_1\cos\beta$$
$$W_x(P_2,Q) = \cos\beta_2\cos\gamma - \cos\gamma_2\cos\beta,$$

then the following inequality holds true

$$\left|M^o_{ik}(P_1,Q) - M^o_{ik}(P_2,Q)\right| \leqslant 2\theta$$

where θ denotes as above the angle between the vector $\overrightarrow{P_1Q}$ and the vector $\overrightarrow{P_2Q}$. Hence, in view of (15) and (16) we obtain the inequality (14).

If the points P_1,P_2 and Q lie on the same straight line, then either the left-hand side of the inequality (14) equals zero, or it is not greater than four, and, moreover, $\rho(P_1,P_2) > \rho(Q,P_2)$ and consequently also in this case the inequality (14) holds true.

We shall next prove that for every $P \in S$ and for every $\varepsilon > 0$ the following inequality is satisfied

$$\left| \int_{S_\varepsilon} \frac{M^o_{ik}(P,Q)}{\rho^2(P,Q)} \, ds_Q \right| \leq 8\pi, \tag{17}$$

$$(i,k = 1,2,3,4),$$

where S_ε denotes the set of all points of the surface S which do not lie inside the sphere with the centre P and radius ε.

Note that

$$\int_{S_\varepsilon} \frac{M^o(P,Q)}{\rho^2(P,Q)} \, q(Q) ds_Q = \int_{S_\varepsilon + \sigma} \frac{M^o(P,Q)}{\rho^2(P,Q)} \, q(Q) ds_Q - \int_{\sigma} \frac{M^o(P,Q)}{\rho^2(P,Q)} q(Q) ds_Q, \tag{18}$$

where σ denotes that part of the surface of the sphere with centre P and radius ε which does not lie in the region D^+. Let $q(Q)$ be a column with the elements $q_1 = 1$, $q_2 = q_3 = q_4 = 0$. Obviously, this column is holomorphic in the whole space. Thus we cam make use of the analogue of the Cauchy integral formula ([1], p.170); hence we have

$$\int_{S_\varepsilon + \sigma} \frac{M^o(P,Q)}{\rho^2(P,Q)} \, q(Q) ds_Q = 4\pi q(P).$$

Thus, in view of (18) we obtain

$$\int_{S_\varepsilon} \frac{M^o_{ik}(P,Q)}{\rho^2(P,Q)} ds_Q = 4\pi \delta_{ik} - \int_{\sigma} \frac{M^o_{ik}(P,Q)}{\rho^2(P,Q)} ds_Q. \tag{19}$$

$$(i,k = 1,2,3,4)$$

Because for every $Q \in \sigma$ we have $\rho(P,Q) = \varepsilon$ therefore, taking into account the inequality (13), we finally get

$$\left| \int_{S_\varepsilon} \frac{M^o_{ik}(P,Q)}{\rho^2(P,Q)} ds_Q \right| \leq 4\pi + \frac{1}{\varepsilon^2} \cdot 4\pi\varepsilon^2 = 8\pi ,$$

and this completes the proof.

3. PROPERTIES OF THE SINGULAR INTEGRAL (6)

Let K be a positive number, and let $h \in (0,1]$. Denote by $H(K;h)$ the set of all columns (3) with elements defined on the surface S and satisfying on it the following inequalities

$$|q_i(P)| < K, \quad |q_i(P_1) - q_i(P_2)| \leqslant K \cdot \rho^h(P_1,P_2) \quad (i = 1,2,3,4). \tag{20}$$

T h e o r e m . If $q(Q) \in H(K;h)$, $0 < h < 1$ and

$$F(P) = \frac{1}{4\pi}\int_S M(P,Q)q(Q)ds_Q \tag{21}$$

then $F(P) \in H(C \cdot K;h)$, where C is a constant not depending on the column $q(Q)$.

Proof. On account of the inequality (7) and the first of inequalities (20) we have

$$|F_i(P)| \leqslant C_1 \cdot K, \quad \text{for} \quad i=1,2,3,4, \tag{22}$$

in which the constant

$$C_1 = \frac{1}{2} + \frac{1}{\pi}\int_S \frac{ds_Q}{\rho^{2-h}(P,Q)},$$

does not depend on the column $q(Q)$. Since the elements of the matrix $M^o(P,Q)$ satisfy the conditions (13), (14) and (17), the assumptions of the theorem 1 of [3] are fulfilled. Thus, the moduli of continuity $\omega_i(r,f_i)$ of the elements $f_i(P)$ of column

$$f(P) = \frac{1}{4\pi}\int_S M(P,Q)\left[q(Q) - q(P)\right]ds_Q \tag{23}$$

and the moduli of continuity $\omega_i(r;q_i)$ of the elements of column $q(Q)$ satisfy the following system of inequalities

$$\omega_i(r;f_i) < C_2 \sum_{j=1}^{4}\left[\omega_j(r;q_j) + \int_0^r \frac{\omega_j(\tau;q_j)}{\tau}d\tau + r\int_r^{\eta} \frac{\omega_j(\tau;q_j)}{\tau^2}d\tau\right] \tag{24}$$

$$(i = 1,2,3,4),$$

where C_2 and η are constants independent of the column $q(Q)$. Since the elements of the column $q(Q)$ satisfy the Hölder condition (20), then $\omega_i(\tau;q_i) \leq K\cdot\tau^h$ and, therefore, in view of (24), we have

$$\omega_i(r;f_i) \leq C_3\cdot K\cdot r^h,$$

$$(i = 1,2,3,4),$$

where the constant C_3 does not depend on the column $q(Q)$. Hence it follows that the elements of the column (23) satisfy on the surface S the Hölder condition with constant $C_3\cdot K$ and with exponent h. Since the column (23) is the second component on the right-hand side of the equality (7), then the elements of the column (21) satisfy on the surface S the Hölder condition with the constant $C_4\cdot K$, where $C_4 = \frac{1}{2} + C_3$, and with the exponent h, i.e.

