J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2025-0082

Journal für die reine und angewandte Mathematik

On the structure of singularities of weak mean curvature flows with mean curvature bounds

By *Maxwell Stolarski* at Coventry

Abstract. This paper studies singularities of mean curvature flows with integral mean curvature bounds $H \in L^\infty L^p_{\mathrm{loc}}$ for some $p \in (n, \infty]$. For such flows, any backward tangent flow is given by the flow of a stationary cone $\mathbf{C}$. When $p = \infty$ and $\mathbf{C}$ is a regular cone, we prove that the backward tangent flow is unique. These results hold for general integral Brakke flows of arbitrary codimension in an open subset $U \subseteq \mathbb{R}^N$ with $H \in L^\infty L^p_{\mathrm{loc}}$. For smooth, codimension-one mean curvature flows with $H \in L^\infty L^\infty_{\mathrm{loc}}$, we also show that, at points where a backward tangent flow is given by an area-minimizing Simons cone, there is an accompanying limit flow given by a smooth Hardt–Simon minimal surface.

1. Introduction

A time-dependent family of embeddings $F\colon M^n \times [0, T) \to \mathbb{R}^N$ is said to evolve by mean curvature flow if $\partial_t F = H$, where $H = \vec{H}_{M_t}$ denotes the mean curvature vector of the embedded submanifold $M_t = F(M \times \{t\}) \subseteq \mathbb{R}^N$. Submanifolds evolving by mean curvature flow often develop singularities in finite time $T < \infty$. Huisken [10] showed that the second fundamental form A of a compact hypersurface M_t evolving by mean curvature flow always blows up at a finite-time singularity $T < \infty$, that is,

$$\limsup_{t \nearrow T} \sup_{x \in M_t} |A| = \infty.$$

Given Huisken's result [10], it is natural to ask if the trace of the second fundamental form, namely the mean curvature $H = \operatorname{tr} A$, must also blow up at finite-time singularities of the mean curvature flow. In [26], we answered this question in the negative by showing the mean curvature flow solutions constructed by Velázquez [30] develop finite-time singularities even though H remains uniformly bounded $\sup_{t \in [0,T)} \sup_{x \in M_t} |H| < \infty$. In dimension $n = 7$, [2] further showed how to extend these mean curvature flow solutions $M_t^7 \subseteq \mathbb{R}^8$ to weak mean curvature flows defined for later times $t \in [0, T + \epsilon)$ in such a way that H remains uniformly bounded and the flow has an isolated singularity at $(\mathbf{0}, T) \in \mathbb{R}^8 \times [0, T + \epsilon)$.

The author is supported by a Leverhulme Trust Early Career Fellowship (ECF-2023-182).

Given that there exist smooth mean curvature flows which develop singularities with bounded mean curvature [2, 26, 30], the focus of this current article is to instead study the singularities of *any* mean curvature flow $M_t^n \subseteq \mathbb{R}^N$ with uniform mean curvature bounds. In this general setting, uniform mean curvature bounds along the flow allow us to incorporate the well-developed theory of varifolds with bounds on their first variation. In particular, we leverage that theory to obtain the following result which holds more generally for weak mean curvature flows of arbitrary codimension with integral mean curvature bounds.

Theorem 1.1. *Let $2 \leq n < N$ and let $U \subseteq \mathbb{R}^N$ be open. Let $(\mu_t)_{t\in(a,b)}$ be an integral n-dimensional Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n, \infty]$. Except for a countable set of times $t \in (a,b)$, $(\mu_t)_{t\in(a,b)}$ equals the Brakke flow $(\mu_{V_t})_{t\in(a,b]}$ of a family of integer rectifiable n-varifolds $(V_t)_{t\in(a,b]}$ that extends to the final time-slice $t = b$.*

For any $(x_0, t_0) \in U \times (a, b]$, any backward tangent flow of $(\mu_t)_{t\in(a,b)}$ at (x_0, t_0) is given by the static flow of a stationary cone $\mathbf{C}$. The stationary cones $\mathbf{C}$ that arise as backward tangent flows to $(\mu_t)_{t\in(a,b)}$ at (x_0, t_0) are exactly the tangent cones to V_{t_0} at x_0.

Note that the stationary cones $\mathbf{C}$ in Theorem 1.1 are generally integer rectifiable n-varifolds that are dilation invariant and have zero first variation (see Lemma 2.7 and Theorem 2.13 for more precise statements of Theorem 1.1).

Given a Brakke flow (μ_t), tangent flows at (x_0, t_0) are, simply speaking, subsequential limits of parabolic rescalings of (μ_t) based at (x_0, t_0). Backward tangent flows refer to the restriction of these limits to negative times (Definition 2.11). Analogously, tangent cones of a varifold V_{t_0} at x_0 are subsequential limits of spatial rescalings of V_{t_0} based at x_0. Thanks to suitable monotonicity formulas and compactness theorems, (backward) tangent flows and tangent cones always exist. However, the uniqueness of tangent cones and (backward) tangent flows, that is, independence of subsequence, is generally an open problem. This uniqueness question is fundamental for singularity analysis and regularity.

Theorem 1.1 in particular shows that, for flows with mean curvature

$$H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b)),$$

uniqueness of the backward tangent *flow* of (μ_t) at (x_0, t_0) is equivalent to uniqueness of the tangent *cone* of V_{t_0} at x_0. Because of this correspondence, we are able to prove the following uniqueness result.

Theorem 1.2. *Let $2 \leq n < N$ and let $U \subseteq \mathbb{R}^N$ be open. Let $(\mu_t)_{t\in(a,b)}$ be an integral n-dimensional Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times (a,b))$. If a backward tangent flow to $(\mu_t)_{t\in(a,b)}$ at $(x_0, t_0) \in U \times (a, b]$ is given by the static flow of a regular cone $\mathbf{C}$ with multiplicity one, then this is the unique backward tangent flow to (μ_t) at (x_0, t_0).*

It is worth mentioning some related uniqueness results. Huisken's monotonicity formula ensures tangent flows are always self-similarly shrinking $\Sigma_t = \sqrt{-t}\,\Sigma_{-1}$ for $t < 0$. There are various uniqueness results depending on $\Sigma = \Sigma_{-1}$. When Σ is compact, [19] proved that if $\sqrt{-t}\,\Sigma$ is a tangent flow of a mean curvature flow (M_t) at (x_0, t_0), then it is the unique tangent flow at (x_0, t_0). In [5], uniqueness of the tangent flow $\sqrt{-t}\,\Sigma$ is proved when $\Sigma = \mathbb{R}^{n-k} \times \mathbb{S}^k$ is a generalized cylinder (see also a generalization in [35]).

