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PARTIAL DIFFERENTIAL EQUATIONS AND INFINITESIMAL PROPERTIES OF NON-MARKOVIAN STOCHASTIC PROCESSES

Let Y_t $(t \geq 0)$ be a stochastic process with real values from some interval I. Let F and H be the conditional probability distribution functions of this process, defined as

$$F(t_o,y_o,t,y) = P(Y_t < y \mid Y_{t_o} = y_o) \tag{1}$$

$$H(t_o,y_o,w,z,t,y) = P(Y_t < y \mid Y_{t_o} = y_o,\ Y_w = z),\quad 0 \leq t_o \leq w < t. \tag{2}$$

From (1) and (2) it follows that

$$\begin{aligned} F(t_o,y_o,t,y) &= \int_J P(Y_t < y \mid Y_{t_o} = y_o,\ Y_w = z)\, d_z P(Y_w < z \mid Y_{t_o} = y_o) = \\ &= \int_J H(t_o,y_o,w,z,t,y)\, d_z F(t_o,y_o,w,z). \end{aligned} \tag{3}$$

Note that the functions $H(t_o,y_o,w,z,t,y)$ is undefined on the set S: $t_o = w$, $y_o \neq z$ since by putting $t_o = w$, $y_o \neq z$ we obtain contradicting conditions in (2). Thus, all subsequent considerations will concern only the values outside S, where the function H will be assumed well defined.

Let us assume that the derivations

$$\frac{\partial F}{\partial y} = f(t_o,y_o,t,y), \tag{4}$$

$$\frac{\partial H}{\partial y} = h(t_o,y_o,w,z,t,y) \tag{5}$$

exist and are continuous with respect to all arguments.

Relations (3) - (5) imply that

$$f(t_0,y_0,t,y) = \int_J h(t_0,y_0,w,z,t,y)f(t_0,y_0,w,z)\,dz. \qquad (6)$$

Equation (6) is a generalization of the Chapman-Kołmogorov equation to the case of densities of non-Markovian processes. The fact that the first factor of the integrend depends on t_0 and y_0 shows that the Markov property does not hold for the process under consideration [1].

In paper [2] a partial differential equation was derived for non-Markovian process, this equation was a generalization of the Kołmogorov equation for Markovian processes. More precisely, it was proved that the following theorem holds.

T h e o r e m 1. If for any $\delta > 0$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-z|<\delta} (y-z)h(t_0,y_0,t,z,t+\Delta t,y)\,dy = a_1^{[h]}(t_0,y_0,t,z), \qquad (7)$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-z|<\delta} (y-z)^2 h(t_0,y_0,t,z,t+\Delta t,y)\,dy = a_2^{[h]}(t_0,y_0,t,z), \qquad (8)$$

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-z|\geq\delta} (y-z)^2 h(t_0,y_0,t,z,t+\Delta t,y)\,dy = 0 \qquad (9)$$

where the convergence in (7)-(9) is uniform in z,

$$\left.\begin{array}{l} \frac{\partial}{\partial t} f(t_0,y_0,t,y), \quad \frac{\partial}{\partial y}\left[a_1^{[h]}(t_0,y_0,t,y)f(t_0,y_0,t,y)\right] \\ \frac{\partial^2}{\partial y^2}\left[a_2^{[h]}(t_0,y_0,t,y)\, f(t_0,y_0,t,y)\right] \end{array}\right\} \qquad (10)$$

exist and are continuous with respect to all arguments, then the following partial differential equation holds

$$\frac{\partial}{\partial t} f(t_0,y_0,t,y) + \frac{\partial}{\partial y}\left[a_1^{[h]}(t_0,y_0,t,y)f(t_0,y_0,t,y)\right] = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[a_2^{[h]}(t_0,y_0,t,y)f(t_0,y_0,t,y)\right]. \qquad (11)$$

Equation (11) corresponds to the prospective equation. Besides, the following fact was proved in [2]. Let $P\left[Y_{t_0}=y_0\right]=1$. Denote by K the class of functions such that

