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Zbigniew Żekanowski

ON CERTAIN GENERALIZATION OF THE CONCEPT OF A SELF-DUAL TENSOR, A HARMONIC TENSOR AND A KILLING TENSOR IN A V_n

PREFACE

It is possible to prove that many properties of the p-vector fields as well as many geometrical, topological or differential facts which can be described by these fields are not only characteristic feature of p-vectors but that they constitute a common feature of a broader class of tensor fields.

The author of this paper deals with a certain sector of this problem, namely with proving that many properties of dual and self-dual p-vector fields, many properties of the harmonic p-vector fields or Killing p-vector fields cover a broader class of the tensor fields.

The solution of this problem reguired first of all the generalization into broader class of the tensor fields the concept of dual and self-dual p-vector and bi-tensor fields, the concept of the harmonic p-vector fields and Killing p-vector fields and finally the examination of certain properties of special tensor fields, thus generalized respectively.

Formally - in a sense - transferring of some concept of p-vector or bi-tensor fields into more generalized class of tensor fields, enabled the author to obtain also a member of new results for the p-vector fields and bi-tensor fields.

It has also occurred that at the harmonic tensor field, generalized in this manner (definition 1) it was possible to give non trivial examples on the existence of the generalized harmonic tensor fields in the Riemannian space (expanded metric tensor $\overset{p}{g}_{i_1\ldots i_p j_1\ldots j_p}$ and expanded curvature tensor $\overset{p}{R}_{i_1\ldots i_p j_1\ldots j_p}$ in the Einstein space V_{2p}).

Solution concerning the generalized self-dual p-vector and bi-tensor field also enabled the author to introduce the concept of a generalized Einstein space V_{2p} and concept of generalized conformal Euclidean space V_{2p} as well as to gave a number of necessary and sufficient conditions that the 2p-dimensional Riemannian space V_{2p} should be the generalized Einstein space or the generalized conformal - Euclidean space.

§1. INTRODUCTION

Let V_n be an n-dimensional Riemannian space with non singular fundamental tensor g_{ij} and reciprocal to it g^{ij}.

Consider a p-vector $f^{i_1\ldots i_p}$ at a fixed point of the examines space V_n. The scalar

$$\|f\|^2=\left(\frac{1}{p!}\right)^2\overset{p}{g}_{i_1\ldots i_p j_1\ldots j_p} f^{i_1\ldots i_p} f^{j_1\ldots j_p} \tag{1}$$

where (2)

$$\overset{p}{g}_{i_1\ldots i_p j_1\ldots j_p} \overset{df}{=} p!\, g_{[i_1[j_1}\cdots g_{i_p]j_p]}$$

is a square of the norm of the p-vector $f^{i_1\ldots i_p}$ [13].

The tensor $\overset{p}{g}_{i_1\ldots i_p j_1\ldots j_p}$ - defined above will be called here inaffer the expanded metric tensor.

If $\|f\|^2 = \pm 1$ then the p-vector $f^{i_1 \dots i_p}$ is called a unit p-vector, if $\|f\|^2 = 0$, the p-vector $f^{i_1 \dots i_p}$ is called singular p-vector.

Denote by $\varepsilon^{i_1 \dots i_n}$ and by $\mathfrak{E}_{i_1 \dots i_n}$ the contravariant and covariant Ricci's symbols - respectively. From the definiton we have

$$\varepsilon^{i_1 \dots i_n} = \mathfrak{E}_{i_1 \dots i_n} \overset{df}{=} \begin{cases} +1 & \text{if } i_1 \dots i_n \text{ is the even permutation} \\ & \text{of the sequence of } 1, 2, \dots n. \\ -1 & \text{if } i_1 \dots i_n \text{ is the odd permutation} \\ & \text{of the sequence of } 1, 2 \dots n, \\ 0 & \text{in the remaining cases.} \end{cases} \tag{2}$$

From the definition of determinant and (2) it follows

$$n!\, g = \varepsilon^{i_1 \dots i_n} \varepsilon^{j_1 \dots j_n} g_{i_1 j_1} \cdots g_{i_n j_n} = \varepsilon^{i_1 \dots i_n} \varepsilon_{i_1 \dots i_n} =$$

$$= x\, \varepsilon^{i_1 \dots i_n} \mathfrak{E}_{i_1 \dots i_n} = n!\, x$$

Hence

$$x = g = \det((g_{ij}))$$

and consequently we get

$$\varepsilon_{i_1 \dots i_n} = g\, \mathfrak{E}_{i_1 \dots i_n} \tag{3}$$

As is known, the Ricci symbole $\varepsilon^{i_1 \dots i_n}$ and $\mathfrak{E}_{i_1 \dots i_n}$ satisfy the identities [18].

$$\varepsilon^{i_1 \dots i_p i_{p+1} \dots i_n} \mathfrak{E}_{i_1 \dots i_p j_{p+1} \dots j_n} = p!\,(n-p)!\, \delta^{i_{p+1} \dots i_n}_{j_{p+1} \dots j_n} \tag{4}$$

where $\delta^{i_1 \dots i_k}_{j_1 \dots j_k}$ is the generalized Kronecker symbol.

From (3) and (4) it follows

$$\varepsilon^{i_1\dots i_p i_{p+1}\dots i_n}\,\varepsilon_{i_1\dots i_p j_{p+1}\dots j_n} = g\,p!(n-p)!\,\delta^{i_{p+1}\dots i_n}_{j_{p+1}\dots j_n} \tag{5}$$

Consider n-vector $J^{i_1\dots i_n}$ defined, in a fixed coordinate system (x^i), by the equations

$$J^{i_1\dots i_n} \overset{df}{\underset{*}{=}} \frac{1}{\sqrt{|g|}}\,\varepsilon^{i_1\dots i_n} \tag{6}$$

where the asterisk denotes that equality taken place in a fixed coordinate system.

From the definition of n-vector $J^{i_1\dots i_n}$ as well as from (5) it follows

$$J^{i_1\dots i_p i_{p+1}\dots i_n}\,J_{i_1\dots i_p j_{p+1}\dots j_n} = \epsilon\,p!(n-p)!\,\delta^{i_{p+1}\dots i_n}_{j_{p+1}\dots j_n} \tag{7}$$

where

$$\epsilon \overset{df}{=} g|g|^{-1} = \operatorname{sgn} g$$

From (6) it follows, that the n-vector $J^{i_1\dots i_n}$ the so-called Ricci's n-vector [3], is a real n-vector and from (7) it follows that it is a unit n-vector.

The unit n-vector $J^{i_1\dots i_n}$ makes it possible to establish a one correspondence between p-vectors and (n-p) vectors, taken at the same point of a V_n, using the formula

$$^*f^{i_{p+1}\dots i_n} \overset{df}{=} \frac{1}{p!}\,J_{i_1\dots i_p}{}^{i_{p+1}\dots i_n}\,f^{i_1\dots i_p}$$

Be means of the n-vector $J^{i_1 \dots i_n}$ we can also establish a one to one correspondence between p-vectors and (n-p)-vectors, taken at the same point of the V_n, using the following formula [5].

$$ {}^{o}f^{i_{p+1} \dots i_n} \stackrel{df}{=} \frac{\varrho}{p!} J_{i_1 \dots i_p}{}^{i_{p+1} \dots i_n} f^{i_1 \dots i_p} \tag{8} $$

where $\varrho = (-1)^{\frac{1}{2}p(n-p)}$, if $g > 0$ and $\varrho = i(-1)^{\frac{1}{2}p(n-p)}$ if $g < 0$, $i = \sqrt{-1}$.

Indeed, applying the formula (8) to the (n-p) vector ${}^{o}f^{i_{p+1} \dots i_n}$ we get [13].

$$ {}^{o}({}^{o}f)^{i_1 \dots i_p} = f^{i_1 \dots i_p} \tag{9} $$

The (n-p)-vector ${}^{o}f^{i_{p+1} \dots i_n}$ defined by the equations (8) is called the dual of $f^{i_1 \dots i_p}$

The one to one mapping $T: \{f\} \to \{{}^{o}f\}$ of the set of all p-vectors f onto the set of all (n-p)-vectors ${}^{o}f$ determined by the formula (8) has, besides its elegance and simplicity, some defect. As it is easy to observe if $g > 0$ and $p(n-p) = 2k+1$ or if $g < 0$ and $p(n-p) = 2k$, the formula (8) establishes a one to one correspondence between real p-vectors and (n-p)-vectors, whose components are imaginare, and vice versa.

When the dimension of the space is even i.e. $n = 2p$ it is possible to consider p-vectors which satisfy the relations

$$ {}^{o}f^{i_1 \dots i_p} = \theta f^{i_1 \dots i_p} \tag{10} $$

where θ is some scalar. It results from (9), that $\theta = \pm 1$

The p-vectors satisfing the relations (10) are called the salf-dual p-vectors [13].

In general the self-dual p-vectors are the p-vectors with complex components, however if $g > 0$ and $p = 2k$, or if $g < 0$ and $p = 2k+1$ there exists the real self-dual p-vectors.

Let $a_{i_1 \dots i_p j_1 \dots j_p}$ be a tensor satisfing the identities

$$a_{[i_1 \dots i_p] j_1 \dots j_p} = a_{i_1 \dots i_p j_1 \dots j_p} \tag{11}$$

as well as

$$a_{i_1 \dots i_p j_1 \dots j_p} = a_{j_1 \dots j_p i_1 \dots i_p} \tag{12}$$

The tensors of the form $a_{i_1 \dots i_p j_1 \dots j_p}$ which satisfy the relations (11) and (12) will be called bi-tensors [6].

When $n \geqslant 4$ and $2 = p \leqslant n-2$ the bi-tensor

$$^{o}a^{i_{p+1} \dots i_n j_{p+1} \dots j_n} \stackrel{df}{=} \frac{\varrho^2}{(p!)^2} \mathcal{J}_{i_1 \dots i_p}{}^{i_{p+1} \dots i_n} a^{i_1 \dots i_p j_1 \dots j_p} \mathcal{J}_{j_1 \dots j_p}{}^{j_{p+1} \dots j_n} \tag{13}$$

where $\varrho = (-1)^{\frac{1}{2}p(n-p)}$, if $g > 0$ and $\varrho = i(-1)^{\frac{1}{2}p(n-p)}$ if $g < 0$ is called the dual of the bi-tensor $a_{i_1 \dots i_p j_1 \dots j_p}$ [13].

From (13) it follows

$$^{o}(^{o}a)^{i_1 \dots i_p j_1 \dots j_p} = a^{i_1 \dots i_p j_1 \dots j_p} \tag{14}$$

If $n = 2p$[1)] the bi-tensors $a^{i_1 \dots i_p j_1 \dots j_p}$ satisfing the identities

$$^{o}a^{i_1 \dots i_p j_1 \dots j_p} = \theta \, a^{i_1 \dots i_p j_1 \dots j_p} \tag{15}$$

where $\theta = \pm 1$ are called the self-dual bi-tensors.

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1) W. Wrona in the paper [13] gives more general definition of the self-dual bi-tensor but for bi-tensors of special constructions.

In the Riemannian space V_n the covariant derivative of the field of the unit n-vector $J^{i_1 \dots i_n}$ is, as is known, identically equal to zero i.e.

$$\nabla_j J^{i_1 \dots i_n}(x) = 0 \tag{16}$$

New, let us take into consideration a field of self-dual p-vectors $f_{i_1 \dots i_p}(x)$ and a field of the self-dual bi-tensors $a_{i_1 \dots i_p j_1 \dots j_p}(x)$ of class C^1 -respectively.

The from (10), (15) and (16) there follow the identities

$$^{o}f_{i_1 \dots i_p ; j} = \theta f_{i_1 \dots i_p ; j} \tag{17}$$

as well as

$$^{o}a_{i_1 \dots i_p j_1 \dots j_p ; w} = \theta a_{i_1 \dots i_p j_1 \dots j_p , w} \tag{18}$$

where the covariant derivative is denoted by semicolon.

Introduction of the concept of a dual bi tensor as well as the concept of a self-dual bi-tensor constituted a natural transfer of the concept of dualism and self-dualism of multivectors on the bi-tensor. The identities (17) and (18) point to the possibility of transformation of those concept on a broader class of the tensors.

Let us pay attention to the fact, that in the definitions of mappings $T: \{f\} \to \{^{o}f\}$ and $T_1: \{a\} \to \{^{o}a\}$ given by the formulas (8) and (13) intervened in essntial manner all indices of the mapped tensors (multivectors and bitensors).

In the sections to follow we shall introduction a one to one mapping of the set of certain tensors onto the set of other tensors in such a way that not all the indices but only a fixed group of indices of the tensors will be used in the definition.

To conclude our considerations of this section we shall quote some more fundamental concept of the Rimannian geometry, which often shall be dealt with in the next sections of this paper.

Let us denote the curvature tensor of the space V_n by R_{ijke} and the scalar curvature of this space by

$$\varkappa \overset{df}{=} \frac{R_{ij}\, g^{ij}}{n(n-1)}$$

where R_{ij} is the Ricci - tensor.

The tensor

$$\overset{p}{R}_{i_1 \dots i_p j_1 \dots j_p} \overset{df}{=} \frac{p!}{2} R_{[i_1 i_2 [j_1 j_2}\, g_{i_3 j_3} \cdots g_{i_p] j_p]} \tag{19}$$

is called the expanded curvature tensor of the V_n, and the scalar [4], [13].

$$\overset{p}{\varkappa} \overset{df}{=} -\frac{1}{(p!)^2 \| f \|^2} \overset{p}{R}_{i_1 \dots i_p j_1 \dots j_p} f^{i_1 \dots i_p} f^{j_1 \dots j_p} \tag{20}$$

is called the scalar curvature of non - singular p-vector $f^{i_1 \dots i_p}$ - respectively.

In a like manner, let us denote the deviation tensor of the V_n by [13]

$$U_{ijkl} \overset{df}{=} R_{ijkl} + \varkappa \overset{2}{g}_{ijkl}$$

then the tensor

$$\overset{p}{U}_{i_1 \dots i_p j_1 \dots j_p} \overset{df}{=} \frac{p!}{2} U_{[i_1 i_2 [j_1 j_2}\, g_{i_3 j_3} \cdots g_{i_p] j_p]} \tag{21}$$

is called the expanded deviation tensor of the V_n, and the scalar [13]

$$\overset{p}{\omega} = -\frac{1}{(p!)^2\|f\|^2} U_{i_1\dots i_p j_1\dots j_p} f^{i_1\dots i_p} f^{j_1\dots j_p} \tag{22}$$

is called the deviation of non-singular p-vector respectively.