$$\left|F_i(P_1) - F_i(P_2)\right| \leq C_4\cdot K\cdot \rho^h(P_1,P_2), \text{ for } i=1,2,3,4. \quad (25)$$

From the inequalities (22) and (25) it follows that $F(P) \in H(C\cdot K;h)$, where $C = \max(C_1,C_4)$, and this completes our proof.

C o n c l u s i o n 1. If $q(Q) \in H(K;1)$, then the column $F(P)$ defined by the equality (21) belongs to the class $H(C_0\cdot K;1-\varepsilon)$, where ε is an arbitrary positive number, and the constant C_0 does not depend on the column $q(Q)$.

This conclusion follows immediately from the inclusion

$$H(K;1) \subset H(C^*\cdot K;1-\varepsilon),$$

where

$$C^* = \sup \rho^{\varepsilon}(P_1,P_2).$$

C o n c l u s i o n 2. If $q(Q) \in H(K;h)$, $0 < h < 1$, then the boundary values $p^+(P)$ and $p^-(P)$ of the column (5) belong to the class $H(\tilde{C}\cdot K;h)$, where $\tilde{C} = 1 + C$.

This conclusion follows from the equality (8) and from our Theorem.

It is known ([1], p.178) that if the column $\varphi(P)$ satisfies on the surface S the Hölder condition, then the integral equation

$$\frac{1}{2\pi}\int_S M(P,Q)\,\phi(Q)ds_Q = \varphi(P) \tag{26}$$

has in the class of columns $\phi(P)$, satisfying on the surface S the Hölder condition, exactly one solution, namely

$$\phi(P) = \frac{1}{2\pi}\int_S M(P,Q)\ \varphi(Q)ds_Q. \tag{27}$$

Let us assume that $\varphi(P) \in H(K;h)$ where $0<h<1$. In view of our Theorem and the equality (27) we obtain

$$\phi(P) \in H(2C\cdot K;h)$$

and then, making use of the equality (26) we find

$$\varphi(P) \in H(4C^2\cdot K;h). \tag{28}$$

C o n c l u s i o n 3. The constant C mentioned in our Theorem is not less than $\frac{1}{2}$.

Indeed, the column f(P) with constant elements equal to unity belongs to the set $H(1;h)$ and, on account of (28), also to the set $H(4C^2;h)$, hence $4C^2 \geqslant 1$, i.e.

$$C \geqslant \frac{1}{2} \tag{29}$$

Since on the basis of the equality (7) we have

$$\frac{1}{4\pi}\int_S M(P,Q)\cdot\begin{Vmatrix}1\\1\\1\\1\end{Vmatrix} ds_Q = \frac{1}{2}\begin{Vmatrix}1\\1\\1\\1\end{Vmatrix}$$

then the estimate (29) cannot be improved.

C o n c l u s i o n 4. If the column q(Q) satisfies on the surface S the Hölder condition and if the column F(P), defined by the formula (21), belongs to the set H(K;h), where $0<h<1$, then $q(P)\in H(4C\cdot K;h)$.

Indeed, on account of the equalities (26) and (27) we have

$$q(P) = \frac{1}{\pi}\int_S M(P,Q)F(Q)ds_Q, \tag{30}$$

therefore $q(P)\in H(4C\cdot K;h)$.

Hence it follows that besides the implication

$$q(Q)\in H(K;h) \Longrightarrow \frac{1}{4\pi}\int_S M(P,Q)q(Q)ds_Q \in H(C\cdot K;h)$$

mentioned in our Theorem, also the implication

$$\frac{1}{4\pi}\int_S M(P,Q)q(Q)ds_Q \in H(K;h) \Longrightarrow q(Q)\in H(4C\cdot K;h)$$

is true with respect to Hölder columns $q(Q)$.

From the equalities (26) and (27) we can then easily conclude that if there exists a column $q(P)$ satisfying on the surface S the Lipschitz condition and such that the column $F(P)$ corresponding to it in transformation (21) does not satisfy the Lipschitz condition (comp. Conclusion 1), then there exists also a column $q^*(P)$ satisfying on the surface S the Hölder condition, but not satishying the Lipschitz condition, such that the column $F^*(P)$ corresponding to it in transformation (21) satisfies the Lipschitz condition.

Obviously, there exist columns $q(P)$, satisfying on the surface S the Lipschitz condition, whose images $F(P)$ in the transformation (21) also satisfy the Lipschitz condition. Such is, for example, the column formed of ones alone.

Our Theorem constitutes a basis for investigation, by the fixed point method, of non-linear boundary problems for holomorphic functions in the space E_3, and of singular integral equations connected with them.

These problems shall be dealt with in our further papers.

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Adress of author: prof.dr Wojciech Żakowski,
Warszawa,ul.Brazylijska 5a m 38