1° elements of K are probability densities

2° elements of K are homogeneous functions of order $-\varkappa$ of arguments $y-y_0$, $(t-t_0)^{1/p}$ where $\varkappa$ is an arbitrary real number different from zero

while

$$\begin{cases} a_1^{[h]}(t_0,y_0,t,y) = \dfrac{2+\alpha-p}{p}\,(y-y_0)^{1/p} \\ a_2^{[h]}(t_0,y_0,t,y) = \dfrac{2}{p}\,(y-y_0)^{2-p}, \end{cases} \tag{12}$$

then the only function f in class K satisfying (11) is for $y-y_0>0$

$$f(t-t_0,y-y_0) = \frac{p^{\frac{p-1-\alpha}{p}}}{\Gamma\left(\frac{\alpha+1}{p}\right)(t-t_0)^{1/p}}\left[\frac{y-y_0}{(t-t_0)^{1/p}}\right]\exp\left[-\frac{1}{p}\left(\frac{y-y_0}{(t-t_0)^{1/p}}\right)^{p}\right] \tag{13}$$

Moreover, if

$$\begin{aligned} a_1^{[h]} &= \frac{1}{p}\,(2+\alpha-p)\,|y-y_0|^{1-p}\operatorname{sgn}(y-y_0) \\ a_2^{[h]} &= \frac{2}{p}|y-y_0|^{2-p} \end{aligned} \tag{14}$$

then the function satisfying (11) is

$$f(t-t_0,y-y_0) = \frac{p^{\frac{p-1-\alpha}{p}}}{\cdot 2\Gamma\left(\frac{\alpha+1}{p}\right)(t-t_0)^{1/p}}\left[\frac{|y-y_0|}{(t-t_0)^{1/p}}\right]^{\alpha}\exp\left[-\frac{1}{p}\left(\frac{|y-y_0|}{(t-t_0)^{1/p}}\right)^{p}\right] \tag{15}$$

where $\alpha > 0$ and p is natural.

In this paper we shall present

I the equation for non-Markovian processes which corresponds to the retrospective equation in case of Markov processes

II A generalization of equation (11) to the case of equations of order higher than two, containing infinitesimal moments of order higher than two

III The infinitesimal operator of functions h and f.

The properties of infinitesimal characteristics, their phisical interpretation and application to stochastic processes are discuss, for example, in [3], [4], [5].

I. We shall prove

T h e o r e m 2. If for any $\delta > 0$ uniformly in y_0

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|<\delta}(y-y_0)f(t_0,y_0,t_0+\Delta t_0,y)dy = a_1^{[f]}(t_0,y_0) \tag{16}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|<\delta}(y-y_0)^2 f(t_0,y_0,t_0+\Delta t_0,y)dy = a_2^{[f]}(t_0,y_0) \tag{17}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|\geq\delta}(y-y_0)^2 f(t_0,y_0,t_0+\Delta t_0,y)dy = 0 \tag{18}$$

there exist continuous and bounded derivatives

$$\frac{\partial h(t_0,y_0,w,z,t,y)}{\partial z},\quad \frac{\partial^2 h(t_0,y_0,w,z,t,y)}{\partial z^2},\quad \frac{\partial h(t_0,y_0,w,z,t,y)}{\partial w} \tag{19}$$

then the following partial differential equation holds

$$2\left.\frac{\partial h(t_0,y_0,w,y_0,t,y)}{\partial w}\right|_{w=t_0} = 2a_1^{[f]}(t_0,y_0)\left.\frac{\partial h(t_0,y_0,t_0,z,t,y)}{\partial z}\right|_{z=y_0} + a_2^{[f]}(t_0,y_0)\left.\frac{\partial^2 h(t_0,y_0,t_0,z,t,y)}{\partial z^2}\right|_{z=y_0} \tag{20}$$

In this equation the symbols

$$\left.\frac{\partial h(t_0,y_0,t_0,z,t,y)}{\partial z}\right|_{z=y_0}, \quad \left.\frac{\partial^2 h(t_0,y_0,t_0,z,t,y)}{\partial z^2}\right|_{z=y_0}$$

denote the limits

$$\left.\frac{\partial h(t_0,y_0,t_0,z,t,y)}{\partial z}\right|_{z=y_0} = \lim_{\Delta t\to 0}\left[\left.\frac{\partial h(t_0-\Delta t,y_0,t_0,z,t,y)}{\partial z}\right|_{z=y_0}\right]$$

$$\left.\frac{\partial^2 h(t_0,y_0,t_0,z,t,y)}{\partial z^2}\right|_{z=y_0} = \lim_{\Delta t\to 0}\left[\left.\frac{\partial^2 h(t_0-\Delta t,y_0,t_0,z,t,y)}{\partial z^2}\right|_{z=y_0}\right]$$

The above explanation is connected with the fact,that the function H, hence also h, is undefined on the set S.