§2. THE r/p - TENSOR IN A V_n

Let $f_{i_1\dots i_p l_1\dots l_r}(x)$ be an arbitrary tensor field in the Riemannian space V_n, where $2 \leqslant p \leqslant n$, $r \geqslant 1$ satisfying relations

$$f_{[i_1\dots i_p]l_1\dots l_r}(x) = f_{i_1\dots i_p l_1\dots l_r}(x) \tag{23}$$

The tensor fields $f_{i_1\dots i_p\, l_r\dots l_r}(x)$ satisfying the relations (23) will be called the r/p-tensor field.

An example of such a field is the r-th covariant derivative of the p-vector field $f_{i_1\dots i_r}(x)$, $p \geqslant 2$ i.e. the field

$$f_{i_1\dots i_p;\, l_1;\dots;\, l_r}(x)$$

It is easy to note that an arbitrary linear combination of a r/p - tensor field is again a r/p - tensor field.

On the other hand the r/p - tensor $f_{i_1\dots i_p l_1\dots l_r}$ has $\binom{n}{p}n^r$ - linearly independent components. Thus, a set of all the r/p - tensors at the fixed point of V_n creates the $\binom{n}{p}n$-- dimensional linear space.

T h e o r e m 1. The r/p tensor $f_{i_1 \ldots i_p l_1 \ldots l_r}$ is identically equal to zero if and only if for each r/p-tensor $h_{i_1 \ldots i_p l_1 \ldots l_r}$ there is the identity

$$f^{i_1 \ldots i_p l_1 \ldots l_r} h_{i_1 \ldots i_p l_1 \ldots l_r} = 0 \tag{24}$$

P r o o f. The necessity of the condition (24) is evident. To prove its sufficiency it is enough to show that if the assumption of the theorem is satisfied, then all the components of the tensor $f_{i_1 \ldots i_p l_1 \ldots l_r}$ are equal to zero. Let us assume that the relations (24) are satisfied for each r/p-tensor $h_{i_1 \ldots i_p l_1 \ldots l_r}$, where $f_{i_1 \ldots i_p l_1 \ldots l_r}$ is a certain r/p-tensor.

Denote an arbitrary, but a fixed sequence of indices, by $\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_r \ldots \underset{\circ}{l}_r$ and consider the r/p-tensor defined as follows

$$h^{\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_1 \ldots \underset{\circ}{l}_r} = 1$$

all the other components of the tensor $h_{i_1 \ldots i_p l_1 \ldots l_r}$, linearly independent of $h^{\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_1 \ldots \underset{\circ}{l}_r}$ are equal to zero.

Thus we have

$$f_{\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_1 \ldots \underset{\circ}{l}_r} h^{\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_1 \ldots \underset{\circ}{l}_r} = 0$$

from which we obtain

$$f_{\underset{\circ}{i}_1 \ldots \underset{\circ}{i}_p \underset{\circ}{l}_1 \ldots \underset{\circ}{l}_r} = 0$$

Due to arbitrariness of the choice of the sequence of indices $i_{o1}\dots i_{op} l_{o1}\dots l_{or}$ and corresponding to it choice of the tensor $h_{i_1\dots i_p l_1\dots l_r}$, we conclude that

$$f_{i_1\dots i_p l_1\dots l_r} = 0$$

which ends the proof.

For an arbitrary r/p-tensor $f_{i_1\dots i_p l_1\dots l_r}$ there are the evident identities (from the definition of alternation operations)

$$(p+1) f_{[i_1\dots i_p l] l_2\dots l_r} = f_{i_1\dots i_p l_1\dots l_r} - f_{l i_2\dots i_p i_1 l_2\dots l_r} - \dots + \\ - f_{i_1\dots i_{p-1} l i_p l_2\dots l_r}$$

Multiplying the last identities by an arbitrary r/p-tensor $h_{i_1\dots i_p l_1\dots l_r}$ and sumining for $i_1\dots i_p l_1\dots l_r$ we get (using its asymetry to the sequence of indices $i_1\dots i_p$)

$$f^{i_2\dots i_p l i l_2\dots l_r} h_{i_2\dots i_p i l l_2\dots l_r} = \\ = \frac{1}{p} f^{i_2\dots i_p i l l_2\dots l_r} h_{i_2\dots i_p i l l_2\dots l_r} - \frac{p+1}{p} f^{[i_1\dots i_p l] l_2\dots l_r} h_{[i_1\dots i_p l] l_2\dots l_r} \tag{25}$$

From (25) and from theorem 1 it results that, if the r/p-tensor $f_{i_1\dots i_p l_1\dots l_r}$ satisfies the identities

$$f_{i_2\dots i_p i l l_2\dots l_r} = f_{i_2\dots i_p l i l_2\dots l_r}$$

it is equal to zero. Particularly we have

C o r o l l a r y 1. The p-vector field $f_{i_1 \ldots i_p}$, $p > 1$, of the class C^1 is a covariant constant field, if

$$f_{i_2 \ldots i_p [i;l]}(x) = 0$$

Let $f_{i_1 \ldots i_p l_1 \ldots l_r}$ be an arbitrary r/p-tensor. The r/(n-p)-tensor

$$^{\circ}f_{i_{p+1} \ldots i_n l_1 \ldots l_r} \overset{df}{=} \frac{\varrho}{p!} J^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_n} f_{i_1 \ldots i_p l_1 \ldots l_r} \tag{26}$$

where $\varrho = (-1)^{\frac{1}{2}p(n-p)}$, if $g>0$ and where $\varrho = (-1)^{\frac{1}{2}p(n-p)} \cdot i$ if $g<0$ $(i = \sqrt{-1})$, is called a dual of the tensor $f_{i_1 \ldots i_p l_1 \ldots l_r}$

From (26) it results immediately that

$$^{\circ}(^{\circ}f)_{i_1 \ldots i_p l_1 \ldots l_r} = f_{i_1 \ldots i_p l_1 \ldots l_r} \tag{27}$$

Let us assume now that the r/p-tensor $f_{i_1 \ldots i_p l_1 \ldots l_r}$ satisfies the conditions

$$g^{il} f_{i i_2 \ldots i_p l l_2 \ldots l_r} = 0 \tag{28}$$

From (28) and (26) it follows that

$$J^{i_{p+1} \ldots i_n l i_2 \ldots i_p} \, {}^{\circ}f_{i_{p+1} \ldots i_n l l_2 \ldots l_r} = 0$$

Multiplying the last identity by n-vector $J_{k_{p+1} \ldots k_n \varrho i_2 \ldots i_p}$ and summing for the indices $i_2 \ldots i_p$ we obtain by wirtue of (7)

$$^{\circ}f_{[k_{p+1} \ldots k_n \varrho] l_2 \ldots l_r} = 0 \tag{29}$$

Thuse: from (28) there results (29). We shall now prove that from (29 there results (28).

Indeed, the identity (28) is equivalent to the following identity

$$\varepsilon\, p!\,(n-p)!\,\delta^{i_{p+1}\dots i_n l}_{k_{p+1}\dots k_n \varrho}\, f_{i_{p+1}\dots i_n l l_2 \dots l_r} = 0$$

or by virtue of (7) to

$$g^{il}\, J_{k_{p+1}\dots k_n \varrho}{}^{i_2\dots i_p}\, J^{i_{p+1}\dots i_n}{}_{i i_2 \dots i_p}\, {}^{\circ}f_{i_{p+1}\dots i_n l l_2\dots l_r} = 0 \qquad (30)$$

From (30) and from (26) we get

$$g^{il}\, J_{k_{p+1}\dots k_n \varrho}{}^{i_2\dots i_p}\, f_{i i_2 \dots i_p l l_2 \dots l_r} = 0$$

Multiplying the last identity by n-vector $J^{k_{p+1}\dots k_n \varrho \varrho_2 \dots \varrho_p}$ and summing for $k_{p+1}\dots k_n$ we get (29).

In this manner we have showed that

$$g^{il}\, f_{i i_2 \dots i_p l l_2 \dots l_r} = 0 \Longleftrightarrow {}^{\circ}f_{[i_{p+1}\dots i_n l]\, l_2 \dots l_r} = 0 \qquad (31)$$

If $f_{i_1\dots i_p}(x)$ is a p-vector field of the class C^1 on the manifold V_n, then the (p-1)-vector field [3]

$$(p+1)\, f_{[i_1\dots i_p ; j]}(x)$$

is called the rotation of the p-vector field $f_{i_1\dots i_p}(x)$,which is written as

$$\mathrm{Rot}\ f_{i_1\dots i_p}(x) \overset{df}{=} (p+1)\, f_{[i_1\dots i_p ; j]}(x) \qquad (32)$$

and the (p-1)-vector field

$$g^{ij} f_{ii_2\ldots i_p;j}(x)$$

is called the divergency of the p-vector field $f_{i_1\ldots i_p}(x)$ which is written as

$$\mathrm{Div}\, f_{i_1\ldots i_p}(x) \overset{df}{=} g^{ij} f_{ii_2\ldots i_p;j}(x) \tag{33}$$

A p-vector field $f_{i_1\ldots i_p}(x)$ of the class C^1 is called a harmonic field [6], if its rotation and divergency are equal to zero.

Putting $r=1$ and $f_{i_1\ldots i_p l} = f_{i_1\ldots i_p;l}$ into (31), by virtue of (32) and (33) we get.

C o r o l l a r y 2. The rotation of the p-vector field $f_{i_1 \ldots i_p}(x)$ of the C^1 class is identically equal to zero if and only if the divergency of dual (n-p)-vector field ${}^{o}f_{i_{p+1}\ldots i_n}$ is identically equal to zero.

Hence, from the definition of the harmonic p-vector field there results:

C o r o l l a r y 3. If a p-vector field $f_{i_1\ldots i_p}(x)$ is a harmonic p-vector field then the dual field to it is a harmonic (n-p)-vector field too.

From (26) it follows immediately that, if r/p-tensor $f_{i_1\ldots i_p l_1\ldots l_r}$ is the product of the p-vector $f_{i_1\ldots i_p}$ and a certain tensor $a_{l_1\ldots l_r}$ i.e. if

$$f_{i_1\ldots i_p l_1\ldots l_r} = \bar{f}_{i_1\ldots i_p}\, a_{l_1\ldots l_r}$$

then the dual tensor ${}^{o}f_{i_{p+1}\ldots i_n l_1\ldots l_r}$ takes the form

$${}^{o}f_{i_{p+1}\ldots i_n l_1\ldots l_r} = {}^{o}\bar{f}_{i_{p+1}\ldots i_n}\, a_{l_1\ldots l_r}$$

The p-vector field $f_{i_1 \dots i_p}(x)$ of the class C^1 in a V_n is called a recurrent field [11] if on the given manifold there is a vector field $v_j(x)$ satisfying the identities

$$f_{i_1 \dots i_p; j}(x) = f_{i_1 \dots i_p}(x)\, v_j(x)$$

From the definition of the recurrent p-vector field and from the above it follows: It the p-vector field $f_{i_1 \dots i_p}(x)$ is a recurrent field then the dual field ${}^{o}f_{i_{p+1} \dots i_n}(x)$ is also a recurrent field.

Let us assume now that the space examined by us is a 2p-dimensional Riemannian space.

The r/p-tensors $f_{i_1 \dots i_p l_1 \dots l_r}$ of a V_{2p} will be called self-dual r/p-tensor, if they satisfy the relations

$$ {}^{o}f_{\underline{i_1 \dots i_p} l_1 \dots l_r} = \theta\, f_{i_1 \dots i_p l_1 \dots l_r} \tag{34}$$

where θ is a scalar.

From (27) it follows that if (34) occurs, then $\theta = \pm 1$ Analogically, as in the case of p-vectors (15), the r/p-tensor $f_{i_1 \dots i_p l_1 \dots l_r}$ satisfying the identities (34) when $\theta = +1$, is called the self-dual r/p-tensor of the 1-st kind and when $\theta = -1$, the self-dual r/p-tensor of the 2-nd kind respectively.

Similarly to §1 we conclude from (6), (26) and (34) that the self-dual r/p-tensors in general are the tensors with complex components. However, if $g > 0$ and $p = 2k$, or if $g < 0$ and $p = 2k+1$, there are also self-dual r/p-tensors with real components.

We shall continue to deal with the examination of the properties of the real self-dual r/p-tensors only, and consequently we assume that the dimension of the space under further examination is $n = 4k$, if $g > 0$ or $n = 4k+2$, if $g < 0$.

From the definition (34) and from (17) it follows that the r-th covatiant derivative of the self-dual p-vector field of class C^r is the self-dual r/p-tensor field.

Let $\underset{1}{f}_{i_1\ldots i_r}(x)$ and $\underset{2}{f}_{i_1\ldots i_p}(x)$ be the fields, of class C^r, of the self - dual p-vector of the 1-st kind and of the 2-nd kind respectively. Moveover, let us assume for example, that

$$\underset{2}{f}_{i_1\ldots i_p;l_1;\ldots;l_r}(x) = 0$$

and consider the field of p-vectors

$$f_{i_1\ldots i_p}(x) \overset{df}{=} \underset{1}{f}_{i_1\ldots i_p}(x) + \underset{2}{f}_{i_1\ldots i_p}(x) \tag{35}$$

The p-vector field (35) in general will not be a p-vector field of the self - dual p-vectors. Differentiating (35) r-times and taking into account the assumption, we obtain:

$$f_{i_1\ldots i_p;l_1;\ldots;l_r}(x) = \underset{1}{f}_{i_1\ldots i_p;l_1;\ldots l_r}(x)$$

Thus: The covariant derivative of the r-th order of the field of non self-dual p-vectors may be a field of self-dual r/p-tensors (upon meeting certain conditions).