In [4], uniqueness of the backward tangent flow is proved when Σ is smooth and asymptotically conical (see also a generalization due to [16]). Importantly, the uniqueness results in [4, 16] require Σ to be smooth and do not apply when $\Sigma = \mathbf{C}$ is a minimal cone for example. The first uniqueness result for backward tangent flows given by non-smooth Σ came in [17] which showed that, for 2-dimensional Lagrangian mean curvature flows $L_t^2 \subseteq \mathbb{C}^2$, backward tangent flows given by a transverse pair of planes $\Sigma = P_1 \cup P_2 \subseteq \mathbb{C}^2$ are unique. The uniqueness result Theorem 1.2 here applies to non-smooth minimal cones $\Sigma = \mathbf{C}$ which arise as backward tangent flows to general integral Brakke flows of any codimension in an open subset $U \subseteq \mathbb{R}^N$, albeit under the assumption of a uniform mean curvature bound $H \in L^\infty L^\infty_{\mathrm{loc}}$. Note in particular that a transverse pair of n-dimensional planes $P_1 \cup P_2$ in $\mathbb{R}^{2n} = \mathbb{C}^n$ is a regular cone. Thus, for the class of Lagrangian mean curvature flows with uniformly bounded mean curvature, Theorem 1.2 gives an alternative proof of the uniqueness result from [17] that generalizes to all dimensions.

Under additional hypotheses to Theorem 1.2, we can also describe an accompanying limit flow that arises as a more general blow-up limit around a singularity.

Theorem 1.3. *Let $\mathcal{M} = (M_t^n)_{t\in(a,b)}$ be a smooth, properly embedded mean curvature flow in an open subset $U \subseteq \mathbb{R}^{n+1}$ with mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times (a,b))$. If the backward tangent flow of $\mathcal{M}$ at $(x,b) \in U \times \{b\}$ is given by the static flow of a generalized Simons cone $\mathbf{C}^n \subseteq \mathbb{R}^{n+1}$ with multiplicity one and $\mathbf{C}$ is area minimizing, then there exist a sequence $(x_i, t_i) \in U \times (a,b)$ with $\lim_{i\to\infty}(x_i, t_i) = (x,b)$ and a sequence $\lambda_i \searrow 0$ such that the sequence of rescaled mean curvature flows $\mathcal{M}_i = \mathcal{D}_{\lambda_i^{-1}}(\mathcal{M} - (x_i, t_i))$ converges to the static flow $\mathcal{M}_\infty = (\widehat{M})_{t\in\mathbb{R}}$ of a smooth Hardt–Simon minimal surface $\widehat{M}$ for all time $t \in \mathbb{R}$.*

We refer the reader to Section 4 for the definitions relevant to Theorem 1.3. For now, we simply note that the mean curvature flow solutions constructed in [30] provide examples of mean curvature flows satisfying Theorem 1.3. Theorem 1.3 states that general mean curvature flows with $H \in L^\infty L^\infty_{\mathrm{loc}}$ in some sense mimic the dynamics of Velázquez's mean curvature flow solutions [30] near Simons cone singularities.

The paper is organized as follows. In Section 2, we establish notation, specify definitions, and obtain some general results used throughout the paper. In particular, Theorem 1.1 is proven here. Section 3 proves the uniqueness of backward tangent flows given by regular stationary cones, Theorem 1.2. In Section 4, we obtain refined dynamics of mean curvature flows near regular stationary cones and prove Theorem 1.3. Finally, Appendix A reviews some well-known results about integral varifolds with generalized mean curvature $H \in L^p_{\mathrm{loc}}$ that are cited throughout the paper.

2. Preliminaries

2.1. Brakke flows with H bounds.

Definition 2.1. Let $2 \le n < N$, let $U \subseteq \mathbb{R}^N$ be open, and $p \in (1,\infty]$. Let V be an integer rectifiable n-varifold in U and let μ_V be the associated measure on U. We say that

V has *generalized mean curvature* $H \in L^p_{\text{loc}}(U)$ if there exists a Borel function $H: U \to \mathbb{R}^N$ with $H \in L^p_{\text{loc}}(U, d\mu_V)$ such that the first variation δV satisfies

$$\delta V(X) = -\int H \cdot X \, d\mu_V \quad \text{for all } X \in C^1_c(U, \mathbb{R}^N). \tag{2.1}$$

Remark 2.2. Observe that, by the definition given in Definition 2.1, if V has generalized mean curvature $H \in L^p_{\text{loc}}(U)$, then V has locally bounded first variation[1] which is absolutely continuous with respect to μ_V and V has no generalized boundary in U. This convention differs somewhat from the existing literature, but we adopt it nonetheless to simplify the statements in the remainder of the paper.

Throughout, we use the following notion of Brakke flow as in [34, Definition 5.1 with empty boundary $\Gamma = \emptyset$].

Definition 2.3. Let $1 \leq n < N$, let $U \subseteq \mathbb{R}^N$ be open, and let $I \subseteq \mathbb{R}$ be a non-empty interval. An *n-dimensional integral Brakke flow* or simply *integral n-Brakke flow* $(\mu_t)_{t \in I}$ *in* U is a family of Radon measures $(\mu_t)_{t \in I}$ on U such that

(1) for almost every $t \in I$, there exists an integer rectifiable n-varifold V_t in U such that $\mu_t = \mu_{V_t}$ and V_t has locally bounded first variation δV_t which is absolutely continuous with respect to $\mu_{V_t} = \mu_t$,

(2) for any $[s,t] \subseteq I$ and any compact $K \subseteq U$,

$$\int_s^t \int_K 1 + |H|^2 \, d\mu_\tau \, d\tau < \infty$$

(where, for almost every $t \in I$, $H = H_t$ is the vector-valued function representing δV_t as in (2.1)), and

(3) for any $[s,t] \subseteq I$ and $f \in C^1_c(U \times [s,t])$ with $f \geq 0$,

$$\mu_t(f) - \mu_s(f) \leq \int_s^t \int \partial_t f + \nabla f \cdot H - |H|^2 f \, d\mu_\tau \, d\tau. \tag{2.2}$$

We refer to inequality (2.2) as *Brakke's inequality*.

Definition 2.4. Let $2 \leq n < N$, let $U \subseteq \mathbb{R}^N$ be open, and $p \in (1, \infty]$. Let $(\mu_t)_{t \in (a,b)}$ be an n-dimensional integral Brakke flow in U. We say the Brakke flow $(\mu_t)_{t \in (a,b)}$ has *generalized mean curvature* $H \in L^\infty L^p_{\text{loc}}(U \times (a,b))$ if, for almost every $t \in (a,b)$, there exists an integer rectifiable n-varifold V_t in U with generalized mean curvature $H_t \in L^p_{\text{loc}}(U)$ such that $\mu_{V_t} = \mu_t$ and[2]

$$\|H\|_{L^\infty L^p(K \times (a,b))} \doteqdot \operatorname*{ess\,sup}_{t \in (a,b)} \|H_t\|_{L^p(K, d\mu_t)} < \infty \quad (\text{for all } K \Subset U).$$

[1] "Locally bounded first variation" should not be confused with "$H \in L^\infty_{\text{loc}}(U)$." See e.g. [29, Definition 1.11] for a precise definition of locally bounded first variation.

[2] Throughout, "$K \Subset U$" means $K \subseteq U$ and K is compact.