Proof. Note that

$$\begin{aligned} H(t_0,y_0,t_0,y_0,t,y) &= P(Y_t < y \mid Y_{t_0} = y_0,\ Y_{t_0} = y_0) \\ &= P(Y_t < y \mid Y_{t_0} = y_0) \end{aligned}$$

Therefore

$$h(t_0,y_0,t_0,y_0,t,y) = f(t_0,y_0,t,y). \tag{21}$$

Expanding h into Taylor's formula, we can write equation (6) in the form

$$\begin{aligned} f(t_0-\Delta t,y_0,t,y) &= \int_J f(t_0-\Delta t,y_0,t_0,z)h(t_0-\Delta t,y_0,t_0,z,t,y)dz = \\ &= \int_J f(t_0-\Delta t,y_0,t_0,z)\ h(t_0-\Delta t,y_0,t_0,y_0,t,y) + \\ &+ (z-y_0)\frac{\partial}{\partial z} h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0} + \end{aligned} \tag{22}$$

$$+\frac{(z-y_0)^2}{2}\frac{\partial^2}{\partial z^2}h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0}+$$
$$+\frac{o(z-y_0)^2}{2}\frac{\partial^2}{\partial z^2}h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0}\Bigg]dz. \tag{22}$$

It follows from (21) and (22) that

$$\frac{1}{\Delta t}\left[h(t_0-\Delta t,y_0,t_0-\Delta t,y_0,t,y)-h(t_0-\Delta t,y_0,t_0,y_0,t,y)\right]= \tag{23}$$
$$=\frac{\partial}{\partial z}h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0}\frac{1}{\Delta t}\int\limits_{|z-y_0|<\delta}(z-y_0)f(t_0-\Delta t,y_0,t_0,z)dz+$$
$$+\frac{1}{2}\frac{\partial^2}{\partial z^2}h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0}\frac{1}{\Delta t}\int\limits_{|z-y_0|<\delta}(z-y_0)^2f(t_0-\Delta t,y_0,t_0,z)dz+$$
$$+\frac{1}{2}\frac{\partial^2}{\partial z^2}h(t_0-\Delta t,y_0,t_0,z,t,y)\Big|_{z=y_0}\frac{1}{\Delta t}\int\limits_{|z-y_0|<\delta}o(z-y_0)^2f(t_0-\Delta t,y_0,t_0,z)dz+$$
$$+o(\Delta t).$$

Next note that

$$\left|\frac{1}{\Delta t}\int\limits_{|z-y_0|<\delta}o(z-y_0)^2f(t_0-\Delta t,y_0,t_0,z)dz\right|= \tag{24}$$
$$=\left|\frac{1}{\Delta t}\int\limits_{|z-y_0|<\delta}o(z-y_0)(z-y_0)^2f(t_0-\Delta t,y_0,t_0,z)dz\right|\leqslant$$
$$<\delta\frac{1}{\Delta t}\int\limits_{|z-y|<\delta}(z-y_0)^2f(t_0-\Delta t,y_0,t_0,z)dz\xrightarrow[\text{when }\delta\to0]{}0.$$

Passing to the limit with $\Delta t\to 0$ in (23) and using (16), (17), (21) and (24) we obtain (20), g.e.d.

II. We shall prove

T h e o r e m 3. If for any $\sigma > 0$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y-z|<\sigma} (y-z)^{2k-1} h(t_o,y_o,t,z,t+\Delta t,y)dy = \overset{[h]}{a}_{2k-1}(t_o,y_o,t,z), \qquad (25)$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y-z|<\sigma} (y-z)^{2k} h(t_o,y_o,t,z,t+\Delta t,y)dy = \overset{[h]}{a}_{2k}(t_o,y_o,t,z), \qquad (26)$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|y-z|\geq\sigma} (y-z)^{2k} h(t_o,y_o,t,z,t+\Delta t,y)dy = 0 \qquad (27)$$