It is easy to note, like as in the case of p-vectors, that each r/p-tensor $f_{i_1\ldots i_p l_1\ldots l_r}$ in a V_{2p} may be expressed in one to one form as a sum of two self-dual r/p-tensors of the 1-st kind and of the 2-nd kind according to the formula:

$$\begin{aligned} f_{i_1\ldots i_p l_1\ldots l_r} &= \frac{1}{2}\left(f_{i_1\ldots i_p l_1\ldots l_r} + {}^{o}f_{i_1\ldots i_p l_1\ldots l_r}\right) \\ &+ \frac{1}{2}\left(f_{i_1\ldots i_p l_1\ldots l_r} - {}^{o}f_{i_1\ldots i_p l_1\ldots l_r}\right) = \\ &= \underset{1}{f}_{i_1\ldots i_p l_1\ldots l_r} + \underset{2}{f}_{i_1\ldots i_p l_1\ldots l_r} \end{aligned} \tag{36}$$

where $\underset{1}{f}=\frac{1}{2}\left(f+{}^{\circ}f\right)={}^{\circ}\underset{1}{f}$ is the self-dual r/p-tensor of the 1-st kind, and $\underset{2}{f}=\frac{1}{2}\left(f-{}^{\circ}f\right)=-{}^{\circ}\underset{2}{f}$ is the self-dual r/p-tensor of the 2-nd kind respectively.

From the definition of self-dual r/p-tensor it follows immediately that an arbitrary combination of self-dual r/p--tensors of the 1-st kind (or the 2-nd king) is also selfdual r/p-tensors of the 1-st kind (or the 2-nd kind respectively).

From the above and (36) as well as from the fact that a set of all r/p-tensors at a fixed point of a V_n creates an $\binom{n}{p}n^r$ - dimensional linear space it follows that:

I. A set all self-dual r/p-tensors of the 1-st kind (analogically of, the 2-nd kind) at a point $x \in V_{2p}$ creates $\frac{1}{2}\binom{2p}{p}(2p)^r$ - dimensional linear space.

II. A linear space of r/p-tensors is a direct sum of the linear sub-space of the self-dual r/p-tensors of the 1-st kind and of the 2-nd kind.

Now, let $f_{i_1\ldots i_p l_1\ldots l_r}$ be an arbitrary self-dual r/p-tensor, i.e.

$$f_{i_1\ldots i_p l_1\ldots l_r} = \theta \frac{\varrho}{p!} J^{j_1\ldots j_p}{}_{i_1\ldots i_p} f_{j_1\ldots j_p l_1\ldots l_r} \tag{37}$$

where $\theta = \pm 1$

Multiplying (37) by an arbitrary tensor $a^{l_1\ldots l_r}$ and summing for $l_1\ldots l_r$ we get the p-vector

$$\bar{f}_{i_1\ldots i_p} \overset{df}{=} f_{i_1\ldots i_p l_1\ldots l_r} a^{l_1\ldots l_r}$$

which is, as it follows from (37), a self-dual p-vector.

Thus we have

C o r o l l a r y 4. Multiplying an arbitrary self-dual r/p-tensor of the 1-st kind (or 2-nd kind) $f_{i_1\ldots i_p l_1\ldots l_r}$ by an arbitrary tensor $a^{l_1\ldots l_r}$ and summing for $l_1\ldots l_r$ we obtain a self-dual p-vector $\bar{f}_{i_1\ldots i_p}$ of the 1-st kind (2-nd kind respectively).

In the paper [15] i.a. there have been proved the following theorems.

I. The p-vector $f_{i_1 \dots i_p}$ is a self-dual p-vector of the 1-st kind (of 2-nd kind) if and only if, for each self-dual p-vector $h_{i_1 \dots i_p}$ of 2-nd kind (of 1-st kind respectively) if p is even, and for each self-dual p-vector $h_{i_1 \dots i_p}$ of the 1-st kind (of the 2-nd kind respectively) if p is odd, there is the identity:

$$f_{i_1 \dots i_p} h^{i_1 \dots i_p} = 0$$

II. The p-vector $f_{i_1 \dots i_p}$ is a self-dual p-vector of the 1-st kind (or of the 2-nd kind) if and only if, for each self-dual p-vector $h_{i_1 \dots i_p}$ of the 2-nd kind (of the 1-st kind respectively) if p is even, and for each self-dual p-vector $h_{i_1 \dots i_p}$ of the 1-st kind (of the 2-nd kind respectively) if p is odd, there are the identities

$$f^{i_2 \dots i_p [i} h_{i_2 \dots i_p}{}^{j]} = 0$$

From corollary 4 and from the above theorems there follow:

Theorem 2. The r/p-tensor $f_{i_1 \dots i_p l_1 \dots l_r}$ is a self-dual r/p-tensor of the 1-st kind (of the 2-nd kind) if and only if, for each self-dual s/p-tensor $h_{i_1 \dots i_p l_1 \dots l_s}$ of the 2-nd kind (of the 1-st kind respectively) if p is even, and for each self-dual s/p-tensor $h_{i_1 \dots i_p l_1 \dots l_s}$ of the 1-st kind (of the 2-nd kind respectively) if p is odd, there are the identities

$$f_{i_1 \dots i_p l_1 \dots l_r} h^{i_1 \dots i_p}{}_{k_1 \dots k_s} = 0 .$$

P r o o f. Let us assume that $f_{i_1\dots i_p l_1\dots l_r}$ is a self-dual r/p-tensor, e.g. of the 1-st kind and $h_{i_1\dots i_p l_1\dots l_s}$ is a self-dual s/p-tensor of the 2-nd kind respectively and p= = 2k.

Multiplying the r/p-tensor $f_{i_1\dots i_p l_1\dots l_r}$ by an arbitrary tensor $a^{l_1\dots l_r}$ as well as the s/p-tensor $h_{i_1\dots i_p l_1\dots l_s}$ by an arbitrary tensor $b^{l_1\dots l_s}$ and summing for $l_1\dots l_r$ and $l_1\dots l_s$ respectively, we get, by virtue of corollary 4, the self-dual p-vectors

$$\bar{f}_{i_1\dots i_p} = f_{i_1\dots i_p l_1\dots l_r} a^{l_1\dots l_r} \quad \text{- of the 1-st kind}$$

$$\bar{h}_{i_1\dots i_p} = h_{i_1\dots i_p l_1\dots l_s} b^{l_1\dots l_s} \quad \text{- of the 2-nd kind respectively}$$

By virtue of the above quoted theorem I, from the paper [15], the self-dual p-vectors $\bar{f}_{i_1\dots i_p}$ and $h_{i_1\dots i_p}$ satisfy the identity

(*)
$$f_{i_1\dots i_p l_1\dots l_r} h^{i_1\dots i_p j_1\dots j_s} a^{l_1\dots l_r} b_{j_1\dots j_s} = 0$$

Since this identity (*) occurs for each pair of tensors $a^{l_1\dots l_r}$ and $b^{l_1\dots l_s}$, thus from (*) there follow the identities

(**)
$$f_{i_1\dots i_p l_1\dots l_r} h^{i_1\dots i_p j_1\dots j_s} = 0$$

Inversely, let us assume that the identities (**) are satisfied for each self-dual s/p-tensor $h_{i_1\dots i_p l_1\dots l_s}$ for example of the 2-nd kind, if $p=2k$ - where $f_{i_1\dots i_p l_1\dots l_r}$ is a certain r/p-tensor.

From the identities (**) there follows

$$(***)\qquad f_{i_1\dots i_p l_1\dots l_r}\, a^{l_1\dots l_r}\, h^{i_1\dots i_p j_1\dots j_s}\, b_{j_1\dots j_s} = 0$$

where $a^{l_1\dots l_r}$ and $b^{l_1\dots l_s}$ are arbitrary tensors.

Since $f_{i_1\dots i_p l_1\dots l_r} a^{l_1\dots l_r}$ is a p-vector, $h_{i_1\dots i_p l_1\dots l_s} b^{l_1\dots l_s}$ is a self-dual p/vector of the 2-nd kind. Hence, by virtue of the above theorem I we conclude from (* * *) that the p-vector $\bar{f}_{i_1\dots i_p} = f_{i_1\dots i_p l_1\dots l_r}\, a^{l_1\dots l_r}$ is a self-dual p-vector of the 1-st kind, i.e.

$$(****)\qquad f_{i_1\dots i_p l_1\dots l_r}\, a^{l_1\dots l_r} = {}^{o}f_{i_1\dots i_p l_1\dots l_r}\, a^{l_1\dots l_r}$$

The last identities are satisfied for each tansor $a^{l_1\dots l_r}$ by virtue of our assumption. Hence, from (****) we obtain

$$f_{i_1\dots i_p l_1\dots l_r} = {}^{o}f_{i_1\dots i_p l_1\dots l_r}$$

which means that the r/p-tensor $f_{i_1\dots i_p l_1\dots l_r}$ is a self-dual r/p-tensor of the 1-st kind.

The further part of the proof is analogical.

T h e o r e m 3. The r/p-tensor $f_{i_1\dots i_p l_1\dots l_r}$ is a self-dual r/p-tensor of the 1-st kind (of the 2-nd kind) if and only if for each self-dual s/p-tensor $h_{i_1\dots i_p l_1\dots l_s}$ of the 2-nd kind (of the 1-st kind respectively) if p is even, and for each self-dual s/p-tensor $h_{i_1\dots i_p l_1\dots l_s}$ of the 1-st kind (of the 2-nd kind respectively) if p is odd, there are the identities:

$$f^{i_2\dots i_p[i}{}_{l_1\dots l_r}\, h_{i_2\dots i_p}{}^{j]}{}_{k_1\dots k_s} = 0$$

Note that the theorems 2 and 3 hold true also if r or s are equal to zero. It is obvious that in this canse the r/p or s/p-tensors are p-vectors.

Now we shall prove

T h e o r e m 4. The r/p-tensor $f_{i_1\ldots i_p l_1\ldots l_r}$ is a self-dual r/p-tensor of the 1-st kind (of the 2-nd kind) if and only if for each self-dual r/p-tensor $h_{i_1\ldots i_p l_1\ldots l_r}$ of the 2-nd kind (of the 1-st kind respectively), if p is even, and for each self-dual r/p-tensor $h_{i_1\ldots i_p l_1\ldots l_r}$ of the 1-st kind (of the 2-nd kind respectively, if p is odd there is the identity:

$$f_{i_1\ldots i_p l_1\ldots l_r} h^{i_1\ldots i_p l_1\ldots l_r} = 0 \tag{38}$$

P r o o f. Let $f_{i_1\ldots i_p l_1\ldots l_r}$ be, for example, a self-dual r/p-tensor of the 1-st kind and $h_{i_1\ldots i_p l_1\ldots l_r}$ a self-dual r/p-tensor of the 2-nd kind respectively.

Additionaly let us assume that p = 2k then at the assumption that s = r, from theorem 2 there follows (38).

Let us assume inversely that there is identity (38) for each self-dual tensor $h_{i_1\ldots i_p l_1\ldots l_r}$ of the 2-nd kind, where $f_{i_1\ldots i_p l_1\ldots l_r}$ is a certain r/p-tensor.

From (38) and by virtue of (36) we have

$$\left(\underset{1}{f}_{i_1\ldots i_p l_1\ldots l_r} + \underset{2}{f}_{i_1\ldots i_p l_1\ldots l_r}\right) h^{i_1\ldots i_p l_1\ldots l_r} = 0$$

and consequently by virtue of the first part of the theorem we get

$$\underset{2}{f}_{i_1\ldots i_p l_1\ldots l_r} h^{i_1\ldots i_p l_1\ldots l_r} = 0 \tag{39}$$

By virtue of the first part of the theorem we also have

$$\underset{2}{f}_{i_1\dots i_p l_1\dots l_r}\,\bar{h}^{i_1\dots i_p l_1\dots l_r} = 0 \tag{40}$$

where $\bar{h}_{i_1\dots i_p l_1\dots l_r}$ is a arbitrary self-dual r/p-tensor of the 1-st kind. Thus from (39) and (40) it follows that

$$\underset{2}{f}_{i_1\dots i_p l_1\dots l_r}\,\bar{\bar{h}}^{i_1\dots i_p l_1\dots l_r} = 0$$

where $\bar{\bar{h}}_{i_1\dots i_p l_1\dots l_r}$ is absolutely arbitrary r/p-tensor. By virtue of theorem 1 we get

$$\underset{2}{f}_{i_1\dots i_p l_1\dots l_r} = 0$$

that is

$$f_{i_1\dots i_p l_1\dots l_r} = \underset{1}{f}_{i_1\dots i_p l_1\dots l_r}$$

thus $f_{i_1\dots i_p l_1\dots l_r}$ is the self-dual r/p-tensor of the 1-st kind. In all the remaining cases the proof runs analogically.

In paper [17] it has been showed i.a. that for each pair of self-dual p-vectors $f_{i_1\dots i_p}$ and $h_{i_1\dots i_p}$ of the same kind, where p is even and of different kind where p is odd there are the identities

$$f^{i_2\dots i_p(i}\,h_{i_2\dots i_p}{}^{j)} = \frac{1}{2p}\,g^{ij} f^{i_1\dots i_p} h_{i_1\dots i_p} \tag{41}$$

From corollary 4 and identity (41) there follows:

T h e o r e m 5. If $f_{i_1\dots i_p l_1\dots l_r}$ and $h_{i_1\dots i_p l_1\dots l_s}$ are respectively the self-dual r/p and s/p-tensors of the same kind when p is even and of different kind, when p is odd, there are the identities

$$f^{i_2\dots i_p(i}{}_{l_1\dots l_r}\, h_{i_2\dots i_p}{}^{j)}{}_{k_1\dots k_s} = \frac{1}{2p} g^{ij} f^{i_1\dots i_p}{}_{l_1\dots l_r} h_{i_1\dots i_p k_1\dots k_s}$$

From identity (25) and from theorems 2 and 4 there follows:

T h e o r e m 6. For each pair of self-dual r/p - ten- $f_{i_1\dots i_p l_1\dots l_r}$ and $h_{i_1\dots i_p l_1\dots l_r}$ of different kind, if p is even, and of the same kind, if p is odd, respectively, there are the identities

$$f^{i_2\dots i_p i\cdot l_2\dots l_p}{}_{i}\, h_{i_2\dots i_p j\cdot l_2\dots l_r}{}^{j} =$$

$$= -\frac{p+1}{p} f^{[i_2\dots i_p ij]\, l_2\dots l_r}\, h_{[i_2\dots i_p ij]\, l_2\dots l_r}$$

Similarly from identity (25) and theorem 5 there follows:

T h e o r e m 7. For each pair of self-dual r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}$ and $h_{i_1\dots i_p l_1\dots l_r}$ of the same kind, if p is even and of different kind, if p is odd respectively, there are the identities:

$$f^{i_2\dots i_p i\cdot l_2\dots l_r}{}_{i}\, h_{i_2\dots i_p j\cdot l_2\dots l_r}{}^{j} =$$

$$= \frac{p+1}{p} f^{[i_2\dots i_p ij]\, l_2\dots l_5}\, h_{[i_2\dots i_p ij]\, l_2\dots l_r}$$

We shall prove for example theorem 6, the proof of theorem 7 runs similarly.