Assumption 2.5. Let $2 \le n < N$, let $U \subseteq \mathbb{R}^N$ be an open subset, $-\infty < a < b < \infty$. Throughout, we assume $(\mu_t)_{t\in(a,b)}$ is an n-dimensional integral Brakke flow in U such that (μ_t) has locally uniformly bounded areas, that is,

$$\sup_{t\in(a,b)} \mu_t(K) < \infty \quad \text{for all } K \Subset U.$$

For simplicity, we will abbreviate this assumption as "$(\mu_t)_{t\in(a,b)}$ is an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas" in the remainder of the paper.

We will also often assume that, for some $p \in (1,\infty]$, $(\mu_t)_{t\in(a,b)}$ has generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$, but this assumption will be indicated in each statement.

Remark 2.6. The assumption that the flow has locally uniformly bounded areas is quite mild. For example, it holds for Brakke flows $(\mu_t)_{t\in(a,b)}$ obtained as restrictions of Brakke flows defined for $t \in [a,b)$. Indeed, this can be seen by using Brakke's inequality with suitably defined spherically shrinking test functions (see e.g. [34, Theorem 5.5] or [6, Proposition 4.9]). In particular, if $(\mu_t)_{t\in(a,b)}$ is an integral n-Brakke flow in U, then for any $0 < \epsilon < b-a$, $(\mu_t)_{t\in(a+\epsilon,b)}$ is an integral n-Brakke flow in U with locally uniformly bounded areas.

On the other hand, the assumption that the flow has generalized mean curvature

$$H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$$

is much more restrictive. Nonetheless, [26] shows that even smooth mean curvature flows $(M^n_t \subseteq \mathbb{R}^{n+1})_{t\in(a,b)}$ with $H \in L^\infty L^\infty(\mathbb{R}^{n+1} \times (a,b))$ can develop singularities at the final time $t = b$. Combined with the work of [2], there are non-smooth Brakke flows with

$$H \in L^\infty L^\infty(\mathbb{R}^N \times (a,b))$$

with mild singularities and small singular sets, informally speaking.

The next lemma shows that Brakke flows with $H \in L^\infty L^p_{\mathrm{loc}}$ can be changed at countably many times to get a Brakke flow which is a varifold with $H \in L^p_{\mathrm{loc}}$ at *every* time. Moreover, the flow naturally extends to the final time-slice.

Lemma 2.7. *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (1,\infty]$. Then, for all $t \in (a,b]$, there exists a unique integer rectifiable n-varifold V_t with generalized mean curvature in $H_{V_t} \in L^p_{\mathrm{loc}}(U)$ such that*

(1) $\mu_t \le \lim_{t'\nearrow t} \mu_t = \mu_{V_t}$ *for all $t \in (a,b)$,*

(2) $\mu_t = \mu_{V_t}$ *for all but countably many $t \in (a,b)$,*

(3) *for all $t \in (a,b]$ and all $K \Subset U$,*

$$\mu_{V_t}(K) \le \sup_{\tau\in(a,b)} \mu_\tau(K) \quad \textit{and} \quad \|H_{V_t}\|_{L^p(K)} \le \|H\|_{L^\infty L^p(K\times(a,b))},$$

(4) *for all $t \in (a,b]$,*

$$\lim_{t'\nearrow t} V_{t'}(f) = V_t(f) \quad \textit{for all } f \in C^0_c(G(n,U)),$$

(5) $(\mu_{V_t})_{t\in(a,b]}$ *is an n-dimensional integral Brakke flow in U.*

Proof. Let $t \in (a,b]$. Since (μ_t) has $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$, there exist a sequence of times $t_j \nearrow t$ (with $t_j < t$) and integer rectifiable n-varifolds $\widetilde{V}_j$ with generalized mean curvature $H_{\widetilde{V}_j}$ in $L^p_{\mathrm{loc}}(U)$ such that $\mu_{t_j} = \mu_{\widetilde{V}_j}$ and

$$\|H_{\widetilde{V}_j}\|_{L^p(K,d\mu_{t_j})} \le \|H\|_{L^\infty L^p(K\times(a,b))} \doteqdot C_K < \infty \quad \text{for all } K \Subset U.$$

By compactness (Lemma A.1), there exist a subsequence (still denoted $\widetilde{V}_j$) and an integer rectifiable n-varifold V_t with locally bounded areas and generalized mean curvature $H_{V_t} \in L^p_{\mathrm{loc}}(U)$ such that $\widetilde{V}_j \rightharpoonup V_t$ as varifolds and

$$\mu_{V_t}(K) \le \sup_{\tau\in(a,b)} \mu_\tau(K) \quad \text{and} \quad \|H_{V_t}\|_{L^p(K,d\mu_{V_t})} \le C_K \quad \text{for all } K \Subset U.$$

This defines the varifold V_t for any $t \in (a,b]$ and proves it satisfies (3).

By [12, Theorem 7.2 (ii)], for any $f \in C^0_c(U, \mathbb{R}_{\ge 0})$ and any $t \in (a,b)$,

$$\mu_t(f) \le \lim_{s\nearrow t} \mu_s(f) = \lim_{j\to\infty} \mu_{\widetilde{V}_j}(f) = \mu_{V_t}(f). \tag{2.3}$$

This proves (1). For (2), simply note that [12, Theorem 7.2 (iii)] implies the first inequality in (2.3) is an equality for all but countably many times $t \in (a,b)$. Note additionally that the equality $\lim_{s\nearrow t}\mu_s = \mu_{V_t}$ in (1) implies the varifold V_t is unique.

To prove (4), let $t \in (a,b]$ and $f \in C^\infty_c(U,\mathbb{R}_{\ge 0})$. Take a sequence $t_j \nearrow t$. By [12, Theorem 7.2 (i)], there exists C_f such that $\mu_t(f) - C_f t$ is decreasing in t. It follows that, for any j,

$$\begin{aligned}
\mu_{V_t}(f) &= \lim_{s\nearrow t}(\mu_s(f) - C_f s) + C_f t \quad \text{(by (1))}\\
&\le \mu_{t_j}(f) - C_f t_j + C_f t\\
&\le \mu_{V_{t_j}}(f) + C_f(t - t_j) \qquad \text{(by (1))}.
\end{aligned}$$

Taking $j \to \infty$ gives $\mu_{V_t}(f) \le \liminf_j \mu_{V_{t_j}}(f)$. For the reverse inequality, let $\epsilon > 0$. Recall $\mu_{V_t}(f) = \lim_{s\nearrow t}\mu_t(f)$ and similarly for the V_{t_j}. Thus there exists $s = s(\epsilon) < t$ such that $|s-t| < \epsilon$ and $|\mu_{V_t}(f) - \mu_s(f)| < \epsilon$. There exists J such that $s < t_j < t$ for all $j > J$. It follows that, for $j > J$,

$$\begin{aligned}
\mu_{V_{t_j}}(f) - \mu_{V_t}(f) &\le \mu_{V_{t_j}}(f) - \mu_s(f) + \epsilon\\
&= \lim_{\sigma\nearrow t_j}(\mu_\sigma(f) - C_f\sigma) + C_f t_j - \mu_s(f) + \epsilon\\
&\le \mu_s(f) - C_f s + C_f t_j - \mu_s(f) + \epsilon\\
&= C_f(t_j - s) + \epsilon.
\end{aligned}$$

Taking $j \to \infty$ then $\epsilon \to 0$ gives $\limsup_j \mu_{V_{t_j}}(f) \le \mu_{V_t}(f)$. Since the sequence $t_j \nearrow t$ and $f \in C^\infty_c(U,\mathbb{R}_{\ge 0})$ were arbitrary, it follows that

$$\lim_{t'\nearrow t}\mu_{V_{t'}}(f) = \mu_{V_t}(f) \quad \text{for all } f \in C^\infty_c(U,\mathbb{R}_{\ge 0}).$$

Convergence as varifolds then follows from Lemma A.2 and completes the proof of (4).