$k = 1,2,\ldots,$ the convergence in (25) - (27) is uniform in z, the order of convergence is, for $r \leq k$, given by formulas

$$\int_{|y-z|<\sigma} (y-z)^{2r-1} h(t_o,y_o,t,z,t+\Delta t,y)dy = \Delta t^r \overset{[h]}{a}_{2r-1}(t_o,y_o,t,z) + o(\Delta t)^k,$$

$$\int_{|y-z|<\sigma} (y-z)^{2r} h(t_o,y_o,t,z,t+\Delta t,y)dy = \overset{[h]}{a}_{2r}(t_o,y_o,t,z)\Delta t^r + o(\Delta t)^k,$$

$$\int_{|y-z|\geq\sigma} (y-z)^{2r} h(t_o,y_o,t,z,t+\Delta t,y)dy = o(\Delta t)^k$$

there exist continuous derivatives $\frac{\partial^k}{\partial t^k} f(t_o,y_o,t,y)$,

$$\frac{\partial^{2k}}{\partial y^{2k}}\left[\overset{[h]}{a}_{2k}(t_o,y_o,t,y)f(t_o,y_o,t,y)\right] \qquad (28)$$

$$\frac{\partial^{2k-1}}{\partial y^{2k-1}}\left[\overset{[h]}{a}_{2k-1}(t_o,y_o,t,y)f(t_o,y_o,t,y)\right]$$

then for any natural k we have

$$\frac{1}{k!}\,\frac{\partial^k f(t_o,y_o,t,y)}{\partial t^k} + \qquad (29)$$

$$+\frac{1}{(2k-1)!}\frac{\partial^{2k-1}}{\partial y^{2k-1}}\left[a_{2k-1}^{[h]}(t_0,y_0,t,y)f(t_0,y_0,t,y)\right] = \tag{29}$$
$$= \frac{1}{(2k)!}\frac{\partial^{2k}}{\partial y^{2k}}\left[a_{2k}^{[h]}(t_0,y_0,t,y)f(t_0,y_0,t,y)\right].$$

Proof. We shall proceed by induction. For $k=1$ the theorem reduces to theorem 1. Let us assume that the theorem holds for $r<k$. We shall show that it holds for $r=k$.

Let a and b be arbitrary real numbers such that $(a,b)\subset I$. Let $R(y)$ denote and arbitrary non-negative function from class $C^{(k)}$ and let

$$R(y) = 0 \quad \text{for} \quad y<a \quad \text{and} \quad y>b.$$

Then

$$R^{(r)}(a) = R^{(r)}(b) = 0 \quad \text{for} \quad r = 1,2,\dots,k.$$

Let us expand f and R into Taylor series

$$f(t_0,y_0,t+\Delta t,y) = f(t_0,y_0,t,y) + \frac{\Delta t}{1!}\frac{\partial f(t_0,y_0,t,y)}{\partial t} + \dots$$
$$+\frac{\Delta t^{k-1}}{(k-1)!}\frac{\partial^{k-1} f(t_0,y_0,t,y)}{\partial t^{k-1}} + \frac{\Delta t^{k}}{k!}\frac{\partial^{k} f(t_0,y_0,t,y)}{\partial t^{k}} + o(\Delta t)^k, \tag{30}$$

$$R(z) = R(y) + \frac{y-z}{1}R(y) + \dots + \frac{(y-z)^{k-1}}{(k-1)!}R^{(k-1)}y + \tag{31}$$
$$+\frac{(y-z)^k}{k!}R^{(k)}y + o(y-z)^k.$$

Using (6), (30), changing the order of integration, and changing the notations of variables of integration, we obtain

$$\int_a^b f(t_0,y_0,t+\Delta t,y)\, R(y) =$$
$$=\int_a^b\left[\sum_{r=0}^{k}\frac{\Delta t^r}{r!}\frac{\partial^r f(t_0,y_0,t,y)}{\partial t^r} + o(\Delta t)^k\right]R(y)dy = \tag{32}$$

$$= \int_a^b dy \int_J f(t_0,y_0,t,z)h(t_0,y_0,t,z,t+\Delta t,y)R(y)dz =$$

$$= \int_a^b dz \int_J f(t_0,y_0,t,z)h(t_0,y_0,t,z,t+\Delta t,y)R(y)dy = \quad (32)$$

$$= \int_J dy \int_{|y-z|<\delta} f(t_0,y_0,t,y)h(t_0,y_0,t,y,t+\Delta t,z)R(z)dz + o(\Delta t)^k.$$