Let us assume that $p = 2k$ and that $f_{i_1\dots i_p l_1\dots l_r}$ and $h_{i_1\dots i_p l_1\dots l_r}$ are the self-dual r/p-vectors of the 1-st kind and of the 2-nd kind respectively.

By virtue of theorem 4 identity (25) is reduced to

$$f^{i_2\dots i_p l i l_2\dots l_r} h_{i_2\dots i_p i l l_2\dots l_r} = -\frac{p+1}{p} f^{[i_1\dots i_p l] l_2\dots l_r} h_{[i_1\dots i_p l] l_2\dots l_r} \tag{41}$$

On the other hand from theorem 3, if $s = r$, we have the relations

$$f^{i_2\dots i_p i l_1\dots l_r} h_{i_2\dots i_p \cdot k_1\dots k_r}^{j} = f^{i_2\dots i_p j l_1\dots l_r} h_{i_2\dots i_p \cdot k_1\dots k_r}^{i} \tag{42}$$

Multiplying (42) by $g^{il_1} g^{jk_1}$ and summing for $ijl_1\dots l_r k_1\dots k_r$ we get

$$f^{i_2\dots i_p i \cdot l_2\dots l_r}_{i} h_{i_2\dots i_p j \cdot l_2\dots l_r}^{j} = f^{i_2\dots i_p j i l_2\dots l_r} h_{i_2\dots i_p i j l_2\dots l_r} \tag{43}$$

From (41) and (43) there follows theorem 6.

§3. The r/p × p-tensors in a V_n

Now, let us take into cosideration any tensor-field $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}(x)$ where $2\leqslant p\leqslant n$ and $r\geqslant 1$, in a V_n, satisfying the conditions

$$a_{[i_1\dots i_p] j_1\dots j_p l_1\dots l_r} = a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} \tag{44}$$

as well as

$$a_{i_1\ldots i_p l_1\ldots l_p j_1\ldots j_r} = a_{l_1\ldots l_p i_1\ldots i_p j_1\ldots j_r} \tag{45}$$

The tensor-field $a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}$ satisfying the identities (44) and (45) shall be called briefly the r/p×p-tensor fields.

The covariant derivative of the r-th order of the expanded curvature tensor $\overset{p}{R}_{i_1\ldots i_p j_1\ldots j_p}$ in a V_n is an example of the r/p×p-tensor-fields.

The r/p×p-tensor - fields constitute - as it may be noted easily - a particular example of more general class of tensor-fields, created by the s/p-tensor - fields $a_{i_1\ldots i_p l_1\ldots l_s}(x)$ where $s=p+r$. For this reason all the properties of the s/p-tensor - fields, where $s=p+r$, examined in §2 are automatically the properties of the r/p×p-tensor, fields.

From theorem 1 and the identity (25), (if we note that it is unimportant in (25) which of the indices l_α, where $\alpha = 1, 2, \ldots r$ are alternated with the group of indices $i_1\ldots i_p$), there follows.

The r/p×p-tensor $a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}$ satisfying the identities

$$a_{i_1\ldots i_p j_2\ldots j_p j l l_2\ldots l_r} = a_{i_1\ldots i_p j_2\ldots j_p l j l_2\ldots l_r}$$

is a tensor equal to zero.

Hence, there follows:

C o r o l l a r y 5. The covariant derivative of the bi-tensor field $a_{i_1\ldots i_p j_1\ldots j_p}(x)$, of class C^1, is equal to zero, if

$$a_{i_1\ldots i_p j_2\ldots j_p[j;l]}(x) = 0$$

From corollary 5 it follows particulary that:
The Riemannian space V_n is a symmetric space, in sens of Cartan [7] if its curvature tensor satisfies the relations

$$R_{ijk[l;m]} = 0$$

It is obvious that due to their specific construction; anti-symmetry with regard to all indices of each group of indices $\{i_\alpha\}$ $\{j_\alpha\}$ (cf. (44)) and symmetry with regard to these whole groups of indices (cf. (45)) the r/p×p-tensor-fields $a_{i_1 \ldots i_p j_1 \ldots j_p l_1 \ldots l_r}$ possess more properties than the r/p-tensor-fields which are of much more general character.

Let us note, first of all, that the propersties (44) and (45) of the r/p×p-tensor-fields allow us, besides the one to one mapping of the from (26) to determine, by the formula

$$ {}^{0}a_{i_{p+1} \ldots i_n j_{p+1} \ldots j_n l_1 \ldots l_r} \overset{df}{=} \frac{\rho^2}{(p!)^2} J^{i_1 \ldots i_p}{}_{i_{p+1} \ldots i_n} J^{j_1 \ldots j_p}{}_{j_{p+1} \ldots j_n} a_{i_1 \ldots i_p j_1 \ldots j_p l_1 \ldots l_r} \tag{46}$$

where $\rho = (-1)^{\frac{1}{2}p(n-p)}$, the second one to one mapping (analogically to (13)) of the set of all r/p×p-tensors onto the set of all r/(n-p)×(n-p) - tensors, taken at the same point of the V_n.

It is easy to note that the above mapping, in contradistrinction to the mapping defined by (26), always establishes a one to one correspondence between the real r/p×p-tensors and the real r/(n-p)×(n-p)-tensors, independently of the index of the space, the dimension of the space and the valency of the r/p×p-tensor under mapping.

The tensor ${}^{0}a_{i_{p+1} \ldots i_n j_{p+1} \ldots j_n l_1 \ldots l_r}$ is called dual[1)] of the tensor $a_{i_1 \ldots i_p j_1 \ldots j_p l_1 \ldots l_r}$.

---

1) Since for the r/p×p-tensors it is possible to determine fundamentally two different dual tensors, one by formula (26) and the other by formula (46), they should be distinguished. The first of them, for example, should be called dual and the other bi-dual.

From 46 there follows immediately

$$^{o}(^{o}a)_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} = a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} \qquad (47)$$

It is easy to state that due to the analogy in the formulas (26) and (46) defining the dual tensors of the given tensors we shall get, at least in part, analogical formulas, corollaries and theorems.

Really it is so, for example for the r/p×p-tensors we prove analogical relations to relations (31)

$$g^{jl} a_{i_1\dots i_p j_2\dots j_p j l l_2\dots l_r} = 0 \iff {}^{o}a_{i_{p+1}\dots i_n [j_{p+1}\dots j_n l] l_2\dots l_r} = 0 \qquad (48)$$

In the same manner it follows imediately from the definition (46) that if the r/p×p-tensor $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ is the product of the bi-tensor $\bar{a}_{i_1\dots i_p j_1\dots j_p}$ and a certain tensor $V_{l_1\dots l_r}$, i.e.

$$a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} = \bar{a}_{i_1\dots i_p j_1\dots j_p} V_{l_1\dots l_r}$$

then its dual tensor ${}^{o}a_{i_{p+1}\dots i_n j_{p+1}\dots j_n l_1\dots l_r}$ takes the form

$${}^{o}a_{i_{p+1}\dots i_n j_{p+1}\dots j_n l_1\dots l_r} = {}^{o}\bar{a}_{i_{p+1}\dots i_n j_{p+1}\dots j_n} V_{l_1\dots l_r}$$

In particular we get.

C o r o l l a r y 6. If the bi-tensor $a_{i_1\dots i_p j_1\dots j_p}$ field is a recurrent field, the dual field of it is also a recurrent field.

Let $a_{i_1\dots i_p j_1\dots j_p}$ be an arbitrary recurrent field of bi-tensors. By a simple calculation it is possible to state that the formula

$$h_{i_1\dots i_p} = \frac{1}{p!} a_{i_1\dots i_p j_1\dots j_p} f^{j_1\dots j_p}$$

etablishes a one correspondence between the recurrent fields of p-vectors $f_{i_1\dots i_p}$ and the recurrent fields of p-vectors $h_{i_1\dots i_p}$.

Let us assume now that the space examined further on is a 2p - dimensional Riemannian manifold.

The r/p×p-tensors $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ satifying the identities

$$\overset{0}{a}_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} = \theta\, a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} \tag{49}$$

where $\theta = \pm 1$, are called the self-dual r/p × p-tensors of the 1-st kind, if $\theta = +1$ and respectively of the 2-nd kind, if $\theta = -1$.

It is easy to note that the covariant derivative of the r-th order of the field of self-dual bi-tensors is a field of self-dual r/p×p-tensors.

It is also evident that analogically to the decomposition (36) each r/p×p-tensor $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ at the point $x \in V_{2p}$ may be expressed in a uniqual form as the sum of to self-dual r/p×p-tensors of the 1-st kind and of the 2-nd kind by the formula

$$a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} = \frac{1}{2}(a + \overset{0}{a}) + \frac{1}{2}(a - \overset{0}{a}) =$$

$$= \underset{1}{a}_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} + \underset{2}{a}_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} \tag{50}$$

where $\underset{1}{a} = \frac{1}{2}(a + \overset{0}{a}) = \overset{0}{\underset{1}{a}}$ and $\underset{2}{a} = \frac{1}{2}(a - \overset{0}{a}) = -\overset{0}{\underset{2}{a}}$

Let $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ be an arbitrary self-dual r/p p-tensor. Multiplying the self-dual r/p×p-tensor $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ by an arbitrary tensor $v^{l_1\dots l_r}$ and summing for $l_1\dots l_r$ we always get, as it follows from (49), a self-dual bi-tensor.

Hence, we have:

C o r o l l a r y 7. Multiplying an arbitrary self-dual r/p×p-tensor $a_{i_1\ldots i_p l_1\ldots l_p j_1\ldots j_r}$ of the 1-st kind (or of the 2-nd kind) by an arbitrary tensor $V^{j_1\ldots j_r}$ and summing for $j_1\ldots j_r$ we get a self-dual bi-tensor

$$\bar{a}_{i_1\ldots i_p j_1\ldots j_p} \overset{df}{=} a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r} V^{l_1\ldots l_r}$$

of the 1-st kind (and of the 2-nd kind respectively).

In particular from (50) and corollary 7 there follows immediately the respective formula [15]

$$a_{i_1\ldots i_p j_1\ldots j_p} = \underset{1}{a}_{i_1\ldots i_p j_1\ldots j_r} + \underset{2}{a}_{i_1\ldots i_p j_1\ldots j_r} \qquad (51)$$

for the decomposition of an arbitrary bi-tensor $a_{i_1\ldots i_p j_1\ldots j_p}$ into the sum of two self-dual bi-tensors of the 1-st kind and of the 2-nd kind.

From the definition of the r/p×p-tensors it follows that an arbitrary linear combination of these tensors gives a tensor of the same type. The same may be said about the salf-dual r/p×p-tensors of the 1-st kind (and of the 2-nd kind respectively). Thus: A set of all the r/p×p-tensors at the point $x \in V_{2p}$ creates an N-dimensional linear space. By virtue of (50) the N-dimensional linear space of the r/p×p-tensors may be decomposed into the simple sum of $\frac{1}{2}$ N-dimensional linear spece of self-dual r/p×p-tensors of the 1-st kind and of the 2-nd kind.

The following theorem have also been proved in paper [15]:

I. The bi-tensor $a_{i_1\ldots i_p j_1\ldots j_p}$ is a self-dual bi-tensor of the 1-st kind (or of the 2-nd kind), if and only if for each self-dual bi-tensor $b_{i_1\ldots i_p j_1\ldots j_p}$ of the 2-nd kind and (of the 1-st kind respectively) there is

$$a_{i_1\ldots i_p j_1\ldots j_p} b^{i_1\ldots i_p j_1\ldots j_p} = 0$$

and analogically, if

$$a^{i_2\dots i_p[i|j_2\dots j_p|[j}\, b_{i_2\dots i_p\cdot j_2\dots j_p\cdot\cdot}{}^{k]\quad l]} = 0$$

II. For each pair of self-dual bi tensors of the same kind $a_{i_1\dots i_p j_1\dots j_p}$ and $b_{i_1\dots i_p j_1\dots j_p}$ there are the identities

$$a^{i_1\dots i_p j_2\dots j_p(j}\, b_{i_1\dots i_p j_2\dots j_p\cdot}{}^{l)} = \frac{1}{2p}\, g^{jl}\, a^{i_1\dots i_p j_1\dots j_p}\, b_{i_1\dots i_p j_1\dots j_p}$$

Using the above theorems and corollary 7 we prove the following theorems:

T h e o r e m 8. The r/p×p-tensor $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ is a self-dual s/p×p-tensor of the 1-st kind (or of the 2-nd kind), if and only if for each self-dual s/p×p-tensor $b_{i_1\dots i_p j_1\dots j_p l_1\dots l_s}$ of the 2-nd kind (and of the 1-st kind respectively) there are the identities

$$a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}\, b^{i_1\dots i_p j_1\dots j_p}{}_{k_1\dots k_s} = 0$$

and analogically, if

$$a^{i_2\dots i_p[i|j_2\dots j_p|[j}{}_{l_1\dots l_r}\, b_{i_2\dots i_p\cdot j_2\dots j_p\cdot k_1\dots k_s}{}^{k]\quad l]} = 0$$

where the alternation concerns independetly the indices "i" and "k" as well as "j" and "l".