To prove (5), it suffices to check Brakke's inequality. Let $a < t_0 < t_1 \le b$ and suppose that $f \in C^1_c(U \times [t_0,t_1])$ with $f \ge 0$. Then

$$\mu_{V_{t_1}}(f) - \mu_{V_{t_0}}(f) \le \lim_{s\nearrow t_1}\mu_s(f) - \mu_{t_0}(f) \quad \text{(by (1))}$$

$$\leq \lim_{s \nearrow t_1} \int_{t_0}^{s} \int \partial_t f + \nabla f \cdot H - |H|^2 f \, d\mu_t \, dt$$
$$= \int_{t_0}^{t_1} \int \partial_t f + \nabla f \cdot H - |H|^2 f \, d\mu_{V_t} \, dt \quad \text{(by (2))}. \qquad \square$$

2.2. Huisken's monotonicity formula and Gaussian density. Let

$$\Phi_{x_0,t_0}(x,t) \doteqdot \frac{1}{(4\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} \quad (t < t_0)$$

denote the backwards heat kernel based at (x_0, t_0). Let

$$\varphi_{x_0,t_0;r}(x,t) \doteqdot \Big(1 - \frac{|x-x_0|^2 - 2n(t_0-t)}{r^2}\Big)_+^3 \quad (t \leq t_0)$$

denote the spherically shrinking localization function based at (x_0, t_0) with scale $r > 0$.

Huisken's monotonicity formula and its localized analogue for Brakke flows states

$$\int \Phi_{x_0,t_0}(\cdot\,,t)\varphi_{x_0,t_0;r}(\cdot\,,t)\,d\mu_t - \int \Phi_{x_0,t_0}(\cdot\,,s)\varphi_{x_0,t_0;r}(\cdot\,,s)\,d\mu_s$$
$$\leq -\int_s^t \int \Big| H + \frac{(x-x_0)^\perp}{2(t_0-\tau)} \Big|^2 \Phi_{x_0,t_0}\varphi_{x_0,t_0;r}\,d\mu_\tau\,d\tau \leq 0 \tag{2.4}$$

for all $s < t < t_0$ and $r > 0$ such that (μ_t) is a Brakke flow on $B_{\sqrt{r^2+2n(t_0-s)}}(x_0) \times [s,t]$. Moreover, for Brakke flows $(\mu_t)_{t\in(a,b)}$ defined in $U \subseteq \mathbb{R}^N$ and points $x_0 \in B_{2r_0}(x_0) \subseteq U$, the Gaussian density

$$\Theta_\mu(x_0,t_0) \doteqdot \lim_{t \nearrow t_0} \int \Phi_{x_0,t_0}(\cdot\,,t)\varphi_{x_0,t_0;r}(\cdot\,,t)\,d\mu_t$$

is well-defined for $t_0 \in (a,b]$ and independent of $r \in (0, r_0)$.

For an n-dimensional Brakke flow (μ_t) and $\lambda > 0$, define the parabolically dilated Brakke flow $\mathcal{D}_{x_0,t_0;\lambda}\mu$ based at (x_0,t_0) to be

$$[\mathcal{D}_{x_0,t_0;\lambda}\mu]_t(A) \doteqdot \lambda^n \mu_{t_0+t\lambda^{-2}}(\lambda^{-1}A + x_0).$$

Often, we omit the basepoint x_0, t_0 and simply write $\mathcal{D}_\lambda\mu$ when the basepoint is clear from context.

2.3. Densities and blow-up limits. Recall the density of a varifold V at $x_0 \in \mathbb{R}^N$ is given by

$$\theta_V(x_0) \doteqdot \lim_{\rho \searrow 0} \frac{\mu_V(B_\rho(x_0))}{\omega_n \rho^n}$$

when the limit exists and where ω_n denotes the volume of the unit n-ball. In the remainder of the article, we use $\eta_{x_0,\lambda}$ to denote the spatial translation and dilation

$$\eta_{x_0,\lambda}\colon \mathbb{R}^N \to \mathbb{R}^N, \quad \eta_{x_0,\lambda}(x) \doteqdot \lambda(x - x_0).$$

The map $\eta_{x_0,\lambda}$ naturally induces a map of integer rectifiable n-varifolds denoted by $(\eta_{x_0,\lambda})_\sharp$. Specifically, if $V = (\Gamma, \theta)$, then $(\eta_{x_0,\lambda})_\sharp V = (\eta_{x_0,\lambda}(\Gamma), \theta \circ \eta_{x_0,\lambda}^{-1})$. When the basepoint x_0 is clear from context or $x_0 = 0$, we shall often write η_λ for simplicity.

When $H \in L^\infty L^p_{\mathrm{loc}}$ with $p > n$, the monotonicity formula (A.3) for varifolds gives a characterization of subsequential blow-ups of time-slices.

Lemma 2.8 (Existence of tangent cones of time-slices). *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas (Assumption 2.5) and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n,\infty]$. Let $(V_t)_{t\in(a,b]}$ be the family of varifolds as in Lemma 2.7. For any $(x_0,t_0) \in U \times (a,b]$ and any sequence $\lambda_i \nearrow +\infty$, there exist a subsequence (still denoted λ_i) and an integer rectifiable n-varifold $\mathbf{C}$ in $\mathbb{R}^N$ such that $(\eta_{x_0,\lambda_i})_\sharp V_{t_0} \rightharpoonup \mathbf{C}$ as varifolds in $\mathbb{R}^N$. Moreover, $\mathbf{C}$ is a stationary, dilation invariant varifold in $\mathbb{R}^N$ with*

$$\frac{\mu_{\mathbf{C}}(B_r(0))}{\omega_n r^n} = \theta_{V_{t_0}}(x_0) = \lim_{\rho\searrow 0} \frac{\mu_{V_{t_0}}(B_\rho(x_0))}{\omega_n \rho^n} \quad \textit{for all } 0 < r < \infty.$$

Any $\mathbf{C}$ that arises as such a limit is called a tangent cone *of V_{t_0} at x_0.*

Proof. By Lemma 2.7, V_{t_0} has generalized mean curvature $H = H_{V_{t_0}} \in L^p_{\mathrm{loc}}(U)$. It follows that the dilations

$$V_i \doteqdot (\eta_{x_0,\lambda_i})_\sharp V_{t_0}$$

have generalized mean curvature $H_i \in L^p_{\mathrm{loc}}(\lambda_i(U - x_0))$ with

$$\|H_i\|_{L^p(K,d\mu_{V_i})} \le \lambda_i^{\frac{n}{p}-1} \|H_{V_{t_0}}\|_{L^p(\overline{B}_R(x_0),d\mu_{V_{t_0}})} \tag{2.5}$$

for all $K \subseteq \mathbb{R}^N$ compact and $i \gg 1$ sufficiently large so that $\frac{1}{\lambda_i}K + x_0 \subseteq \overline{B}_R(x_0) \Subset U$. In particular, $\lim_{i\to\infty}\|H_i\|_{L^p(K,d\mu_{V_i})} = 0$.