It follows from (25), (26), 31 and (32) that

$$\int_a^b \left[\sum_{r=0}^{k} \frac{\Delta t^r}{r!} \frac{\partial^r f(t_0,y_0,t,y)}{\partial t^r} + o(\Delta t)^k\right] R(y)dy = \quad (33)$$

$$= \int_J f(t_0,y_0,t,y)dy \int_{|y-z|<\delta} h(t_0,y_0,t,y,t+\Delta t,z) \left[\sum_{r=0}^{2k} \frac{(y-z)^r}{r!} R^{(r)}(y) + o(y-z)^{2k}\right] dz + o(\Delta t) =$$

$$= \int_J f(t_0,y_0,t,y)\Big\{R(y) + R'(y)\Big[\Delta t \, a_1^{[h]}(t_0,y_0,t,y) + o(\Delta t)^k + \ldots$$

$$+ \frac{1}{(2k-1)!} R^{(2k-1)}(y)\Big[\Delta t^k \, a_{2k-1}^{[h]}(t_0,y_0,t,y) + o(\Delta t)^k +$$

$$+ \frac{1}{2k} R^{(2k)}(y)\Big[\Delta t^k \, a_{2k}^{[h]}(t_0,y_0,t,y) + o(\Delta t)^k +$$

$$+ \int_{|y-z|<\delta} o(y-z)^{2k} h(t_0,y_0,t,y,t+\Delta t,z)dz\Big\} dy + o(\Delta t)^k.$$

Integrating by parts and using properties of function R we obtain

$$\int_a^b f(t_0,y_0,t,y)a_{2k-1}^{[h]}(t_0,y_0,t,y)R^{(2k-1)}(y)dy = \quad (34)$$

$$= -\int_a^b R(y)\,\frac{\partial^{2k-1}}{\partial y^{2k-1}}\left[\overset{[h]}{a}_{2k-1}(t_o,y_o,t,y)f(t_o,y_o,t,y)\right]dy. \tag{34}$$

$$\begin{aligned}&\int_a^b f(t_o,y_o,t,y)\overset{[h]}{a}_{2k}(t_o,y_o,t,y)R^{(2k)}(y)dy = \\ &= \int_a^b R(y)\,\frac{\partial^{2k}}{\partial y^{2k}}\left[\overset{[h]}{a}_{2k}(t_o,y_o,t,y)f(t_o,y_o,t,y)\right]dy.\end{aligned} \tag{35}$$

Grouping in (33) terms with equal powers of Δt and using (34) and (35) we obtain

$$\begin{aligned}&\int_a^b \Bigg\{\Delta t\left[\frac{\partial}{\partial t} f(t_o,y_o,t,y) + \frac{\partial}{\partial y}\left(\overset{[h]}{a}_1 f\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\overset{[h]}{a}_2 f\right)\right] + \ldots + \\ &+ \Delta t^{k-1}\left[\frac{\partial^{k-1} f}{\partial t^{k-1}}\frac{1}{(k-1)!} + \frac{\partial^{2(k-1)-1}}{\partial y^{2(k-1)-1}}\left(\overset{[h]}{a}_{2(k-1)-1} f\right)\frac{1}{[2(k-1)-1]!} + \right. \\ &\left. + \frac{\partial^{2(k-1)}}{\partial y^{2(k-1)}}\left(\overset{[h]}{a}_{2(k-1)} f\right)\frac{1}{[2(k-1)]!}\right] + \\ &+ \Delta t^{k}\left[\frac{\partial^{k} f}{\partial t^{k}}\frac{1}{k!} + \frac{\partial^{2k-1}}{\partial y^{2k-1}}\left(\overset{[h]}{a}_{2k-1} f\right)\frac{1}{(2k-1)!} + \right. \\ &\left.\left. - \frac{\partial^{2k}}{\partial y^{2k}}\left(\overset{[h]}{a}_{2k} f\right)\frac{1}{(2k)!}\right]\right\}R(y)dy + W = 0\end{aligned} \tag{36}$$

where

$$\begin{aligned}W = &\int_a^b f(t_o,y_o,t,y)\left\{o(\Delta t)^k\left[\frac{R'(y)}{1!} + \frac{R''(y)}{2!}\right] + \ldots + \right. \\ &+ o(\Delta t)^k\left[\frac{R^{(2k-1)}(y)}{(2k-1)!} + \frac{R^{(2k)}(y)}{(2k)!}\right. dy +\end{aligned} \tag{37}$$

$$+\int_a^b f(t_0,y_0,t,y)dy \int_{|y-z|<\delta} o(y-z)^{2k} R^{(2k)}(y)h(t_0,y_0,t,y,\Delta t+\Delta t,z)dz +$$
$$+ o(\Delta t)^k = I_1 + I_2 + o(\Delta t)^k.$$