T h e o r e m 9. For each pair of self-dual r/p×p-tensors $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ and s/p×p-tensors $b_{i_1\dots i_p j_1\dots j_p l_1\dots l_s}$ of the same kind there are the identities:

$$a^{i_1\dots i_p j_2\dots j_p (j}{}_{l_1\dots l_r}{}^{l)} b_{i_1\dots i_p j_2\dots j_p \cdot k_1\dots k_s} =$$

$$= \frac{1}{2p} g^{jl} a^{i_1\dots i_p j_1\dots j_p}{}_{l_1\dots l_r} b_{i_1\dots i_p j_1\dots j_p k_1\dots k_s}$$

Theorems 8 and 9 also hold true if. $r=0$ or $s=0$. In this case the r/p×p-tensors become just the bi-tensors.

It is easy to note that the r/p×p-tensors $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ may be used to establish a certain correspodence between p-vectors and r/p-tensors.

Indeed, the formula

$$h_{i_1\dots i_p} = \frac{1}{p!} a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} f^{j_1\dots j_p l_1\dots l_r} \tag{52}$$

establishes a certain correspondence betwen the r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}$ and the p-vectors $h_{i_1\dots i_p}$.

Similarly, the formula

$$f_{i_1\dots i_p l_1\dots l_r} = \frac{1}{p!} a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r} h^{i_1\dots i_p} \tag{53}$$

establishes other correspondence between the p-vectors $h_{i_1\dots i_p}$ and the r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}$.

Assuming that $g>0$ and $p=2k$ or $g<0$ and $p=2k+1$ (i.e. that we refer only our consideration to real self-dual p-vectors and r/p-tensors) we shall prove:

T h e o r e m 10. The r/p×p-tensor $a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$ is the self-dual r/p×p-tensor of the 1-st kind (or of the 2-nd kind), if and only if the formula (52) and analogically (53) establish a correspondence between self-dual r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}$ and the self-dual p-bector $h_{i_1\dots i_p}$ according to the following table:

Table 1

		$a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}$	$h_{i_1\ldots i_p}$	$f_{i_1\ldots i_p l_1\ldots l_r}$
(*)	$p=2k$	1-st kind	1 (2) kind	1 (2) kind
		2-nd kind	1 (2) kind	2 (1) kind
	$p=2k+1$	1-st kind	1 (2) kind	2 (1) kind
		2-nd kind	1 (2) kind	1 (2) kind

Proof. Let us assume that $a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}$ is a self-dual $r/p\times p$-tensor, that is

$$a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}=\theta\,\frac{\varrho^2}{(p!)^2}\,\mathcal{J}_{\alpha_1\ldots\alpha_p i_1\ldots i_p}\mathcal{J}_{\beta_1\ldots\beta_p j_1\ldots j_p}\,a^{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_p}{}_{l_1\ldots l_r}$$

where $\theta=\pm1$ and that $f_{i_1\ldots i_p l_1\ldots l_r}$ is a self-dual r/p-tensor, that is

$$f_{i_1\ldots i_p l_1\ldots l_r}=\tilde{\theta}\,\frac{\varrho}{p!}\,\mathcal{J}_{\gamma_1\ldots\gamma_p i_1\ldots i_p}\,f^{\gamma_1\ldots\gamma_p}{}_{l_1\ldots l_r}$$

where $\tilde{\theta}=\pm1$. By virtue of the above from (52) it follows that

$$h_{i_1\ldots i_p}=\frac{1}{p!}\,a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}\,f^{j_1\ldots j_p l_1\ldots l_r}=$$

$$=\theta\,\tilde{\theta}\,\frac{\varrho^3}{(p!)^4}\mathcal{J}_{\alpha_1\ldots\alpha_p i_1\ldots i_p}\mathcal{J}_{\beta_1\ldots\beta_p j_1\ldots j_p}\,a^{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_p}{}_{l_1\ldots l_r}\,\mathcal{J}^{\gamma_1\ldots\gamma_p j_1\ldots j_p}f_{\gamma_1\ldots\gamma_p}{}^{l_1\ldots l_r}=$$

$$=\theta\,\tilde{\theta}\,\frac{\varrho^3}{(p!)^2}\,\mathcal{J}_{\alpha_1\ldots\alpha_p i_1\ldots i_p}\,a^{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_p}{}_{l_1\ldots l_r}\,f_{\beta_1\ldots\beta_p}{}^{l_1\ldots l_r}=$$

$$= \theta\tilde{\theta}\,\frac{\varrho^3}{p!}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,\frac{1}{p!}\,a_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_p l_1\ldots l_r}\,f^{\beta_1\ldots\beta_p l_1\ldots l_r} =$$

$$= \theta\tilde{\theta}\,\varrho^2\,\frac{\varrho}{p}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,h_{\alpha_1\ldots\alpha_p} = \theta\,\tilde{\theta}\,\varrho^2\,{}^{o}h_{i_1\ldots i_p} =$$

$$= \underset{1}{\theta}\,{}^{o}h_{i_1\ldots i_p}$$

where $\underset{1}{\theta} = \theta\,\tilde{\theta}\varrho^2 = \pm 1$. Thus, the p-vector $h_{i_1\ldots i_p}$ is the self-dual p-vector, according to table (1).

Let us assume inversely that for each self-dual r/p-tensor $f_{i_1\ldots i_p l_1\ldots l_r}$ the p-vector $h_{i_1\ldots i_p}$ defined by relations (52) is a self-dual p-vector.

Then, by virtue of the assumption, from (52) we have

$$\frac{1}{p!}a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r}f^{j_1\ldots j_p l_1\ldots l_r} = \theta\,\frac{\varrho}{p!}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,\frac{1}{p!}\,a_{\alpha_1\ldots\alpha_p j_1\ldots j_p l_1\ldots l_r}f^{j_1\ldots j_p l_1\ldots l_r}$$

or equivalently

$$\left(\frac{1}{p!}\,a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r} - \frac{\theta\varrho}{(p!)^2}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,a_{\alpha_1\ldots\alpha_p j_1\ldots j_p l_1\ldots l_p}\right)f^{j_1\ldots j_p l_1\ldots l_r} = 0 \tag{54}$$

By virtue of theorem 4 it follows from (54) that

$$\frac{1}{p!}\,a_{i_1\ldots i_p j_1\ldots j_p l_1\ldots l_r} - \frac{\theta\varrho}{(p!)^2}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,a_{\alpha_1\ldots\alpha_p j_1\ldots j_p l_1\ldots l_r} =$$

$$= -\tilde{\theta}\,\frac{1}{p!}\,J^{\beta_1\ldots\beta_p}{}_{j_1\ldots j_p}\left(\frac{1}{p!}\,a_{i_1\ldots i_p\beta_1\ldots\beta_p l_1\ldots l_r} - \frac{\theta\varrho}{(p!)^2}\,J^{\alpha_1\ldots\alpha_p}{}_{i_1\ldots i_p}\,a_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_p l_1\ldots l_r}\right)$$

or equivlently

$$\frac{1}{p!} a_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r} - \frac{\theta \varrho}{(p!)^2} J^{\alpha_1 \dots \alpha_p}{}_{i_1 \dots i_p} a_{\alpha_1 \dots \alpha_p j_1 \dots j_p l_1 \dots l_r} = \tag{55}$$

$$= -\theta \tilde{\theta} \frac{1}{p!} \overset{o}{a}_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r} + \frac{\tilde{\theta} \varrho}{(p!)^2} J^{\beta_1 \dots \beta_p}{}_{j_1 \dots j_p} a_{i_1 \dots i_p \beta_1 \dots \beta_p l_1 \dots l_r}$$

Finally, from (55) it follows (it is sufficient to consider seperate cases, for example:f-is the self-dual r/p-tensor of the 1-st kind and h-is a self-dual p-vector of the 1-st kind and $\rho = 2k$ and so on)

$$a_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r} = -\theta \tilde{\theta} \overset{o}{a}_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r}$$

i.e. that the r/p×p-tensor $a_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r}$ is the self-dual r/p×p-tensor of the kind defined in table (1).

The remaining part of the proof runs analogically.

From theorems 10, 2, 3 and 4 and from theorems I and II cited in §2, there immediately follows:

C o r o l l a r y 8. Each of the following identities

$$a_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r} f^{i_1 \dots i_p} h^{j_1 \dots j_p} = 0 \tag{56}$$

$$f^{i_1 \dots i_p} a_{i_1 \dots i_p j_2 \dots j_p \cdot l_1 \dots l_r}{}^{[j} h^{k] j_2 \dots j_p} = 0 \tag{57}$$

$$a_{i_1 \dots i_p j_1 \dots j_p l_1 \dots l_r} f^{i_1 \dots i_p} h^{j_1 \dots j_p l_1 \dots l_r} = 0 \tag{58}$$

$$f^{i_1\dots i_p l_1\dots l_r}\, a_{i_1\dots i_p j_1\dots j_p m_1\dots m_r}\, h^{j_1\dots j_p k_1\dots k_s} = 0 \qquad (59)$$

$$f^{i_1\dots i_p l_1\dots l_r}\, a_{i_1\dots i_p j_2\dots j_p}{}^{[j}{}_{\cdot\, m_1\dots m_r}\, h^{k] j_2\dots j_p k_1\dots k_s} = 0 \qquad (60)$$

$$f^{i_2\dots i_p [i}\, a_{i_2\dots i_p}{}^{j]}{}_{\cdot\, j_1\dots j_p l_1\dots l_r}\, h^{j_1\dots j_p l_1\dots l_r} = 0 \qquad (61)$$

(independently one of the other),if is satisfied by an arbitrary pair of self-dual: p-vectors $f_{i_1\dots i_p}$ and $h_{i_1\dots i_p}$, r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}$ and $h_{i_1\dots i_p l_1\dots l_r}$ or the p-vector $f_{i_1\dots i_p}$ and the r/p-tensor $h_{i_1\dots i_p l_1\dots l_r}$ according to the following table 2

Table 2

		$a_{i_1\dots i_p j_1\dots j_p l_1\dots l_r}$	$f_{i_1\dots i_p}$	$h_{i_1\dots i_p}$	$f_{i_1\dots i_p l_1\dots l_r}$	$h_{i_1\dots i_p l_1\dots l_r}$
$p=2k$	I-st kind	I (II)	I (II)	I (II)	I (II)
	II-nd kind	II (I)	I (II)	II (I)	I (II)
$p=2k+1$	I-st kind	I (II)	I (II)	I (II)	I (II)
	II-nd kind	II (I)	I (II)	II (I)	I (II)

expresses the necessary and sufficient condition that the r/p×p-tensor $a_{i_1\dots i_p l_1\dots l_p j_1\dots j_r}$ be the self-dual r/p×p-tensor of the kind defined by the table above.

§4. The r/p harmonic and Killing tensors in a V_n

In §2 we have taken into consideration a certain class of tensor-fields which we have named the r/p-tensor - fields.

The r/p-tensor-fields - in a sense - may be regarded as a certain kind of generalized p-vector fields.

By analogy from p-vector fields, for the "generalized" p-vector fields we have introduced the concept of a dual field to the given field and the concept of a self-dual field, and the remaining part of §2 has been devoted to examinations of certain properties of these fields.

In §3 we were dealing with a certain sub-class of r/p-tensor-fields, called the r/p×p-tensor-fields.

In this paragraph we return to the general class of r/p-tensor-fields to apply to them the concept of the harmonic p-vectors and Killing p-vectors, taken from p-vector-fields.

However, here again we shall begin our considerations from citing the fundamental concepts and facts from the theory of the harmonic and Killing p-vector - fields [6].

In this connection let us assume that the space under examination is an n-dimensional compact Riemannian manifold of class C^3 with a positive definite metric $ds^2 = g_{ij}\,dx^i dx^j$.

Moreover, we assume that all the tensor-field examined on the given manifold of this section are of class C^2.

Let $f_{i_1 \ldots i_p}(x)$ be an arbitrary tensor-field of class C^2 on the examined manifold.

The Laplacian of the scalar-function

$$\Phi(x) = f_{i_1 \ldots i_p} f^{i_1 \ldots i_p}$$

is given by the formula

$$\Delta\Phi(x) = 2\left(f^{i_1 \ldots i_p;j} f_{i_1 \ldots i_p;j} + f^{i_1 \ldots i_p} g^{jk} f_{i_1 \ldots i_p;j;k}\right)$$

From the assumption concerning the metric of V_n it follows that the form

$$\omega = f^{i_1 \dots i_p ; j} f_{i_1 \dots i_p ; j}$$

is a positive definite form of $f_{i_1 \dots i_p ; j}$

Therefore, if the tensor-field $f_{i_1 \dots i_p}(x)$ satisfies the deferential equations of the form

$$g^{jk} f_{i_1 \dots i_p ; j ; k} = T_{i_1 \dots i_p j_1 \dots j_p} f^{j_1 \dots j_p} \tag{62}$$

where $T_{i_1 \dots i_p j_1 \dots j_p}$ is a certain tensor-field and if the quadratic form

$$T = T_{i_1 \dots i_p j_1 \dots j_p} f^{i_1 \dots i_p} f^{j_1 \dots j_p} \geqslant 0 \tag{63}$$

then

$$\Delta \Phi \geqslant 0$$

and by virtue of Bochner's Lemma 1 [2] we have

$$\Delta \Phi = 0$$

Consequently we get

$$f_{i_p \dots i_p ; j} = 0$$

as well as

$$T_{i_1 \dots i_p j_1 \dots j_p} f^{i_1 \dots i_p} f^{j_1 \dots j_p} = 0$$

If the form T is the positive definite form, then from the equation $T=0$ we conclude that

$$f_{i_1 \dots i_p} = 0$$

Thus we obtain 6.