As $p > n$, there is the monotonicity formula (A.3) for integral n-varifolds with $H \in L^p_{\mathrm{loc}}$ (Proposition A.3; see also [23, Chapter 4, §4]). In particular, the density

$$\theta_{V_{t_0}}(x_0) \doteqdot \lim_{\rho\searrow 0} \frac{\mu_{V_{t_0}}(B_\rho(x_0))}{\omega_n \rho^n}$$

is well-defined and the blow-up sequence V_i has uniform local area bounds.

It now follows from compactness (Lemma A.1) that there are a subsequence (still denoted V_i) and an integer rectifiable n-varifold $\mathbf{C}$ in $\mathbb{R}^N$ such that $V_i \rightharpoonup \mathbf{C}$. By (2.5) and Lemma A.1,

$$\|H_{\mathbf{C}}\|_{L^p(K,d\mu_{\mathbf{C}})} \le \limsup_i \|H_{V_i}\|_{L^p(K,d\mu_{V_i})} = 0,$$

which implies $\mathbf{C}$ is stationary. Moreover, the convergence $V_i \rightharpoonup \mathbf{C}$ implies

$$\frac{\mu_{\mathbf{C}}(B_r(0))}{\omega_n r^n} = \theta_{V_{t_0}}(x_0) \quad \text{for all } 0 < r < \infty.$$

The monotonicity formula (A.3) for stationary varifolds finally implies $\mathbf{C}$ is dilation invariant, that is, $(\eta_{0,\lambda})_\sharp\mathbf{C} = \mathbf{C}$ for all $0 < \lambda < \infty$ (see e.g. [23, Chapter 8, §5]). □

We will show in Lemma 2.10 below that Huisken's monotonicity formula (2.4) similarly allows us to extract tangent *flows* from *space-time* blow-ups. First, we show local uniform area bounds imply local uniform area *ratio* bounds. These area ratio bounds are necessary for the blow-up argument in the proof of Lemma 2.10 below.

Note that, for general Brakke flows defined for $t \in [a,b)$, local uniform area ratio bounds at the initial time $t = a$ imply local uniform area ratio bounds for all $t \in [a,b)$. We emphasize that, for integral Brakke flows with locally uniformly bounded areas and $H \in L^\infty L^p_{\mathrm{loc}}$, Lemma 2.9 below obtains local uniform area ratio bounds *without assuming* local uniform area ratio bounds at the initial time.

Lemma 2.9 (Area ratio bounds). *Suppose that $(\mu_t)_{t\in(a,b)}$ is an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas (Assumption 2.5) and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n,\infty]$. For any $K \Subset U$, there exists C such that*

$$\sup_{t\in(a,b)} \sup_{B_r(x)\subseteq K} \frac{\mu_t(B_r(x))}{r^n} \le C.$$

Proof. The proof will proceed by combining the monotonicity formula (A.3) with the local uniform area bounds. Given $K \Subset U$, there exists $\epsilon > 0$ such that

$$K' \doteqdot \overline{B_\epsilon(K)} = \overline{\bigcup_{x\in K} B_\epsilon(x)} \Subset U.$$

Let

$$R_0 \doteqdot \sup\{r > 0 : B_r(x) \subseteq K \text{ for some } x\}, \quad R_1 \doteqdot \max\{R_0, 1\}.$$

For any $B_r(x) \subseteq K$, there exists $R \in [\max\{\epsilon, r\}, R_1]$ such that $B_r(x) \subseteq B_R(x) \subseteq K'$. Let $(V_t)_{t\in(a,b]}$ be the family of varifolds as in Lemma 2.7 and let $t \in (a,b)$. The monotonicity formula (A.3) implies

$$\frac{\mu_{V_t}(B_r(x))}{r^n} \le \frac{\mu_{V_t}(B_r(x))}{r^n} e^{\frac{\|H_{V_t}\|}{1-n/p} r} + e^{\frac{\|H_{V_t}\|}{1-n/p} r} \le \frac{\mu_{V_t}(B_R(x))}{R^n} e^{\frac{\|H_{V_t}\|}{1-n/p} R} + e^{\frac{\|H_{V_t}\|}{1-n/p} R},$$

where $\|H_{V_t}\| = \|H_{V_t}\|_{L^p(B_R(x), d\mu_{V_t})}$. It then follows that

$$\begin{aligned}
\frac{\mu_t(B_r(x))}{r^n} &\le \frac{\mu_{V_t}(B_r(x))}{r^n} \le \frac{\mu_{V_t}(B_R(x))}{R^n} e^{\frac{\|H_{V_t}\|_{L^p(K')}}{1-n/p} R} + e^{\frac{\|H\|_{L^p(K')}}{1-n/p} R} \\
&\le \frac{\mu_{V_t}(K')}{\epsilon^n} e^{\frac{\|H_{V_t}\|_{L^p(K')}}{1-n/p} R_1} + e^{\frac{\|H\|_{L^p(K')}}{1-n/p} R_1} \\
&\le \frac{\sup_{\tau\in(a,b)} \mu_\tau(K')}{\epsilon^n} e^{\frac{\|H\|_{L^\infty L^p(K'\times(a,b))}}{1-n/p} R_1} + e^{\frac{\|H\|_{L^\infty L^p(K'\times(a,b))}}{1-n/p} R_1} < \infty.
\end{aligned}$$

Taking the supremum over $t \in (a,b)$ and $B_r(x) \subseteq K$ completes the proof. □

Lemma 2.10 (Existence of tangent flows). *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature*

$$H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b)) \quad \text{for some } p \in (n,\infty].$$

For any $(x_0, t_0) \in U \times (a,b]$ and any sequence $\lambda_i \nearrow +\infty$, there exist a subsequence (still denoted λ_i) and an integer rectifiable n-varifold $\mathbf{C}$ in $\mathbb{R}^N$ such that, for any $t < 0$,

$$(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t \rightharpoonup \mu_{\mathbf{C}} \quad \text{as } i \to \infty \text{ (weakly as measures).}$$

