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How To Make Pure Mathematical Ideas Clear

Abstract: Peirce expressed his pragmatic maxim in the 1870s. If, as Peirce maintained, this original definition is a maxim of logic, it is mainly a maxim of the logic of science, as the title "Illustrations of the Logic of Science" indicates. Pure mathematical conceptions, and the logic of mathematics, if not totally excluded, have at least not been emphasized. During his years at Johns Hopkins University, pure mathematics became his subject of most concern, while logic was also conceived as semiotics during this time. So around the turn of the century, when the popular movement of pragmatism began with James' "Berkeley Address", Peirce found that the main difficulty with his original definition of the pragmatic maxim was how to make pure mathematical conceptions clear. He mentioned this problem repeatedly but only gave a tentative solution admitting that, at least according to his original definition, some meanings of pure mathematical conceptions cannot be clarified. This, I believe, is the most important reason for Peirce's renaming and redefining the pragmatic maxim in semiotic terms. If other pragmatists, and scholars of pragmatism, had noticed this, then most criticisms of pragmatism could have been avoided and the history of pragmatism may have taken a different direction.

Keywords: incommensurables; infinitesimals; linguistic expression; semiotics; pragmatism; synechism

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1 Introduction

In 1901 Peirce wrote the entry in Baldwin's *Dictionary of Philosophy and Psychology* for "Pragmatic and Pragmatism" in which he found some difficulties in his original expression of the pragmatic maxim:

> The writer subsequently saw that the principle might easily be misapplied, so as to sweep away the whole doctrine of incommensurables, and, in fact, the whole Weierstrassian way of regarding the calculus. In 1896 William James published his *Will to Believe*, and later his *Philosophical Conceptions and Practical Results*, which pushed this method to such extremes as must tend to give us pause. The doctrine appears to assume that the end of

> man is action – a stoical axiom that, to the present writer at the age of sixty, does not recommend itself so forcibly as it did at thirty. If it be admitted, on the contrary, that action wants an end, and that that end must be something of a general description, then the spirit of the maxim itself, which is that we must look to the upshot of our concepts in order right to apprehend them, would direct us towards something different from practical facts, namely, to general ideas, as the true interpreters of our thought. Nevertheless, the maxim has approved itself to the writer, after many years of trial, as of great utility in leading to a relatively high grade of clearness of thought. He would venture to suggest that it should always be put into practice with conscientious thoroughness, but that, when that has been done, and not before, a still higher grade of clearness of thought can be attained by remembering that the only ultimate good which the practical facts to which it directs attention can subserve is to further the development of concrete reasonableness; so that the meaning of the concept does not lie in any individual reactions at all, but in the manner in which those reactions contribute to that development. Indeed, in the article of 1878 above referred to, the writer practiced better than he preached; for he applied the stoical maxim most unstoically, in such a sense as to insist upon the reality of the objects of general ideas in their generality. (CP 5.3)

We should notice that Peirce mentioned here that he had found two difficulties in his original expression: the first one concerns the ascertaining or clarifying of pure mathematical conceptions, e.g., the doctrine of incommensurables, or what are called “irrationals”, etc. If the statement of this difficulty were changed into the active mood, it would read, “We might easily misapply the principle, so as to sweep away the whole doctrine of incommensurables, and, in fact, the whole Weierstrassian way of regarding the calculus”. What Peirce meant was that we must avoid such a misapplication, and employ the pragmatic maxim to clarify these mathematical conceptions.

We should also notice that this difficulty is so serious that it was found by Peirce himself – upon his self-reflection, we might say – during the progress of his investigation, unlike the second one, i.e., it is a maxim of behaviorism, which was found upon James’ push.

The second difficulty, as everybody knows, has already been paid enough attention by Peirce scholars, although Peirce wrote the following passage in 1906:

> ... I myself expressed (Baldwin's Dictionary Article, Pragmatism) after a too hasty rereading of the forgotten magazine paper, that it expressed a stoic, that is, a nominalistic,

> materialistic, and utterly philistine state of thought, was quite mistaken ("Consequences of Pragmaticism" 1906.[1] (CP 5.402n3)

As to the first difficulty, limited as I am in my knowledge of Peircean scholarship, I have not yet seen it draw any attention. I suspect the reason for this neglect is that this one seems, in the eyes of Peirce scholars, only a little particular, even just an individual problem, compared with the second problem since Peirce only mentioned it in passing. But I must say, this view is totally wrong.