T h e o r e m I. If, in compact V_n with positive definite metric $ds^2 = g_{ij}dx^i dx^j$ the tensor-field $f_{i_1 \dots i_p}(x)$ satisfies the relations (62) and (64) then it also satisfies the identities

$$f_{i_1 \dots i_p ; j}(x) = 0 \tag{65}$$

and

$$T_{i_1 \dots i_p j_1 \dots j_p} f^{i_1 \dots i_p} f^{j_1 \dots j_p} = 0 \tag{66}$$

If the quadratic form (63) is a positive definite form, then from (66) it follows that

$$f_{i_1 \dots i_p}(x) = 0$$

Let us take into consideration an arbitrary r/p-tensor - field of class C^2, i.e. the tensor field satisfying the relations

$$f_{[i_1 \dots i_p] l_1 \dots l_r}(x) = f_{i_1 \dots i_p l_1 \dots l_r} \qquad (p \geqslant 2; \ r \geqslant 1)$$

The anti-symetric part of the second covariant derivative of a r/p-tensor field is given by the formula

$$f_{i_1\dots i_p l_1\dots l_r;j;k} - f_{i_1\dots i_p l_1\dots l_r;k;j} =$$

$$= -f_{a i_2\dots i_p l_1\dots l_r} R^{a}{}_{i_1 jk} - f_{i_1 a i_3\dots i_p l_1\dots l_r} R^{a}{}_{i_2 jk} - \dots +$$

$$- f_{i_1\dots i_{p-1} a l_1\dots l_r} R^{a}{}_{i_p jk} - \dots - f_{i_1\dots i_p l_1\dots l_{r-1} a} R^{a}{}_{l_r jk}$$

According to the above, we have the identities

$$f_{j i_2\dots i_p l_1\dots l_r; i_1; k} - f_{j i_2\dots i_p l_1\dots l_r; k; i_1} + f_{i_1 j i_2\dots i_p l_1\dots l_r; i_2; k} +$$

$$- f_{i_1 j i_2\dots i_p l_1\dots l_r; k; i_2} + \dots + f_{i_1\dots i_{p-1} j l_1\dots l_r; i_p; k} +$$

$$- f_{i_1\dots i_{p-1} j l_1\dots l_r; k; i_p} = - f_{a i_2\dots i_p l_1\dots l_r} R^{a}{}_{j i_1 k} - f_{j a i_3\dots i_p l_1\dots l_r} R^{a}{}_{i_2 i_1 k} +$$

$$\dots - f_{j i_2\dots i_{p-1} a l_1\dots l_r} R^{a}{}_{i_p i_1 k} - f_{j i_2\dots i_p a l_2\dots l_r} R^{a}{}_{l_1 i_1 k} - \qquad (67)$$

$$\dots - f_{j i_2\dots i_p l_1\dots l_{r-1} a} R^{a}{}_{l_r i_1 k} - f_{a j i_3\dots i_p l_1\dots l_r} R^{a}{}_{i_1 i_2 k} -$$

$$\dots - f_{i_1 j i_3\dots i_p l_1\dots l_{r-1} a} R^{a}{}_{l_r i_2 k} - \dots - f_{a i_2\dots i_{p-1} j l_1\dots l_r} R^{a}{}_{i_1 i_p k} +$$

$$\dots - f_{i_1\dots i_{p-1} j l_1\dots l_{r-1} a} R^{a}{}_{l_r i_p k}$$

Adding to and subtracting from the left side of (67) the expression $f_{i_1\dots i_p l_1\dots l_r; j; k}$ and next multiplying both sides of (67) by the tensor g^{jk} and summing for jk as well as taking into account the identity $\mathcal{R}^{a}{}_{[bcd]}=0$ we get (cf. paper [12])

$$g^{jk} f_{i_1\dots i_p l_1\dots l_r;j;k} - g^{jk}\Big(f_{i_1\dots i_p l_1\dots l_r;j} - f_{j i_2\dots i_p l_1\dots l_r;i_1} - \dots$$

$$- f_{i_1\dots i_{p-1} j l_1\dots l_r;i_p}\Big)_{;k} - \Big(f^{k}{}_{i_2\dots i_p l_1\dots l_r;k;i_1} - f^{k}{}_{i_1 i_3\dots i_p l_1\dots l_r;k;i_2}$$

$$\dots - f^{k}{}_{i_2\dots i_{p-1} i_1 l_1\dots l_r;k;i_p}\Big) = \sum_{s=1}^{p} R^{a}{}_{i_s} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_p l_1\dots l_r} \quad (68)$$

$$+ \sum_{\substack{s,t=1\\ s<t}}^{p} R^{ab}{}_{i_s i_t} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_{t-1} b i_{t+1}\dots i_p l_1\dots l_r} +$$

$$- \sum_{s=1}^{p} \sum_{t=1}^{r} R^{a}{}_{i_s i_t}{}^{b} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_p l_1\dots l_{t-1} b l_{t+1}\dots l_r}$$

It follows from (68) that if the r/p-tensor field $f_{i_1\dots i_p l_1\dots l_r}(x)$ satisfies the relations

$$g^{jk}\Big(f_{i_1\dots i_p l_1\dots l_r;j} - f_{j i_2\dots i_p l_1\dots l_r;i_1} - \dots - f_{i_1\dots i_{p-1} j l_1\dots l_r;i_p}\Big)_{;j} +$$

$$-\Big(f^{k}{}_{i_2\dots i_p l_1\dots l_r;k;i_1} - f^{k}{}_{i_1 i_3\dots i_p l_1\dots l_r;k;i_2} - \dots - f^{k}{}_{i_2\dots i_{p-1} i_1 l_1\dots l_r;k;i_p}\Big) = 0 \quad (69)$$

it also satisfies the relations

$$g^{jk} f_{i_1\dots i_p l_1\dots l_r;k;j} = \sum_{s=1}^{p} R^{a}{}_{i_s} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_p l_1\dots l_r} +$$

$$+ \sum_{\substack{s,t=1\\ s<t}}^{p} R^{ab}{}_{i_s i_t} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_{t-1} b i_{t+1}\dots i_p l_1\dots l_r} + \quad (70)$$

$$-\sum_{s=1}^{p} \sum_{t=1}^{r} R^{a}{}_{i_s i_t}{}^{b} f_{i_1\dots i_{s-1} a i_{s+1}\dots i_p l_1\dots l_{t-1} b l_{t+1}\dots l_r}$$

and consequently the relations

$$g^{jk} f^{i_1\dots i_p l_1\dots l_r} f_{i_1\dots i_p l_1\dots l_r;j;k} = p R_{ij} f^{ii_2\dots i_p l_1\dots l_r} f^{j}{}_{i_2\dots i_p l_1\dots l_r} +$$

$$- \frac{p(p-1)}{p} R_{ijkl} f^{iji_3\dots i_p l_1\dots l_r} f^{kl}{}_{i_3\dots i_p l_1\dots l_r} + \qquad (71)$$

$$- p \sum_{t=1}^{r} R^{k}{}_{aij} f^{ii_2\dots i_p l_1\dots l_{t-1} a l_{t+1}\dots l_r} f^{j}{}_{i_2\dots i_p l_1\dots l_{t-1} k l_{t+1}\dots l_r}$$

Introducing the denotation

$$F\left(f_{i_1\dots i_p l_1\dots l_r}\right) \overset{df}{=} R_{ij} f^{ii_2\dots i_p l_1\dots l_r} f^{j}{}_{i_2\dots i_p l_1\dots l_r} +$$

$$- \frac{p-1}{2} R_{ijkl} f^{iji_3\dots i_p l_1\dots l_r} f^{kl}{}_{i_3\dots i_p l_1\dots l_r} + \qquad (72)$$

$$- \sum_{t=1}^{r} R^{k}{}_{aij} f^{ii_2\dots i_p l_1\dots l_{t-1} a l_{t+1}\dots l_r} f^{j}{}_{i_2\dots i_p l_1\dots l_{t-1} k l_{t+1}\dots l_r}$$

we can give a more compact form to (71), namely

$$g^{jk} f^{i_1\dots i_p l_1\dots l_r} f_{i_1\dots i_p l_1\dots l_r;j;k} = p F\left(f_{i_1\dots i_p l_1\dots l_r}\right) \qquad (73)$$

From theorem I of this section and from the above considerations there follows:

T h e o r e m 11. If in an n-dimensional compact Riemannian space with a positive definite metric the r/p-tensor field $f_{i_1\dots i_p l_1\dots l_r}(x)$ satisfies the identities (69) and if

$$F\left(f_{i_1\dots i_p l_1\dots l_r}\right) > 0$$

then

$$\nabla_j f_{i_1 \ldots i_p l_1 \ldots l_r}(x) = 0 \tag{74}$$

and consequently

$$F\left(f_{i_1 \ldots i_p l_1 \ldots l_r}\right) = 0 \tag{75}$$

If the form $F\left(f_{i_1 \ldots i_p l_1 \ldots l_r}\right)$ is a positive definite form, then from (75) it follows immediately that

$$f_{i_1 \ldots i_p l_1 \ldots l_r}(x) = 0$$

Now, adding to and subtracting from the left side of (67) the expression $p f_{i_1 \ldots i_p l_1 \ldots l_r; j; k}$ and next multiplying both sides of this identity by g^{jk}, summing for jk and taking into account the identity $R^{a}_{[bcd]} = 0$ we get

$$- p g^{jk} f_{i_1 \ldots i_p l_1 \ldots l_r; j; k} + g^{jk}\Big(p f_{i_1 \ldots i_p l_1 \ldots l_r; j} + f_{j i_2 \ldots i_p l_1 \ldots l_r; i_1} + \cdots$$

$$+ f_{i_1 \ldots i_{p-1} j l_1 \ldots l_r; i_p}\Big)_{;k} - \left(f^{k}{}_{i_2 \ldots i_p l_1 \ldots l_r; k; i_1} - \cdots - f^{k}{}_{i_2 \ldots i_{p-1} i_1 l_1 \ldots l_r; k; i_p}\right) =$$

$$= \sum_{s=1}^{p} R^{a}{}_{i_s} f_{i_1 \ldots i_{s-1} a i_{s+1} \ldots i_p l_1 \ldots l_r} + \tag{76}$$

$$+ \sum_{\substack{s,t=1 \\ s<t}}^{p} R^{ab}{}_{i_s i_t} f_{i_1 \ldots i_{s-1} a i_{s+1} \ldots i_{t-1} b i_{t+1} \ldots i_p l_1 \ldots l_r} +$$

$$- \sum_{p=1}^{p} \sum_{t=1}^{r} R^{a}{}_{i_s i_t}{}^{b} f_{i_1 \ldots i_{s-1} a i_{s+1} \ldots i_p l_1 \ldots l_{t-1} b l_{t+1} \ldots l_r}$$

From (76) it follows that if the r/p-tensor field $f_{i_1 \ldots i_p l_1 \ldots l_r}(x)$ satisfies the relations:

$$g^{jk}\left(pf_{i_1\ldots i_p l_1\ldots l_r;j}+f_{ji_2\ldots i_p l_1\ldots l_r;i_1}+\cdots+f_{i_1\ldots i_{p-1}jl_1\ldots l_r;i_p}\right)_{;k}+$$

$$-\left(f^{k}{}_{i_2\ldots i_p l_1\ldots l_r;k;i_1}-\cdots-f^{k}{}_{i_2\ldots i_{p-1}i_1 l_1\ldots l_r;k;i_p}\right)=0 \tag{77}$$

then it also satisfies the relations

$$g^{jk}f_{i_1\ldots i_p l_1\ldots l_r;j;k}+\frac{1}{p}\sum_{s=1}^{p}R^{a}{}_{i_s}f_{i_1\ldots i_{s-1}ai_{s+1}\ldots i_p l_1\ldots l_r}+$$

$$-\frac{1}{p}\sum_{s,t=1}^{p}R^{ab}{}_{i_s i_t}f_{i_1\ldots i_{s-1}ai_{s+1}\ldots i_{t-1}bi_{t+1}\ldots i_p l_1\ldots l_r}+$$

$$-\frac{1}{p}\sum_{s=1}^{p}\sum_{t=1}^{r}R^{a}{}_{i_s i_t}{}^{b}f_{i_1\ldots i_{s-1}ai_{s+1}\ldots i_p l_1\ldots l_{s-1}bl_{s+1}\ldots l_r}$$

and consequently it satisfies the relations

$$g^{jk}f^{i_1\ldots i_p l_1\ldots l_r}f_{i_1\ldots i_p l_1\ldots l_r;j;k}=-F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right)$$

where $F(f_{i_1\ldots i_p l_1\ldots l_r})$ is the form defined by (72).

Thus from the above considerations and from theorem I of this section there follows:

T h e o r e m 12. If in a compact V_n with a positive definite metric the r/p-tensor field $f_{i_1\ldots i_p l_1\ldots l_r}(x)$ satisfies the identities (77) and if

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right)<0$$

then

$$f_{i_1\ldots i_p l_1\ldots l_r;j}(x)=0 \tag{78}$$

and consequently $$F\left(f_{i_1\dots i_p l_1\dots l_r}\right) = 0 \tag{79}$$

If the form $F\left(f_{i_1\dots i_p l_1\dots l_r}\right)$ is a negative definite form, then from (79) it follows that

$$f_{i_1\dots i_p l_1\dots l_r}(x) = 0$$

Now we introduce the definitions

D e f i n i t i o n 1. The r/p-tensor field $f_{i_1\dots i_p l_1\dots l_r}(x)$ is called a harmonic r/p-tensor field, if

$$\nabla_{[j} f_{i_1\dots i_p] l_1\dots l_r}(x) = \nabla_j f_{i_1\dots i_p l_1\dots l_r} \tag{80}$$

as well as $$g^{jk}\nabla_j f_{k i_2\dots i_p l_1\dots l_r}(x) = 0 \tag{81}$$

D e f i n i t i o n 2. The r/p-tensor field is called the Killing r/p-tensor field, if

$$\nabla_{[j} f_{i_1\dots i_p] l_1\dots l_r}(x) = 0 \tag{82}$$

For the Killing r/p-tensor fields it follows from (82) that

$$g^{jk}\nabla_j f_{k i_2\dots i_p l_1\dots l_r}(x) = 0 \tag{83}$$

It is easy to note that the harmonic r/p-tensor fields satisfy the relations (69), and consequently from therem 11 there follows the

C o r o l l a r y 9. If in a compact n-dimensional Riemannian space with a positive definite metric, the harmonic field of the r/p-tensors $f_{i_1\dots i_p l_1\dots l_r}(x)$ satisfies the inequalities

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) < 0$$

then this field is a covariant constant field, that is

$$f_{i_1\ldots i_p l_1\ldots l_r;j}(x) = 0$$

and consequently

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) = 0 \tag{84}$$

If the quadratic form $F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right)$ is a positive definite form, then - as it follows from (84) - there do not exist the harmonic r/p-tensor fields different from zero.