Moreover, $\mathbf{C}$ is a stationary, dilation invariant varifold in $\mathbb{R}^N$ with

$$\Theta_\mu(x_0, t_0) = \frac{1}{(4\pi)^{n/2}} \int e^{-\frac{|x|^2}{4}}\, d\mu_{\mathbf{C}}.$$

Any $\mathbf{C}$ that arises as such a limit is called a (backward) tangent flow *of $(\mu_t)_{t\in(a,b)}$ at (x_0, t_0) (see Definition 2.11 below).*

Proof. Let us fix $(x_0, t_0) \in U \times (a, b]$. By Lemma 2.9, (μ_t) has local uniform area ratio bounds in a neighborhood of (x_0, t_0). It follows from Huisken's monotonicity formula (2.4) and the compactness of Brakke flows with local uniform area bounds that there exist a Brakke flow $(\mu_t^\infty)_{t<0}$ on $\mathbb{R}^N$ and a subsequence (still denoted λ_i) such that

$$(\mathcal{D}_{\lambda_i}\mu)_t = (\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t \rightharpoonup \mu_t^\infty \quad \text{for all } t < 0$$

(see [11, Lemma 8] or [31]). Moreover, μ^∞ is a shrinker for $t < 0$ in the sense that

$$\mu_t(A) = (-t)^{n/2}\mu_{-1}(A/\sqrt{-t}) \quad \text{for all } t < 0 \tag{2.6}$$

and, for any $t < 0$, μ_t^∞ is an integer rectifiable n-varifold with

$$H + \frac{x^\perp}{2(-t)} = 0 \quad \text{for } \mu_t^\infty\text{-a.e. } x \in \mathbb{R}^N. \tag{2.7}$$

Additionally,

$$\Theta_\mu(x_0, t_0) = \frac{1}{(4\pi(-t))^{n/2}} \int e^{-\frac{|x|^2}{4(-t)}}\, d\mu_t^\infty \quad \text{for all } t < 0.$$

Because (μ_t) is a Brakke flow with $H \in L^\infty L^p_{\text{loc}}$, there exists $t < 0$ such that, for all i, $(\mathcal{D}_{\lambda_i}\mu)_t$ is represented by an integer rectifiable n-varifold V_i with $H_{V_i} \in L^p_{\text{loc}}(\lambda_i U)$ and, for any $K \Subset \mathbb{R}^N$,

$$\|H_{V_i}\|_{L^p(K, d\mu_{V_i})} \le \lambda_i^{\frac{n}{p}-1} \|H\|_{L^\infty L^p(B_{r_0}(x_0)\times(a,b))}$$

for all $i \gg 1$ sufficiently large such that $\lambda_i^{-1}K + x_0 \subseteq B_{r_0}(x_0)$ and $\overline{B_{r_0}(x_0)} \Subset U$. As $p > n$,

$$\lim_{i\to\infty} \|H_{V_i}\|_{L^p(K, d\mu_{V_i})} = 0 \quad \text{for all } K \Subset \mathbb{R}^N.$$

Since $\mu_{V_i} \rightharpoonup \mu_t^\infty$, the compactness of varifolds with mean curvature bounds (Lemma A.1 and Lemma A.2) imply that $\mu_t^\infty = \mu_{\mathbf{C}}$ for some integer rectifiable n-varifold $\mathbf{C}$ in $\mathbb{R}^N$ which is stationary ($H_{\mathbf{C}} = 0$). It then follows from (2.7) that $x^\perp = 0$ $\mu_{\mathbf{C}}$-a.e. and thus $\mathbf{C}$ is dilation invariant (see e.g. [23, Chapter 8, §5]). Finally, (2.6) implies $\mu_t^\infty = \mu_{\mathbf{C}}$ for all $t < 0$. This completes the proof. □

2.4. Remarks on (backward) tangent flows. Lemma 2.10 ensures subsequential limits converge $(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t \rightharpoonup \mu_{\mathbf{C}}$ weakly as measures for all *negative* times $t < 0$. Area ratio bounds as in Lemma 2.9 do generally imply that space-time blow-ups $(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t$ of Brakke flows subsequentially converge to Brakke flows $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ for all times $t \in \mathbb{R}$ and $(\tilde{\mu}_t)$ is self-similarly shrinking for $t < 0$ (see e.g. [11, Lemma 8] or [31]). These limiting flows $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ defined for all $t \in \mathbb{R}$ are sometimes called "tangent flows" in the literature. We, on the other hand, shall principally be concerned with the behavior of these space-time blow-up limits for negative times $t < 0$ only. To distinguish these notions and ensure precise statements, we adopt the following definitions throughout the paper.

Definition 2.11. Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$.

An integral n-Brakke flow $(\tilde{\mu}_t)_{t<0}$ is called a *backward tangent flow* or simply a *tangent flow of* $(\mu_t)_{t\in(a,b)}$ *at* $(x_0, t_0) \in U \times (a, b]$ if there exists a sequence $\lambda_i \nearrow +\infty$ such that, for all $t < 0$,

$$(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t \rightharpoonup \tilde{\mu}_t \quad \text{as } i \to \infty \text{ (weakly as measures).}$$

An integral n-Brakke flow $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ is said to be a *full tangent flow of* $(\mu_t)_{t\in(a,b)}$ *at* $(x_0, t_0) \in U \times (a,b)$ if there exists a sequence $\lambda_i \nearrow +\infty$ such that, for all $t \in \mathbb{R}$,

$$(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t \rightharpoonup \tilde{\mu}_t \quad \text{as } i \to \infty \text{ (weakly as measures).}$$

A backward tangent flow $(\tilde{\mu}_t)_{t<0}$ of $(\mu_t)_{t\in(a,b)}$ at (x_0,t_0) is said to be *static* if $\tilde{\mu}_t = \tilde{\mu}_{-1}$ for all $t < 0$. A full tangent flow $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ of $(\mu_t)_{t\in(a,b)}$ at (x_0,t_0) is said to be *static* if $\tilde{\mu}_t = \tilde{\mu}_{-1}$ for all $t \in \mathbb{R}$. A full tangent flow $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ of $(\mu_t)_{t\in(a,b)}$ at (x_0,t_0) is said to be *quasistatic* if $\tilde{\mu}_t = \tilde{\mu}_{-1}$ for all $t < 0$ but not all $t \in \mathbb{R}$.

If an integral n-Brakke flow $(\tilde{\mu}_t)$ is a static backward tangent flow, then we may abuse notation and say the underlying varifold V (with $\mu_V = \tilde{\mu}_t$ for all t) is a static backward tangent flow (cf. Lemma 2.10 above). Analogous conventions hold in the case of static and quasistatic full tangent flows.

Clearly, the restriction of a full tangent flow to negative times $t < 0$ is a backward tangent flow. Restricting a static or quasistatic full tangent flow to negative times gives a static backward tangent flow. Since backward tangent flows evolve self-similarly, there is no equivalent notion of "quasistatic" for backward tangent flows. In the setting of Lemma 2.10, backward tangent flows are necessarily static. The next example shows full tangent flows may be quasistatic even in the setting of Lemma 2.10.