By the inductive assumption, the terms in the first k-1 brackets are equal 0. We shall show that

$$\lim_{\Delta t\to 0} \frac{W}{\Delta t^k} = 0. \tag{38}$$

We shall first evaluate the integral $\frac{I_1}{\Delta t^k}$. From properties of function R it follows that there exist a constant M such that

$$\left|R^{(r)}(y)\right| \leq M \text{ for } r = 1,2,\dots,2k.$$

Next using the fact that

$$\sum_{r=0}^{\infty} \frac{1}{r!} = e$$

we obtain the following estimate for $\frac{I_1}{\Delta t^k}$

$$\frac{1}{\Delta t^k}|I_1| = \left|\frac{1}{\Delta t^k}\int_a^b f(t_0,y_0,t,y)\left\{o(\Delta t)^k\left[\frac{R'(y)}{1!} + \frac{R''(y)}{2!}\right] + \dots + \right.\right. \tag{39}$$
$$\left.\left. + o(\Delta t)^k\left[\frac{R^{(2k-1)}(y)}{(2k-1)!} + \frac{R^{(2k)}(y)}{(2k)!}\right]\right\}dy\right| \leq \frac{o(\Delta t)^k M e}{\Delta t^k} \xrightarrow[\text{When } \Delta t\to 0]{} 0.$$

Estimating the integral $\frac{I_2}{\Delta t^k}$ in a maner similar to that used for (24) and using (39) we obtain (38).

Relation (36), (38) and inductive assumption lead directly to the assertion of the theorem, that is to formula (29) g.e.d.

The reasoning which was used in proving theorem 3 is similar to that used in proving the corresponding theorem for Markov processes in [6].

In the case of Markov processes Y_t, the basic problem connected with Kołmoforov equation is that of determining function f given the functions $a_1^{[f]}$ and $a_2^{[f]}$. On the other hand, given function f, the functions $a_1^{[f]}$ and $a_2^{[f]}$ can be determined directly from formulas (16) and (17).

In the case of non-Markovian Y_t one could volve analogous problem, that is, determine the function f from equation (11) given the function $a_1^{[h]}$ and $a_2^{[h]}$ defined by (7) and (8). There exist, however, a converse problem, namely that of determining functions $a_1^{[h]}$, $a_2^{[h]}$ from (29) given the function f, that is, the problem of determining some quantities characterizing function h, given the function f. As an example, consider the case when Y_t is homogeneous in time and space, that is

$$f(t_0,y_0,t,y) = f(t-t_0,y-y_0)$$

$$h(t_0,y_0,t,z,t+\Delta t,y) = h(t-t_0,z-y_0,\Delta t,y-z)$$

$$a_r^{[h]}(t_0,y_0,t,y) = a_r^{[h]}(t-t_0,y-y_0), \quad r = 1,2.$$

Suppose, in addition, that f is given by (15), and denote

$$t-t_0 = s, \quad y-y_0 = x \tag{40}$$

under these assumptions, we shall determine functions $a_1^{[h]}$ and $a_2^{[h]}$ from equation (11).

Substituting function (15) into (11) and using (40) we obtain

$$\frac{\alpha+1}{p}\cdot\frac{|x|^{\alpha}}{s^{\frac{\alpha+1+p}{p}}}+\frac{|x|^{\alpha+p}}{ps^{\frac{\alpha+1+2p}{p}}}+\frac{|x|^{\alpha}}{s^{\frac{\alpha+1}{p}}}\frac{\partial a_1^{[h]}(s,x)}{\partial x}+$$
$$+\left[\frac{\alpha|x|^{\alpha-1}}{s^{\frac{\alpha+1}{p}}}-\frac{|x|^{\alpha+p-1}}{s^{\frac{\alpha+1+p}{p}}}\right]a_1^{[h]}(s,x)\operatorname{sgn} x = \tag{41}$$