Similary, the Killing r/p-tensor fields satisfy the identities (77), from theorem 12 there follows:

C o r o l l a r y 10. If a compact n-dimensional Riemannian space with a positive definite metric the Killing r/p-tensor field $f_{i_1\ldots i_p l_1\ldots l_r}(x)$ satisfies the identities:

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) \leq 0$$

then this field is a covariant constant field, i.e.

$$f_{i_1\ldots i_p l_1\ldots l_r;j}(x) = 0$$

and conseqeently

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) = 0 \tag{85}$$

If, however, the quadratic form $F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right)$ is a negative definite form, then - as it follows from (85) there do not exist the Killing r/p-tensor fields other than equal to zero.

In section 2 we have stated that the field of a covariant derivative of the r-th order of the p-vector field $f_{i_1 \dots i_p}(x)$, i.e.

$$f_{i_1 \dots i_p; l_1; \dots; l_r}(x) \tag{86}$$

is a particular case of the r/p-tensor field.

It is easy to note that if we restricted our considerations of this section to the particular case of r/p-tensor fields of the form (86), a part of our considerations and results would be the same as the corresponding considerations and results of R.Srivastava published in paper [12].

There may be noted something more, namely that if in nearly all places of paper [12] we substitute the tensor $f_{i_1 \dots i_p; l_1; \dots; l_r}$ by a more general r/p-tensor $f_{i_1 \dots i_p l_1 \dots l_r}$ respectively, we shall get similar results.

This enable us (while refraining from the proof which would be analogical to the corresponding proof of paper [12]) to formulate the following two theorems which constitute a natural generalization of the corresponding theorems of paper [12]:

T h e o r e m 13. In a compact orientable n-dimensional Riemannian manifold with a positive definite metric, the r/p-tensor field $f_{i_1 \dots i_p l_1 \dots l_r}$ is a harmonic field of the r/p-tensors, if and only if it satisfies the identities

$$g^{jk} f_{i_1 \dots i_p l_1 \dots l_r; j; k} = \sum_{s=1}^{p} R^{a}{}_{i_s} f_{i_1 \dots i_{s-1} a i_{s+1} \dots i_p l_1 \dots l_r} +$$

$$+ \sum_{\substack{s,t=1 \\ s<t}}^{p} R^{ab}{}_{i_s i_t} f_{i_1 \dots i_{s-1} a i_{s+1} \dots i_{t-1} b i_{t+1} \dots i_p l_1 \dots l_r} +$$

$$- \sum_{s=1}^{p} \sum_{t=1}^{r} R^{a}{}_{i_s i_t}{}^{b} f_{i_1 \dots i_{s-1} a i_{s+1} \dots i_p l_1 \dots l_{t-1} b l_{t+1} \dots l_r}$$

T h e o r e m 14. In a compact orientable n-dimensional Riemannian manifold with a positive definite metric, the r/p-tensor field $f_{i_1\ldots i_p l_1\ldots l_r}$ is the Killing r/p-tensor field, if and only if it satisfies the identities

$$p g^{jk} f_{i_1\ldots i_p l_1\ldots l_r;j;k} + \sum_{s=1}^{p} R^{a}{}_{i_s} f_{i_1\ldots i_{s-1} a i_{s+1}\ldots i_p l_1\ldots l_r} +$$

$$+ \sum_{s,t=1}^{p} R^{ab}{}_{i_s i_t} f_{i_1\ldots i_{s-1} a i_{s+1}\ldots i_{t-1} b i_{t+1}\ldots i_p l_1\ldots l_r} +$$

$$- \sum_{s=1}^{p} \sum_{t=1}^{r} R^{a}{}_{i_s i_t}{}^{b} f_{i_1\ldots i_{s-1} a i_{s+1}\ldots i_p l_1\ldots l_{t-1} b l_{t+1}\ldots l_r} = 0$$

and the identities

$$f^{k}{}_{i_2\ldots i_p l_1\ldots l_r;k}(x) = 0$$

Let V_n continue to be a compact orientable Riemannian manifold with a positive definite metric and $f_{i_1\ldots i_p l_1\ldots l_r}(x)$ an arbitrary field of r/p-tensors $f_{i_1\ldots i_p l_1\ldots l_r}$ in the given manifold.

Like in the case of the p-vector fields, we define the inner global product of the r/p-tensor fields by the formula

$$(f;h) = \int_{V_n} f_{i_1\ldots i_p l_1\ldots l_r} h^{i_1\ldots i_p l_1\ldots l_r} dv$$

where:

$$dv \overset{df}{=} \sqrt{g}\, dx^1 \cdots dx^n > 0; \quad g = \det((g_{ij}))$$

Since from the assumption the metric is positive definite, we always hawe $(f,f) \geqslant 0$, where $(f,f) = 0$ if $f = 0$.

To simplify the further calculations we shall establish new symbols

$$\underset{R}{\varphi} = \underset{R}{\varphi}_{ji_1\dots i_p l_1\dots l_r} \overset{df}{=} (p+1)\, \nabla_{[j}\, \varphi_{i_1\dots i_p]\, l_1\dots l_r} \tag{87}$$

and

$$\underset{D}{\varphi} = \underset{D}{\varphi}_{i_2\dots i_p l_1\dots l_r} \overset{df}{=} \nabla_j\, \varphi^{j}{}_{i_2\dots i_p l_1\dots l_r} \tag{88}$$

New, if $\varphi_{i_1\dots i_p l_1\dots l_r}(X)$ is a harmonic field of the r/p-tensors, in accordance with the definition 1 as well as with the above denotations, we have

$$\underset{R}{\varphi} = 0 \qquad \underset{D}{\varphi} = 0$$

Let $f_{i_1\dots i_p l_1\dots l_r}(x)$ and $h_{ji_1\dots i_p l_1\dots l_r}(X)$ continue to be arbitrary r/p and r/(p+1) tensor fields respectively.

Let us consider the vector field

$$u^{j} \overset{df}{=} f_{i_1\dots i_p l_1\dots l_r}\, h^{ji_1\dots i_p l_1\dots l_r}$$

On the basis of Green s theorem [2] we get

$$0=\int_{V_n}\nabla_j u^j dv = \int_{V_n}\Big(\nabla_{[j} f_{i_1\dots i_p]l_1\dots l_r} h^{ji_1\dots i_p l_1\dots l_r} + f_{i_1\dots i_p l_1\dots l_r} h^{ji_1\dots i_p l_1\dots l_r}{}_{;j}\Big) \tag{89}$$

From (89) as well as from (87) and (88) it follows

$$\left(\underset{R}{f}; h\right) + (p+1)\left(f; \underset{D}{h}\right) = 0 \tag{90}$$

If $\varphi_{i_1 \ldots i_p l_1 \ldots l_r}$ is a harmonic field of the r/p-tensors, then $\underset{R}{\varphi} = 0$ and $\underset{D}{\varphi} = 0$

$$\underset{\Delta}{\varphi} \overset{\mathrm{df}}{=} \left(\underset{\substack{R\\D}}{\varphi}\right) + \left(\underset{\substack{D\\R}}{\varphi}\right) = 0 \tag{91}$$

Let us assume now that the r/p-tensor field satisfies the relations (91). Putting $f = \underset{D}{\varphi}$ and $h = \varphi$ into (90) we get

$$\left(\underset{R}{\varphi}; \underset{R}{\varphi}\right) + (p+1)\left(\varphi; \left[\underset{\substack{R\\D}}{\varphi}\right]\right) = 0 \tag{92}$$

If we put $f = \underset{D}{\varphi}$ and $h = \varphi$ into (90) we get

$$\left(\left(\underset{\substack{D\\R}}{\varphi}\right); \varphi\right) + p\left(\underset{D}{\varphi}; \underset{D}{\varphi}\right) = 0 \tag{93}$$

From (91), (92) and (93) we get

$$\left(\varphi; \underset{\Delta}{\varphi}\right) = \left(\varphi; \underset{\substack{R\\D}}{\varphi} + \underset{\substack{D\\R}}{\varphi}\right) = \left(\varphi; \underset{\substack{R\\D}}{\varphi}\right) + \left(\varphi; \underset{\substack{D\\R}}{\varphi}\right) =$$
$$= -\frac{1}{p+1}\left(\underset{R}{\varphi}; \underset{R}{\varphi}\right) - p\left(\underset{D}{\varphi}; \underset{D}{\varphi}\right) = 0$$

from which there follows immediately

$$\left(\underset{R}{\varphi}; \underset{R}{\varphi}\right) = 0 \qquad \text{i.e.} \qquad \underset{R}{\varphi} = 0$$

and

$$\left(\underset{D}{\varphi}; \underset{D}{\varphi}\right) = 0 \qquad \text{i.e.} \qquad \underset{D}{\varphi} = 0$$

In this manner we have proved:

T h e o r e m 15 (De Rahm's and Kodaire's [9]).

In a compact orientable Riemannian manifold V_n with a positive definite metric, the r/p tensor field is a harmonic r/p-tensor field, if and only if it satisfies the identities:

$$\underset{\Delta}{f} = \underset{\substack{R\\D}}{f} + \underset{\substack{D\\R}}{f} = 0$$

Let $\bar{f}_{i_1\ldots i_p l_1\ldots l_r}(x)$ be a harmonic field of r/p-tensors, satisfying the identities

$$\bar{f}_{i_1\ldots i_p l_1\ldots l_r}(x) = p\,\nabla_{[i_1}\bar{h}_{i_2\ldots i_p]l_1\ldots l_r} \qquad (94)$$

i.e. according to the denotations of (87) and (88)

$$\bar{f} = \underset{R}{\bar{h}}$$

From the definition of the harmonic field of the r/p-tensors as well as from the assumptions (94),(87) and (88) there follows

$$\underset{R}{\bar{f}} = \Big(\underset{\substack{R\\R}}{\bar{h}}\Big) = 0 \qquad \text{and} \qquad \underset{D}{\bar{f}} = \Big(\underset{\substack{R\\D}}{\bar{h}}\Big) = 0 \qquad (95)$$

Putting $f=\bar{h}$ and $h=\underset{R}{\bar{h}}$ into (90) we get

$$\Big(\underset{R}{\bar{h}};\underset{R}{\bar{h}}\Big) + p\Big(\bar{h};\ \underset{\substack{R\\D}}{\bar{h}}\Big) = 0$$

and by virtue of (95) we have

$$\underset{R}{\bar{h}} = \bar{f} = 0$$

Thus we have proved that if the harmonic field $f_{i_1\ldots i_p l_1\ldots l_r}(x)$ of the r/p-tensors is linked by (94) with the field

$h_{i_2 \dots i_p l_1 \dots l_r}(x)$ of the r/(p-1)-tensors, it is equal to zero.

It is known from [7] that if a V_n admits a one - parametre group of motions generated by an infinitesimal transformation

$$'x^i = x^i + v^i(x)\,dt$$

the Lie derivative of the metric tensor with respect to this motion disappears in V_n, i.e. $\underset{v}{\mathcal{L}} g_{ij}=0$ and consequently the operator of the covariant differentiation ∇_j, in this case is interchangeable with the operator $\underset{v}{\mathcal{L}}$ of the Lie derivative.

Let us assume now that there is a harmonic field of the r/p tensors $f_{i_1 \dots i_p l_1 \dots l_r}(x)$ in a V_n.

Then from the definition we have the identities

$$\nabla_{[j} f_{i_1 \dots i_p] l_1 \dots l_r}(x) = 0 \tag{96}$$

and

$$g^{jk} \nabla_j f_{k i_2 \dots i_p l_1 \dots l_r}(x) = 0 \tag{97}$$

By virtue of the assumption, that the Lie derivative of the metric tensor g_{ij} turns into zero and that the differentiation operators $\nabla_j ; \underset{v}{\mathcal{L}}$ are interchangeable, from (96) and (97) there follows

$$\nabla_{[j} \underset{v}{\mathcal{L}} f_{i_1 \dots i_p] l_1 \dots l_r}(x) = 0$$

and

$$g^{jk} \nabla_j \underset{v}{\mathcal{L}} f_{k i_2 \dots i_p l_1 \dots l_r}(x) = 0$$

Hence: If the Riemannian space V_n admits infinitesimal motions generated by an infinitesimal transformation $'x^i = x^i + v^i(x)dt$ the Lie derivative $\underset{v}{\mathcal{L}}$ with respect to v of the harmonic field of the r/p-tensors is also the harmonic field of the r/p-tensors.

From the definition of the harmonic fields of the r/p-tensors as well as from the definition of the dual-tensors with respect to the r/p tensors and from (31) it follows immediately that: if the r/p-tensor field is a harmonic field of the r/p-tensors, its dual is also a harmonic field of the r/(n-p)-tensors. Further it follows that: The $h_{i_1\ldots i_p l_1\ldots l_r}(x)$ tensor - field in V_{2p} is a harmonic field of the r/p-tensors if and only if the self-dual tensor-fields in decomposition (36) are harmonic fields of the r/p tensors.

Let us also assume that the space under our consideration is a 2p-dimensional Riemannian space ($p \geqslant 2$).

Consider the form $F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right)$ as defined in the identity (72), i.e.

$$\begin{aligned} F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) &= R_{ij}\, f^{i i_2\ldots i_p l_1\ldots l_r} f^{j}{}_{i_2\ldots i_p l_1\ldots l_r} + \\ &- \frac{p-1}{p} R_{ijkl}\, f^{ij i_3\ldots i_p l_1\ldots l_r} f^{kl}{}_{i_3\ldots i_p l_1\ldots l_r} + \\ &- \sum_{t=1}^{r} R^{k}{}_{aij}\, f^{i i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} f^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} \end{aligned} \tag{98}$$

Using the decomposition (36) we have

$$\begin{aligned} R^{k}{}_{aij}\, f^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r} f^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} &= \\ = R^{k}{}_{aij}\left(\underset{1}{f}{}^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r} + \underset{2}{f}{}^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r}\right). \end{aligned} \tag{99}$$

$$\cdot\left(\underset{1}{f}{}^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} + \underset{2}{f}{}^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r}\right) =$$

$$= R^{k}{}_{aij}\, \underset{1}{f}{}^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r}\, \underset{1}{f}{}^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} +$$

$$+ 2R^{k}{}_{aij}\, \underset{1}{f}{}^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r}\, \underset{2}{f}{}^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r} +$$

$$+ R^{k}{}_{aij}\, \underset{2}{f}{}^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r}\, \underset{2}{f}{}^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} k l_{t+1}\ldots l_r}$$

where $\underset{1}{f}$ and $\underset{2}{f}$ are self-dual r/p -tensors of the 1-st and of the 2-nd kind respectively.