Example 2.12. Consider the n-plane $P = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^N$ and the integral n-Brakke flow $(\mu_t)_{t\in\mathbb{R}}$ given by

$$\mu_t = \begin{cases} 3\mathcal{H}^n \llcorner P & \text{if } t < 0, \\ 2\mathcal{H}^n \llcorner P & \text{if } t = 0, \\ \mathcal{H}^n \llcorner P & \text{if } t > 0, \end{cases}$$

where $\mathcal{H}^n$ is n-dimensional Hausdorff measure. Then $(\mu_t)_{t\in\mathbb{R}}$ is an integral n-Brakke flow in $\mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature

$$H \equiv 0 \in L^\infty L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}).$$

Take $(x_0,t_0) = (\mathbf{0},0) \in \mathbb{R}^N \times \mathbb{R}$. For any sequence $\lambda_i \nearrow +\infty$ and any $t \in \mathbb{R}$,

$$(\mathcal{D}_{\mathbf{0},0;\lambda_i}\mu)_t \rightharpoonup \mu_t \quad \text{as } i \to \infty \text{ (weakly as measures).}$$

Clearly, the full tangent flow $(\mu_t)_{t\in\mathbb{R}}$ is quasistatic and μ_{-1}, μ_0, μ_1 are distinct.

Observe from Definition 2.11 that tangent flows of (μ_t) at (x_0,t_0) are rescaling limits $(\mathcal{D}_{x_0,t_0;\lambda_i}\mu)_t$ along *some* choice of sequence $\lambda_i \nearrow +\infty$. In general, it is an open question whether a tangent flow to (μ_t) at (x_0,t_0) is unique, that is, independent of the choice of sequence $\lambda_i \nearrow +\infty$. This uniqueness question applies to both full and backward tangent flows, and it is fundamental to understanding regularity of the flow (μ_t) near (x_0,t_0).

Observe that uniqueness of full tangent flows implies uniqueness of backward tangent flows. For compact or cylindrical tangent flows as in [5,19,35], uniqueness of backward tangent flows is equivalent to uniqueness of full tangent flows. Indeed, suppose a cylinder is the unique backward tangent flow to (μ_t) at (x_0, t_0) and let $(\tilde{\mu}_t)_{t\in\mathbb{R}}, (\tilde{\mu}'_t)_{t\in\mathbb{R}}$ be full tangent flows to (μ_t) at (x_0, t_0). Then $\tilde{\mu}_t, \tilde{\mu}'_t$ must both be equal to the self-similar flow of the cylinder for negative times $t < 0$, and therefore disappear as $t \nearrow 0$. It then follows that $\tilde{\mu}_t, \tilde{\mu}'_t$ are both identically the 0 measure for all $t \geq 0$, which thereby proves uniqueness of the full tangent flow. In fact, this argument shows moreover that if the restriction of a full tangent flow to negative times is given by a self-similarly shrinking compact submanifold or cylinder, then this backward tangent flow completely determines the full tangent flow. Thus full tangent flows can be canonically identified with backward tangent flows in this case, and there is no ambiguity in simply saying "compact or cylindrical tangent flow" to refer to either the backward or full tangent flow.

Often in the literature, authors study the uniqueness of tangent flows to smooth mean curvature flows $(M_t)_{t\in[0,T)}$ at points (x_0, T) at the first singular time $T < \infty$ or to Brakke flows $(\mu_t)_{t\in(a,b)}$ at points (x_0, b) at the final time $b < \infty$ (see e.g. [4, Theorem 1.1], [17, Theorem 1.1], and [16, Theorem 0.1]). In this case, only the notion of *backward* tangent flow makes sense since the parabolic rescalings $\lambda_i(M_{T+t\lambda_i^{-2}} - x_0)$ and $(\mathcal{D}_{x_0,T;\lambda_i}\mu)_t$ are undefined for $t \geq 0$. In particular, the tangent flow uniqueness results from [4, 16, 17] are strictly speaking backward tangent flow uniqueness results in terms of our notation from Definition 2.11. It is unclear if these results yield any associated uniqueness results for full tangent flows, except in perhaps some trivial settings such as when the Brakke flow $(\mu_t)_{t\in(a,b)}$ is extended by 0 for times $t \geq b$. Nonetheless, uniqueness of backward tangent flows yields important applications to understanding the dynamics of the flow (μ_t) near (x_0, t_0) for times $t < t_0$ (see e.g. [4, Corollary 1.2] and Corollary 3.9 below).

2.5. Equivalent densities and blow-up limits. A priori, the tangent *cones* from Lemma 2.8 could be entirely unrelated to the backward tangent *flows* from Lemma 2.10. Indeed, the two limiting objects arise from entirely different blow-up sequences. The next theorem, however, proves that the tangent cones from Lemma 2.8 exactly correspond to backward tangent flows from Lemma 2.10.

Theorem 2.13. *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n, \infty]$. Let $(V_t)_{t\in(a,b]}$ be the family of varifolds as in Lemma* 2.7.

For any $(x_0, t_0) \in U \times (a, b]$,

$$\theta_{V_{t_0}}(x_0) = \Theta_\mu(x_0, t_0).$$

For any sequence $\lambda_i \nearrow +\infty$, there exists a subsequence (still denoted λ_i) such that there are limiting integral n-varifolds $\mathbf{C}, \tilde{\mathbf{C}}$ as in Lemmas 2.8, 2.10 *respectively, and in fact $\mathbf{C} = \tilde{\mathbf{C}}$.*

Proof. By translation, we can assume without loss of generality that

$$(x_0, t_0) = (\mathbf{0}, 0) \in U \times (a, b].$$

For any sequence $\lambda_i \nearrow +\infty$, denote the rescalings

$$V_0^i \doteqdot (\eta_{\lambda_i})_\sharp V_0, \quad \mu_t^i \doteqdot (\mathcal{D}_{\lambda_i}\mu)_t.$$

Lemmas 2.8 and 2.10 imply there exists a subsequence (still denoted by index i) such that

$$V_0^i \rightharpoonup \mathbf{C}, \quad \text{and} \quad \mu_t^i \rightharpoonup \mu_{\widetilde{\mathbf{C}}} \quad (\text{for all } t < 0),$$

where $\mathbf{C}, \widetilde{\mathbf{C}}$ are stationary, dilation invariant, integral n-varifolds. Additionally,

$$\begin{aligned} \theta_{V_0}(\mathbf{0}) &= \frac{\mu_{\mathbf{C}}(B_r(0))}{\omega_n r^n} && (\text{for all } 0 < r < \infty), \\ \Theta_\mu(\mathbf{0}, 0) &= \frac{1}{(4\pi(-t))^{n/2}} \int e^{-\frac{|x|^2}{4(-t)}}\, d\mu_{\widetilde{\mathbf{C}}} && (\text{for all } t < 0). \end{aligned}$$