First of all, it is far from the case that Peirce talked of this problem only this one time. He talked of it several times in his 1903 Harvard Lectures on Pragmatism. And we might infer that if it were not for James's advice, he would have addressed it even more often.[2] Also on September 10, 1906, he wrote to Schiller:

> Let me rather note that some of the ends which you mention as going to the meanings of concepts seem to me to form no part of those meanings. What the hundredth decimal figure of π means consists to my mind in just what any other figure means. For it is quite conceivable that it should be an important practical quantity. It is one of the beauties of pragmatism that it gives some symbols much more meaning than others, and the hundredth figure of π certainly has precious little. A much better question is, What on pragmatist principles is the difference between a rational and an irrational quantity or what it means to say that the diagonal of a square is incommensurable with its side? It is interpreted in the conduct of the arithmetician as such. (CP 8.323)

Secondly, according to Peirce's classification of sciences from 1903 onward, pure mathematics is placed above all of the other sciences and provides a foundation for them. Now, if the pragmatic maxim cannot be applied to pure mathematical concepts, then the meanings of concepts in all the other sciences would lose their foundation for clarity.

Thirdly, the incommensurables form the bulk of the continuum, which is the heart of Peirce's mature philosophy of synechism. And synechism is the synthesis of tychism and pragmati(ci)sm.

Fourthly, if it cannot be applied to mathematical concepts, the proof of pragmatism in strict sense would be inconceivable. But to provide such a proof for pragmatism is a very important task for Peirce's later philosophy.

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1 Of course this does not mean that Peirce now turned to approve stoicism, but only that he said that his previous interpretation (that is, the second difficulty mentioned in that dictionary definition) is incorrect.

2 See Turrisi 1997: 6 – 9.

Fifthly, the original expression of pragmatism was located in the context of scientific logic; until then Peirce had not yet paid much attention to mathematical logic. But the situation had changed greatly by 1879–1884.

Lastly, the rise of the second difficulty is at least partly due to the neglect of mathematics, which is typical of the less mature sciences.

2 Appeal to Linguistic Expression

The problem is how to ascertain or clarify the meanings of pure mathematical concepts. How do we apply the pragmatic maxim to them if we want to avoid this misapplication?

A very simple way of applying the pragmatic maxim to incommensurables is to resort to linguistic expressions. This does not seem so absurd nowadays – in the 1950s, Wittgenstein，Austin and Searle showed that saying is doing. A little later, the French structuralists believed all behavior is language. The difference between practice and language seems disappeared - as it did a century earlier, but Peirce had the brilliance to try it anyway: "Of course you can say if you like that the act of expressing a quantity as a rational fraction is a piece of conduct and that it is in itself a practical difference that one kind of quantity can be so expressed and the other not" (CP 5.33). But Peirce put aside this way immediately: "But a thinker must be shallow indeed if he does not see that to admit a species of practicality that consists in one's conduct about words and modes of expression is at once to break down all the bars against the nonsense that pragmatism is designed to exclude" (CP 5.33).

It does not seem very clear from Peirce's 1878 original expression of his maxim why "pragmatism is designed to exclude" this way of reasoning, but if we remember the following passage, we will see Peirce is quite right by saying so:

> A better rule for avoiding the deceits of language is this: Do things fulfill the same function practically? Then let them be signified by the same word. Do they not? Then let them be distinguished. If I have learned a formula in gibberish which in any way jogs my memory so as to enable me in each single case to act as though I had a general idea, what possible utility is there in distinguishing between such a gibberish formula and an idea?

> Why use the term a general idea in such a sense as to separate things which, for all experiential purposes, are the same (CP 8.33)?[3]

Thus Peirce had already established a principle of meaning that was later adopted by the logical positivists: the same word signifies the same meaning while different meanings are signified by different words.

The justification for this is really quite involved. Peirce claims: "What the pragmatist has his pragmatism for is to be able to say: here is a definition and it does not differ at all from your confusedly apprehended conception because there is no practical difference" (CP 5.33). It appears linguistic expressions are quite fitted for this job. But Peirce then asks, "...what is to prevent his opponent from replying that there is a practical difference which consists in his recognizing one as his conception and not the other? That is, one is expressible in a way in which the other is not expressible" (CP 5.33). However, he then answers his own question: "Pragmatism is completely volatilized if you admit that sort of practicality" (CP 5.33).

This, we may comment, is the difference between the spirits of scientists and artists. From the very first day of his research to the end of his life, Peirce maintained the scientific spirit throughout.