Using the theorems 3 and 5 as well as identities $R^{k}{}_{a[ij]} = R^{k}{}_{aij}$ it is easy to show that

$$R^{k}{}_{aij}\, f^{i i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r}\, f^{j}{}_{i_2\ldots i_p l_1\ldots l_{t-1} a l_{t+1}\ldots l_r} = 0$$

Thus, we have

C o r o l l a r y 10. The form $F(f_{i_1\ldots i_p l_1\ldots l_r})$ - defined by the identity (72) - in a 2p-dimensional Riemannian space assumes the form:

$$F\left(f_{i_1\ldots i_p l_1\ldots l_r}\right) = R_{ij}\, f^{i i_2\ldots i_p l_1\ldots l_r}\, f^{j}{}_{i_2\ldots i_p l_1\ldots l_r} + \frac{p-1}{2} R_{ijk}\, f^{i j i_3\ldots i_p l_1\ldots l_r}\, f^{ik}{}_{i_3\ldots i_r l_1\ldots l_r} \tag{100}$$

Now, using corollaries 9 and 10 as well as the theorem 3.4. from [6] it is easy to prove:

C o r o l l a r y 11. In a compact 4p-dimensional Riemannian space V_{4p} with a positive definite metric there are no harmonic fields of p-vectors if and only if there do not exist harmonic fields of the r/p-tensors in this space.

Let us take now into consideration an arbitrary harmonic field of the p/p-tensors $a_{i_1 \dots i_p j_1 \dots j_p}(x)$ which are bi-tensor field at the same time.

Thus from the assumption this field satisfies the identites

$$a_{[i_1 \dots i_p] j_1 \dots j_p}(x) = a_{i_1 \dots i_p j_1 \dots j_p}(x)$$

as well as

$$a_{i_1 \dots i_p j_1 \dots j_p}(x) = a_{j_1 \dots j_p i_1 \dots i_p}(x)$$

as the bi-tensor field, and

$$a_{i_1 \dots i_p [j_1 \dots j_p ; k]}(x) = 0$$

as well as

$$a^{k}{}_{i_2 \dots i_p j_1 \dots j_p ; k}(x) = 0$$

as a harmonic field of the p/p -tensors.

It is easy to note that having a given bi-tensor field $a_{i_1 \dots i_p j_1 \dots j_p}(x)$ it is possible to form a new bi-tensor field according to the formula

$$\bar{a}_{i_2 \dots i_p l_2 \dots l_p} \overset{df}{=} g^{ij} a_{i i_2 \dots i_p j j_2 \dots j_p} \tag{101}$$

We shall prove.

T h e o r e m 16. If a bi-tensor field $a_{i_1\dots i_p j_1\dots j_p}(x)$ is a harmonic field of the p/p bi-tensors, the field of the bi-tensors $\bar{a}_{i_2\dots i_p j_2\dots j_p}(x)$ defined by formula (101) is also a harmonic field of bi-tensors.

Proof. Let us assume that $a_{i_1\dots i_p j_1\dots j_p}(x)$ is a harmonic field of the r/p tensors, then

$$a_{i_1\dots i_p l_1\dots l_p;k} - a_{i_1\dots i_p k l_2\dots l_p;l_1} - \dots - a_{i_1\dots i_p l_1\dots l_{p-1} k;i_p} = 0 \qquad (102)$$

as well as

$$a^{k}{}_{i_2\dots i_p l_1\dots l_p;k} = 0 \qquad (103)$$

Multiplying the identities (102) by $g^{i_1 l_1}$ and summing for $i_1 l_1$ and taking into account (101) and (103) we get

$$\bar{a}_{i_2\dots i_p[l_2\dots l_p;k]} = 0$$

From the properties of the bi-tensor fields and from (103) it follows that

$$\bar{a}^{k}{}_{i_3\dots i_p l_2\dots l_p;k} = 0$$

Consequently a field of the bi-tensors $\bar{a}_{i_2\dots i_p j_2\dots j_p}$ is harmonic field of the p/p -tensors.

We prove analogically

T h e o r e m 17. A bi-tensor field $a_{i_1\dots i_p j_1\dots j_p}(x)$ satisfying the identities

$$a_{[i_1\dots i_p|l_1\dots l_p|;k]} = 0$$

is a harmonic field of the p/p -tensors, if and only if the bi-tensor field

$$\bar{a}_{i_2\dots i_p l_2\dots l_p} \overset{df}{=} g^{ij} a_{ii_2\dots i_p jj_2\dots j_p}$$

is a harmonic field of the (p-1)/(p-1)-tensors.

Let the bi-tensor a_{ijkl} be a harmonic field of 2/2 tensors. Then we have

$$a_{ijkl;\alpha} + a_{ij\alpha k;l} + a_{ijl\alpha;k} = 0 \tag{104}$$

as well as

$$a^{\alpha}{}_{jkl;\alpha} = 0 \tag{105}$$

Multiplying (104) by $g^{il}g^{jk}$ and summing for $ijkl$ and taking into consideration (105) we get

$$a_{;\alpha} = 0$$

where

$$a \overset{df}{=} a_{ijkl}\, g^{il} g^{jk}$$

Thus we have:

C o r o l a r y 11'. If the bi-tensor field a_{ijkl} is a harmonic field of 2/2 tensors, then the scalar field

$$a(x) \overset{df}{=} g^{il} g^{jk} a_{ijkl}(x)$$

is a constant field.

§5. APPLICATION AND FINAL CONCLUSIONS

In his paper [13], W.Wrona has proved i.a. that the identities

$$ {}^{\circ}\overset{p}{U}_{i_1\ldots i_p j_1\ldots j_p} = \theta \overset{p}{U}_{i_1\ldots i_p j_1\ldots j_p} \tag{106}$$

where $\theta=+1$ if $p=2k$ or $\theta=-1$ if $p=2k+1$ (or where $\theta=-1$ if $p=2k$ or $\theta=+1$ if $p=2k+1$), express the nacessary and sufficient conditions that the Riemannian space V_{2p} should be the Einstein space (and conformal - Euclidean space respectively).

From (106) it may be concluded [15] that the identities

$$ {}^{\circ}\overset{p}{R}_{i_1\ldots i_p j_1\ldots j_p} = \theta \overset{p}{R}_{i_1\ldots i_p j_1\ldots j_p} \tag{107}$$

where $\theta=+1$ if $p=2k$ or $\theta=-1$ if $p=2k+1$, also satisfy the necessary and sufficient conditions that the Riemannian space V_{2p} should be the Einstein space.

The expanded curvature tensor $\overset{p}{R}_{i_1\ldots i_p j_1\ldots j_p}$ and the expanded deviation tensor $\overset{p}{U}_{i_1\ldots i_p j_1\ldots j_p}$ in the Einstein space V_{2p} satisfy the identities

$$ \overset{p}{R}_{i_1\ldots i_p[l_1\ldots l_p;k]} = 0 \tag{108}$$

as well as

$$ \overset{p}{U}_{i_1\ldots i_p[l_1\ldots l_p;k]} = 0 \tag{109}$$

From (48), (106), (107), (108) and (109) it follows that the expanded curvature tensor in the Einstein space V_{2p} and the expanded deviation tensor, also satisfy the identities

$$\overset{p}{R}{}^{k}{}_{i_2 \dots i_p l_1 \dots l_p ; k} = 0 \tag{110}$$

as well as

$$\overset{p}{U}{}^{k}{}_{i_2 \dots i_p l_1 \dots l_p ; k} = 0 \tag{111}$$

From definition 1 and the identities (108), (109), (110) and (111) there is

C o r o l l a r y 12. A field of the expanded curvature tensor $\overset{p}{R}_{i_1 \dots i_p j_1 \dots j_p}$ as well as a field of the expanded deviation tensor $\overset{p}{U}_{i_1 \dots i_p j_1 \dots j_p}$ in the Einstein space V_{2p} are harmonic fields of the p/p -tensors.

From theorem 16 and corollary 12 we have

C o r o l l a r y 13. The curvature tensor R_{ijkl} and the Ricci tensor R_{ij} in the Einstein space V_{2p} are harmonic p/p-tensors.

It does not follow from the fact that the curvature tensor of the Riemannian space V_{2p} is a harmonic 2/2 tensor that it is also the Einstein space.

It is easy to note that, if the curvature tensor R_{ijkl} in V_4 satisfies the condition

$$R_{ijkl} = {}^{0}R_{ijkl} + T_{ijkl} \tag{112}$$

where $T_{ijkl;\alpha}=0$ the V_4 ceases to be the Einstein space (cf. (107)) however the curvature tensor R_{ijkl} continues be a harmonic p/p-tensor. Thus, the Riemannian space V_n whose curvature tensor R_{ijkl} is a harmonic 2/2 tensor, is a more general space than the Einstein space.

From corollary 11'' it follows

C o r o l l a r y 14. The Riemannian space whose curvature tensor R_{ijkl} is a harmonic 2/2 tensor has a constant scalar curvature.

From (106) and (107) there follow immediately the identities:

$$\overset{o\,p}{U}_{i_1 \ldots i_p j_1 \ldots j_p ; k} = \theta \overset{p}{U}_{i_1 \ldots i_p j_1 \ldots j_p ; k} \tag{113}$$

as well as

$$\overset{o\,p}{R}_{i_1 \ldots i_p j_1 \ldots j_p ; k} = \theta \overset{p}{R}_{i_1 \ldots i_p j_1 \ldots j_p ; k} \tag{114}$$

from which by virtue of the definition of the r/p×p -tensors (cf.(49)) and from (106) and (107) we get

C o r o l l a r y 15. The covariant derivative of the expanded curvature tensor and the expanded deviation tensor in a 2p-dimensional Einstein space are the self-dual fields of r/p×p -tensors of the 1-st kind if $p=2k$ and of the 2-nd kind, if $p=2k+1$

C o r o l l a r y 16. The covariant derivative of the expanded field of a deviation tensor in a 2p-dimensional conformal - Eucliden space is a self-dual field of the r/p×p-tensor of the 2-nd kind, if $p=2k$ and of the 1-st kind, if $p=2k+1$

It is also evident that from (106) and (107) there follow the identities (113) and (114) but in general (as a rule) there do not follow inverse conclusions (cf. (112)). Thus the Riemannian space V_{2p} in which the curvature tensor $\overset{p}{R}_{i_1 \ldots i_p j_1 \ldots j_p}$ satisfies the conditions (114) if $\theta=+1$ and $p=2k$ or if $\theta=-1$ and $p=2k+1$ is as a more general space than the Einstein space V_{2p}.

Similary, the Riemannian space V_{2p} in which the expanded deviation tensor $\overset{p}{U}_{i_1 \ldots i_p j_1 \ldots j_p}$ satisfies the identity (113) if $\theta=-1$ and $p=2k$ of if $\theta=+1$ and $p=2k+1$ is a more general space that the conformal - Euclidean space.

Thus, let us assume the following definitions:

D e f i n i t i o n 3. The Riemannian space V_{2p} in which the curvature tensor $\overset{p}{R}_{i_1 \ldots i_p j_1 \ldots j_p}$ satisfies the identities

$$\overset{o}{\overset{p}{R}}_{i_1\dots i_p j_1\dots j_p ; k} = \theta \overset{p}{R}_{i_1\dots i_p j_1\dots j_p ; k}$$

where $\theta=+1$ and $p=2k$ and where $\theta=-1$ and $p=2k+1$ is called a generalized Einstein space V_{2p}

D e f i n i t i o n. 4. The Riemannian space V_{2p} in which the expanded deviation tensor $\overset{p}{U}_{i_1\dots i_p j_1\dots j_p}$ satisfies the identities

$$\overset{o}{\overset{p}{U}}_{i_1\dots i_p j_1\dots j_p ; k} = \theta \overset{p}{U}_{i_1\dots i_p j_1\dots j_p ; k}$$

where $\theta=-1$ if $p=2k$ or $\theta=+1$ if $p=2k+1$ is called a generalized conformal - Euclidean space.

To conclude, let us pay attention to the fact that all the theorems for self-dual $r/p^x p$ -tensors proved in section 3 and applied to tensors $\overset{p}{R}_{i_1\dots i_p j_1\dots j_p}$ $\left(\overset{p}{U}_{i_1\dots i_p j_1\dots j_p}\right)$ enable us to state whether the given Riemannian space V_{2p} is a generalized Einstein space, or the conformal - Euclidean space, or wheter it is neither of the two.

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O PEWNYM UOGÓLNIENIU POJĘCIA SAMO-DUALNEGO TENSORA, TENSORA HARMONICZNEGO I TENSORA KILLINGA W V_n

S t r e s z c z e n i e

Niniejsza praca składa się z pięciu paragrafów związanych ze sobą tematycznie.

W § 1-szym autor przypomina znane pojęcia i fakty z geometrii riemannowskiej.

W § 2-gim i 3-cim autor uogólnia znane pojęcie dualnego i samo-dualnego multiwektora, oraz bi-tensora na szerszą klasę pól tensorowych, a następnie bada pewne własności tych pól.

W § 4-ym autor uogólnia pojęcie pola harmonicznego p-wektorów i pola p-wektorów Killinga na klasę pól tensorowych rozważanych w §-fach 2-gim i 3-cim, podając również szereg własności tych pól tensorowych.

W ostatnim paragrafie autor podaje próbę stosowania wprowadzonych pojęć do badania pewnych własności specjalnych przestrzeni Riemanna.

Praca stanowi pewne uogólnienie rozważań zawartych w pracach: J. Haantjes i W. Wrona [4], W. Wrona [13], K. Yano i S. Bochner [6], R.Ch. Srivastava [12] oraz we wcześniejszych pracach autora [15]; [16]; [17].

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Adress of author: dr Zbigniew Żekanowski, Warszawa, ul. Marszałkowska 140 m 86