$$= \frac{1}{2}\frac{|x|^{\alpha}}{s^{\frac{\alpha+1}{p}}}\cdot\frac{\partial^2 a_2^{[h]}(s,x)}{\partial x^2} + \left[\frac{\alpha|x|^{\alpha-1}}{s^{\frac{\alpha+1}{p}}} - \frac{|x|^{\alpha+p-1}}{s^{\frac{\alpha+1+p}{p}}}\right]\frac{\partial a_2^{[h]}(s,x)}{\partial x}\operatorname{sgn} x +$$

$$+ \frac{1}{2}\left[\frac{\alpha(\alpha-1)|x|^{\alpha-2}}{s^{\frac{\alpha+1}{p}}} - \frac{(2\alpha+p-1)|x|^{\alpha+p-2}}{s^{\frac{\alpha+1+p}{p}}} + \frac{|x|^{\alpha+2p-2}}{s^{\frac{\alpha+1+2p}{p}}}\right]a_2^{[h]}(s,x).$$

By assumption (15) concerning the form of f it follows that $a_1^{[h]}$ and $a_2^{[h]}$ must satisfy equation (41). This equation does not determine uniquely functions $a_1^{[h]}$ and $a_2^{[h]}$ and we must add some conditions. Let us assume that

$$a_1^{[h]}(s,x) = a_1^{[h]}(x), \qquad a_2^{[h]}(s,x) = a_2^{[h]}(x) \tag{42}$$

i.e. we assume independence of s. It follows from (42) that the coefficients at s with equal powers in (41) must be equal, thus, comparing the coefficients of

$$s^{-\frac{\alpha+1+2p}{p}},\quad s^{-\frac{\alpha+1+p}{p}},\quad s^{-\frac{\alpha+1}{p}}$$

we obtain the relations

$$\frac{1}{p} - \frac{1}{2}|x|^{p-2}a_2^{[h]}(x) = 0 \tag{43}$$

$$-\frac{\alpha+1}{p}|x|^2 - |x|^{p+1}\operatorname{sgn} x\, a_1^{[h]}(x) + |x|^{p+1}\operatorname{sgn} x\frac{da_1^{[h]}x}{dx} + \\ + (2\alpha+p-1)|x|^p a_2^{[h]}(x) = 0 \tag{44}$$

$$x^2\frac{da_1^{[h]}(x)}{dx} + \alpha|x|\operatorname{sgn} x\, a_1^{[h]}(x) - \frac{1}{2}x^2\frac{d^2a_2^{[h]}(x)}{dx^2} + \\ - \alpha|x|\operatorname{sgn} x\frac{da_2^{[h]}(x)}{dx} - \frac{1}{2}\alpha(\alpha-1)a_2^{[h]}(x) = 0. \tag{45}$$

From (43) - (45) it follows that

$$a_1^{[h]}(x) = \frac{1}{p}(2+\alpha-p)|x|^{1-p}\operatorname{sgn} x \tag{46}$$

$$a_2^{[h]}(x) = \frac{2}{p}|x|^{2-p} \tag{47}$$

Thus, if we assume a specific form of f, and make additional assumption (42) we obtain explicitely functions $a_1^{[h]}$ and $a_2^{[h]}$. Note that formulas (46), (47) are identical with formulas (12). Thus, the result obtained can be treated as a theorem converse to the result in [2], expressed by the pair of formulas (14), (15). In a similar way we can show that if f is given by (12) and $a_1^{[h]}$, $a_2^{[h]}$ satisfy (42), then these functions are given by (12).

III. Let g(z) be a continuous function defined in I. Under fairly general assumptions, the infinitesimal operator for homogeneous Markov proces is

$$A\, g(z) = \lim_{\Delta t\to 0} \frac{\int_J P(t,z,t+\Delta t,dy)g(y) - g(z)}{\Delta t} \tag{48}$$

This operation is connected with the transition function $P(t,z,t+\Delta t,dy)$.