For Peirce, resorting to linguistic expressions is not only not a feasible way of application, but a typical misapplication. Peirce himself made such a mistake in his 1878 article, concerning the hardness or softness of a diamond:

> Let us illustrate this rule by some examples; and, to begin with the simplest one possible, let us ask what we mean by calling a thing hard. Evidently that it will not be scratched by many other substances. The whole conception of this quality, as of every other, lies in its conceived effects. There is absolutely no difference between a hard thing and a soft thing so long as they are not brought to the test. Suppose, then, that a diamond could be crystallized in the midst of a cushion of soft cotton, and should remain there until it was finally burned up. Would it be false to say that that diamond was soft? This seems a foolish question, and would be so, in fact, except in the realm of logic. There such questions are often of the greatest utility as serving to bring logical principles into sharper relief than real discussions ever could. In studying logic we must not put them aside with hasty answers, but must consider them with attentive care, in order to make out the principles involved. We may, in the present case, modify our question, and ask what prevents us from saying that all hard bodies remain perfectly soft until they are touched, when their hardness increases with the pressure until they are scratched. Reflection will show that the reply is this: there would be no falsity in such modes of speech. They would involve a modification of our present usage of speech with regard to the words hard and

3 Arthur Burks added a footnote here: "This is an early anticipation of Peirce's pragmatism,....."

> soft, but not of their meanings. For they represent no fact to be different from what it is; only they involve arrangements of facts which would be exceedingly maladroit. This leads us to remark that the question of what would occur under circumstances which do not actually arise is not a question of fact, but only of the most perspicuous arrangement of them. (CP 5.403)

Peirce's dilemma here is that according to his original pragmatic maxim, if things had no practical, experiential difference, then we should use one word to signify them. And since the two qualities of hardness and softness of a diamond, if not being pressed, make no experiential difference, then we should signify them by one word. We may choose hard or soft, but not both. In addition to the difficulty of the original expression of pragmatism, there was a more fundamental mistake in Peirce's philosophy, which was that Peirce, in Max Fisch's words, was then a one-categorical realist. He maintained the reality of thirdness (law), but not that of firstness (qualities), nor that of secondness (reactions). By 1885, Peirce became a two-categorical realist. He did not accept the reality of firstness until 1897.[4] As we will see below, pure mathematical concepts belong essentially to the category of firstness.

In 1906, when Peirce talked again of this matter, his understanding of realism had grown. He now saw the necessity for the reality of qualities.

> Let us now take up the case of that diamond which, having been crystallized upon a cushion of jeweler's cotton, was accidentally consumed by fire before the crystal of corundum that had been sent for had had time to arrive, and indeed without being subjected to any other pressure than that of the atmosphere and its own weight. The question is, was that diamond ***really*** hard? It is certain that no discernible ***actual*** fact determined it to be so. But is its hardness not, nevertheless, a ***real*** fact? To say, as the article of January 1878 seems to intend, that it is just as an arbitrary "usage of speech" chooses to arrange its thoughts, is as much as to decide against the reality of the property, since the real is that which is such as it is regardless of how it is, at any time, thought to be. Remember that this diamond's condition is not an isolated fact. There is no such thing; and an isolated fact could hardly be real. It is an unsevered, though precise part of the unitary fact of nature. Being a diamond, it was a mass of pure carbon, in the form of a more or less transparent crystal (brittle, and of facile octahedral cleavage, unless it was of an unheard-of variety), which, if not trimmed after one of the fashions in which diamonds may be trimmed, took the shape of an octahedron, apparently regular (I need not go into minutiæ), with grooved edges, and probably with some curved faces. Without being subjected to any considerable pressure, it could be found to be insoluble, very highly refractive, showing under radium rays (and perhaps under "dark light" and X-rays) a peculiar bluish phosphorescence, having as high a specific gravity as realgar or orpiment, and giving off during its combustion less heat than any other form of carbon would have

4 See Fisch 1986: 184–200.

> done. From some of these properties hardness is believed to be inseparable. For like it they bespeak the high polemerization (sic-sp) of the molecule. But however this may be, how can the hardness of all other diamonds fail to bespeak ***some*** real relation among the diamonds without which a piece of carbon would not be a diamond? Is it not a monstrous perversion of the word and concept ***real*** to say that the accident of the non-arrival of the corundum prevented the hardness of the diamond from having the ***reality*** which it otherwise, with little doubt, would have had? (CP 5.457)

But until 1906, other pragmatists, F.C.S. Schiller for example, still have such misapplication, and Peirce's attitude about this was:

> When you say that Logical consequences cannot be separated from psychological effects, etc. in my opinion you are merely adopting a mode of expression highly inconvenient which cannot help, but can only confuse, any sound argumentation. It is a part of nominalism which is utterly antipragmatistic, as I think, and mere refusal to make use of valuable forms of thought. (CP 8.326)

3 Peirce's Solution

It must be noticed that Peirce did not want to give a mathematical definition of incommensurables; such definitions had been given by Cantor and Dedekind and other mathematicians. What Peirce wanted to do was to apply the pragmatic maxim to such pure mathematic concepts, that is, to ascertain, or clarify, their meanings, - to reach a still higher degree of clearness than abstract definition, - from a pragmatic point of view.

> What possible effect upon conduct can it have, for example, to believe that the diagonal of a square is incommensurable with the side? ... The proposition that the diagonal is incommensurable has stood in the textbooks from time immemorial without ever being assailed and I am sure that the most modern type of mathematician holds to it most decidedly. Yet it seems quite absurd to say that there is any objective practical difference between commensurable and incommensurable? (CP 5.32)

Peirce's premise for his solution is Weierstrass's definition of equality: "Name a discrepancy e, no matter how small, and the diagonal differs from a rational quantity by much less than that" (CP 5.32).[5]

In his seventh lecture on pragmatism in 1903, Peirce gave us the following solution:

5 This premise was also the premise of Cantor and Dedekind's definition. See Klein 1945: 31–36.

> But now let us look at something else that those logicians would be obliged to admit. Namely, suppose any regular polygon to have all its vertices joined by straight radii to its centre. Then if there were any particular finite number of sides for a regular polygon with radii so drawn, which had the singular property that it should be impossible to bisect all the angles by new radii equal to the others and by connecting the extremities of each new radius to those of the two adjacent old radii to make a new polygon of double the number of angles – if, I say, there were any finite number of sides for which this could not be done – it may be admitted that we should be able to find it out. The question I am asking supposes arbitrarily that they admit that. Therefore these logicians of the third class would have to admit that all such polygons could so have their sides doubled and that consequently there would be a polygon of an infinite multitude of sides which could be, on their principles, nothing else than the circle. But it is easily proved that the perimeter of that polygon, that is, the circumference of the circle, would be incommensurable, so that an incommensurable measure is real, and thence it easily follows that all such lengths are real or possible. But these exceed in multitude the only multitude those logicians admit. Without any geometry, the same result could be reached, supposing only that we have an indefinitely bisectible quantity. (CP 5.203)

See Figure 1.

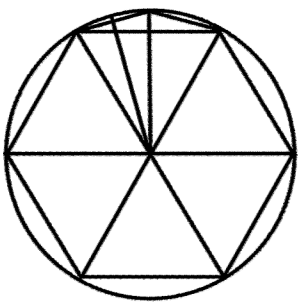

Figure 1: Infinite bisecting a regular polygon approximates a circle

Such is the conceivable effects of $l/2r = \pi$, which is an incommensurable and has practical bearings. This exactly agrees with the original pragmatic maxim. And, compared with the first grade of clearness (familiarity) and second grade of clearness (distinctiveness, abstract definition), do we not reach a higher grade of clear understanding of the incommensurables?

We may say that Peirce then successfully resolved the first difficulty. But this leads to even more difficult problems.

4 More Difficulties

Peirce's solution has at least two other premises:

The first premise is, the bisecting of the regular polygons should be continued infinitely until the sides, or angles of them, become infinitesimal. In short, this is the theory of infinitesimals.

The second premise is that the last regular polygon must be continuous with the circle. In short, this is the theory of continuity or the continuum. As we know, this is the heart of Peirce's mature thought.

Unfortunately, both the theory of infinitesimals and the doctrine of continuum were then yet to be accepted.

Now the question is, can we apply the pragmatic maxim to the concepts of infinitesimals and continuity?

In 1906, Peirce wrote to Schiller:

> As for Cantor's cardinal transfinites, though called numbers by him, they are not properly so called but are ***multitudes***, or many-nesses of infinite collections. The first is the multitude of the objects of an endless series of objects. I call it the ***denumeral*** multitude. The next is the multitude of all collections of objects involved in an endless series (by collection I mean simply a plural). The rest are each the multitude of all collections involved in a collection of the next preceding multitude. I call these the ***abnumerable*** multitudes. (CP 8.324)

Peirce then confessed:

> These abnumerable multitudes are describable intelligibly and exactly, but only in general terms. No precise idea can be formed of the simplest of them; and they increase in difficulty at a frightful rate (that is in the characters that ***would*** make difficulty if it were surmountable at all). *If anything violates the principle of pragmatism it is these. But I have no doubt whatever of the validity of the concepts.* They are interpretable in the conduct of the logician or logico-mathematician in dealing with them. If they were not exact, so as [to] lay definite logical obligations upon him they would be meaningless, or without definite meaning.[6] (CP 8.325)

If this really is the case, then, at least as far as the original definition of the pragmatic maxim is concerned, this is its limit, beyond which it will lose its validity.