Without assuming Markov property, let us define the following operators

$$A^{[f]} g(z) = \lim_{\Delta t\to 0} \frac{\int_J f(t,z,t+\Delta t,y)g(y)dy - g(z)}{\Delta t} \tag{49}$$

$$A^{[h]} g(z) = \lim_{\Delta t\to 0} \frac{\int_J h(t_0,y_0,t,z,t+\Delta t,y)g(y)dy - g(z)}{\Delta t} \tag{50}$$

We shall show that if $g(z) \leqslant C^2$ and the relations (16) - (18) hold then

$$A^{[f]}g(z) = g'(z)a_1^{[f]}(t,z) + \frac{1}{2} g''(z)a_2^{[f]}(t,z), \tag{51}$$

and if $g(z) \in C^2$ and relations (7) - (9) hold, then

$$A^{[h]}g(z) = g'(z)a_1^{[h]}(t_0,y_0,t,z) + \frac{1}{2} g''(z)a_2^{[h]}(t_0,y_0,t,z). \tag{52}$$

To prove (51), let us expand $g(z)$ into Taylor series

$$g(y) = g(z) + \frac{y-z}{1} g'(z) + \frac{(y-z)^2}{2!} g''(z) + o(y-z)^2. \tag{53}$$

Using succesively (49), (53) and then (7) - (9), we obtain

$$A^{[f]}g(z) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{J} f(t,z,t+\Delta t,y)g(y)dy - g(z)\right] =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\int_{|y-z|<\delta} f(t,z,t+\Delta t,y)g(y)dy + \int_{|y-z|>\delta} f(t,z,t+\Delta t,y)g(y)dy + \right.$$

$$\left. - g(z)\right\} =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\int_{|y-z|<\delta} f(t,z,t+\Delta t,y)\left[g(z)+(y-z)g'(z) + \frac{1}{2}(y-z)^2 g''(z) + \right.\right.$$

$$\left.\left. + o(y-z)^2\right]dy + \int_{|y-z|>\delta} f(t,z,t+\Delta t,y)g(y)dy - g(z)\right\} =$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ g'(z)\int_{|y-z|<\delta} (y-z)f(t,z,t+\Delta t,y)dy + \right.$$

$$\left. + \frac{1}{2} g''(z)\int_{|y-z|<\delta} (y-z)^2 f(t,z,t+\Delta t,y)dy + o(\Delta t)\right\} =$$

$$= g'(z)a_1^{[f]}(t,z) + \frac{1}{2} g''(z)a_2^{[f]}(t,z).$$

This proves relation (51). In a similar way we can prove (52).

Examples. 1° Let f be given by (15) for $p = 2$. Then

$$a_1^{[f]} = 0, \quad a_2^{[f]} = \alpha+1$$

$$A^{[f]} g(z) = \frac{1}{2} (\alpha+1) g''(z). \tag{54}$$

Putting $\alpha=0$ in (54) we obtain the result of example 2.16 from paper [4].

2°. Let f be given by (13) and $p = 1$. Then

$$a_1^{[f]}(t,z) = \alpha+1, \quad a_2^{[f]}(t,z) = 0$$

$$A^{[f]} g(z) = (\alpha+1)\, g'(z).$$

Besides infinitesimal operator and infinitesimal moments we can also introduce other infinitesimal characteristics for instance the infinitesimal characteristic function. Let

$$\psi(u,t_0,y_0,t,z) = 1 + iu\Delta t a_1^{[h]}(t_0,y_0,t,z) + \frac{(iu)^2}{2!} \Delta t a_2^{[h]}(t_0,y_0,t,z) + \frac{(iu)^3}{3!} \Delta t^2 a_3^{[h]}(t_0,y_0,t,z) + \frac{(iu)^4}{4!} \Delta t^2 a_4^{[h]}(t_0,y_0,t,z) + \dots \tag{55}$$

provided the series on the right hand side converges. Function defined by (55) has the following properties

$$\psi(0,t_0,y_0,t,z) = 1, \quad \psi(-u,t_0,y_0,t,z) = \overline{\psi(u,t_0,y_0,t,z)}$$

$$\frac{\partial^{2k} \psi(0,t_0,y_0,t,z)}{\partial u^{2k}} = i^{2k} \Delta t^k a_{2k}^{[h]},$$

$$\frac{\partial^{2k-1} \psi(0,t_0,y_0,t,z)}{\partial u^{2k-1}} = i^{2k-1} \Delta t^k a_{2k-1}^{[h]},$$

$$\lim_{\Delta t \to 0} \frac{\psi(u,t_0,y_0,t,z) - \varphi(u,t_0,y_0,t,z,t+\Delta t)}{\Delta t} = 0$$

and φ is a characteristic function corresponding to h, that is

$$\varphi(u,t_0,y_0,t,z,t+\Delta t) = \int_J e^{iuy} h(t_0,y_0,t,z,t+\Delta t,y)dy.$$

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