6 The italics are mine.

5 Influences upon the Theory and Practice of Pragmatism

Peirce's application of the pragmatic maxim to the clarification of incommensurables and related pure mathematical concepts has profound effects upon the development of pragmatism, both theoretically and practically.

First, it provided another typical case for Peirce's proof of pragmatism. In the 1878 article, "How to Make our Ideas Clear", Peirce used four examples to illustrate his original maxim. Then in another article in the series *Illustrations of the Logic of Science*, "The Doctrine of Chances", Peirce applied the pragmatic maxim to the concept of probability. And in 1897, Peirce applied the maxim to the concept of relatives.[7] There was only one area in the theoretical sciences left which waited for the reach of the pragmatic maxim. And this area was the most important of all the sciences. It was the majesty of mathematical logic, or pure mathematics. The incommensurables are obviously the hardest concepts in this area. How can we make pure mathematical ideas clear? In the history of mathematics, the incommensurables were familiar for more than two thousand years. But their abstract definition had just been given by Dedekind and Cantor a few years previously. Peirce knew very well how many mathematicians clearly understood these definitions, and if understood, to what degree. Peirce's intention was to employ his pragmatic maxim to give a higher degree of clearness, and at the same time, apply the maxim to a brand new and most important area. This certainly provided another example for the illustration of pragmatism.

Secondly, the application of the pragmatic maxim to pure mathematical concepts could clarify the significance of Peirce's statement that pragmatism is "a maxim of logic". The 1878 article, in which Peirce first proposed his pragmatic maxim, is the second item in the series *Illustrations of the Logic of Science*. From this title, we may infer reasonably that what Peirce meant by "a maxim of logic" is just a maxim of scientific logic, mathematical logic being excluded. Now Peirce was applying this maxim to mathematical concepts, and at the same time, he began to view abduction, deduction, and induction as three stages of one inferential process. So Peirce now wanted to consider pragmatism as "a maxim of logic" including mathematical logic.

Thirdly, the incommensurables or irrationals are interrelated to the pure mathematical concepts of infinitesimals, limits, and continua. As we know,

7 "The Logic of Relatives", CP 3.457–482.

continuity is the focus of Peirce's whole philosophy. Applying the pragmatic maxim to these pure mathematical ideas would make pragmatism a part of, or a link to, Peirce's philosophy.

In his 1906 presentation to the National Academy of Science entitled "An Improvement on the Gamma Graphs", Peirce wrote:

> We here reach a point at which novel considerations about the constitution of knowledge and therefore of the constitution of nature burst in upon the mind with cataclysmal multitude and resistlessness. It is that synthesis of tychism and of pragmatism for which I long ago proposed the name, Synechism, to which one thus returns; but this time with stronger reasons than ever before. (CP 4.584)

Last, but not least, applying the pragmatic maxim to pure mathematical concepts broadened out the region of pragmatism on the one hand but at the same time narrowed it down. At the time when Peirce first set forth the pragmatic maxim, he said nothing about its domain of definition. According to this fact, we might infer that he thought it applies to all conceptions, or all symbols, including pure mathematical conceptions. But on the other hand, as the title of the series, *Illustrations of the Logic of Science* suggested, Peirce was then illustrating scientific logic, i.e., induction and abduction, so we may suspect that he excluded deductive – mathematical - conceptions. But now as he, in a sense, successfully applied the maxim to incommensurables, pragmatism extended to a new area. But at the same time, as he confessed that some of these pure mathematical ideas violate the maxim, he narrowed its scope. Looking back on the entry of pragmatism in *Baldwin's Dictionary*, we can see the two difficulties are interrelated. Peirce then discovered that his original definition of pragmatism was a maxim of behaviorism, just because he saw that he had understood conduct in a too limited sense during the 1870's, that is, conduct excluded the intellectual behavior typical of mathematical practice.

I believe all of these considerations led Peirce to limit the function of the pragmatic maxim, renaming and redefining pragmaticism in semiotic terms: "The entire intellectual purport of any symbol consists in the total of all general modes of rational conduct which, conditionally upon all the possible different circumstances and desires, would ensue upon the acceptance of the symbol" (CP 5.438).

If other pragmatists and scholars of pragmatism had noticed these changes in Peirce's thought, many criticisms of pragmatism could have been avoided, and the history of pragmatism might have taken on a different look.

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