We first claim that $\mu_{\mathbf{C}} \le \mu_{\widetilde{\mathbf{C}}}$. Let $f \in C_c^1(\mathbb{R}^N)$ with $f \ge 0$. Say $\operatorname{supp} f \subseteq B_R$. It follows that

$$\begin{aligned} \mu_{\mathbf{C}}(f) &= \lim_{i\to\infty} \mu_{V_0^i}(f) \\ &= \lim_{i\to\infty} \lim_{t'\nearrow 0} \mu_{t'}^i(f) && (\text{Lemma 2.7}) \\ &\le \lim_{i\to\infty} \lim_{t'\nearrow 0} \Big(\mu_{-1}^i(f) + \int_{-1}^{t'} \int H_i \cdot \nabla f - |H_i|^2 f \, d\mu_t^i \, dt \Big) \\ &\le \lim_{i\to\infty} \mu_{-1}^i(f) + \lim_{i\to\infty} \int_{-1}^0 \int |H_i||\nabla f| \, d\mu_t^i \, dt && (f \ge 0) \\ &= \mu_{\widetilde{\mathbf{C}}}(f) + \lim_{i\to\infty} \int_{-1}^0 \int |H_i||\nabla f| \, d\mu_t^i \, dt. \end{aligned}$$

For any $0 < \delta \ll 1$, the integral term can be estimated as follows:

$$\begin{aligned} &\lim_{i\to\infty} \int_{-1}^0 \int |H_i||\nabla f| \, d\mu_t^i \, dt \\ &\quad \le \lim_{i\to\infty} \|f\|_{C^1} \int_{-1}^0 \int_{B_R} |H_i| \, d\mu_t^i \, dt \\ &\quad = \lim_{i\to\infty} \|f\|_{C^1} \lambda_i^{n-1} \int_{-1}^0 \int_{B_{R\lambda_i^{-1}}} |H| \, d\mu_{t\lambda_i^{-2}} \, dt \\ &\quad = \lim_{i\to\infty} \|f\|_{C^1} \lambda_i^{n+1} \int_{-\lambda_i^{-2}}^0 \int_{B_{R\lambda_i^{-1}}} |H| \, d\mu_\tau \, d\tau \quad (\tau = t\lambda_i^{-2}) \\ &\quad \le \lim_{i\to\infty} \|f\|_{C^1} \lambda_i^{n+1} \int_{-\lambda_i^{-2}}^0 \Big(\int_{B_{R\lambda_i^{-1}}} |H|^p \, d\mu_\tau \Big)^{1/p} \mu_\tau(B_{R\lambda_i^{-1}})^{\frac{p-1}{p}} \, d\tau \\ &\quad \le \|f\|_{C^1} \|H\|_{L^\infty L^p(B_\delta \times (a,b))} \lim_{i\to\infty} \lambda_i^{n-1} \big(\sup_{-\lambda_i^{-2} \le \tau \le 0} \mu_\tau(B_{R\lambda_i^{-1}}) \big)^{\frac{p-1}{p}} \\ &\quad = \|f\|_{C^1} \|H\|_{L^\infty L^p(B_\delta \times (a,b))} R^{\frac{n(p-1)}{p}} \lim_{i\to\infty} \lambda_i^{-1+\frac{n}{p}} \Big(\sup_{-\lambda_i^{-2} \le \tau \le 0} \frac{\mu_\tau(B_{R\lambda_i^{-1}})}{R^n \lambda_i^{-n}} \Big)^{\frac{p-1}{p}}. \end{aligned}$$

By Lemma 2.9, the

$$\frac{\mu_\tau(B_{R\lambda_i^{-1}})}{R^n \lambda_i^{-n}}$$

term is uniformly bounded above by say $C_0 < \infty$. Since $p > n$,

$$\mu_{\mathbf{C}}(f) \leq \mu_{\widetilde{\mathbf{C}}}(f) + C_0 \|f\|_{C^1} \|H\|_{L^\infty L^p(B_\delta \times (a,b))} R^{\frac{n(p-1)}{p}} C_0 \lim_{i \to \infty} \lambda_i^{-1+\frac{n}{p}} = \mu_{\widetilde{\mathbf{C}}}(f).$$

Thus $\mu_{\mathbf{C}}(f) \leq \mu_{\widetilde{\mathbf{C}}}(f)$ for all $f \in C_c^1(\mathbb{R}^N)$ with $f \geq 0$. It then follows from a limiting argument that $\mu_{\mathbf{C}} \leq \mu_{\widetilde{\mathbf{C}}}$.

Note that, since $\mathbf{C}$ is dilation invariant,

$$\frac{\mu_{\mathbf{C}}(B_r(\mathbf{0}))}{\omega_n r^n} = \frac{1}{(4\pi(-t))^{n/2}} \int e^{-\frac{|x|^2}{4(-t)}} \, d\mu_{\mathbf{C}} \quad \text{for all } 0 < -t, r < \infty$$

and similarly for $\widetilde{\mathbf{C}}$. Then $\mu_{\mathbf{C}} \leq \mu_{\widetilde{\mathbf{C}}}$ implies

$$\frac{\mu_{\mathbf{C}}(B_r(\mathbf{0}))}{\omega_n r^n} \leq \frac{\mu_{\widetilde{\mathbf{C}}}(B_r(\mathbf{0}))}{\omega_n r^n} \quad \text{for all } 0 < r < \infty.$$

We claim that the reverse inequality

$$\frac{\mu_{\mathbf{C}}(B_r(\mathbf{0}))}{\omega_n r^n} \geq \frac{\mu_{\widetilde{\mathbf{C}}}(B_r(\mathbf{0}))}{\omega_n r^n}$$

also holds (for some or equivalently all $0 < r < \infty$). First, observe there exists a time $\tau < 0$ such that $\mu_{\tau\lambda_i^{-2}}$ is an integral n-varifold with generalized mean curvature $H_{\tau\lambda_i^{-2}} \in L^p_{\mathrm{loc}}(U)$ for all i and

$$\|H_{\tau\lambda_i^{-2}}\|_{L^p(K)} \leq \|H\|_{L^\infty L^p(K \times (a,b))} < \infty \quad (\text{for all } K \Subset U),$$

where as usual H denotes the mean curvature of the flow (μ_t). Fix $R > 0$ such that

$$\overline{B}_R = \overline{B}_R(\mathbf{0}) \Subset U.$$

Let

$$F_i(\rho) \doteqdot e^{\|H_{\tau\lambda_i^{-2}}\|_{L^p(B_R)} \frac{1}{1-\frac{n}{p}} \rho^{1-\frac{n}{p}}} \quad \text{and} \quad \overline{F}(\rho) \doteqdot e^{\|H\|_{L^\infty L^p(B_R \times (a,b))} \frac{1}{1-\frac{n}{p}} \rho^{1-\frac{n}{p}}}$$

as in the monotonicity formula (A.3). Note $1 \leq F_i(\rho) \leq \overline{F}(\rho)$ for all $\rho > 0$. It follows from the monotonicity formula (A.3) that, for any $0 < \rho < R$ and any $r > 0$,

$$\begin{aligned}
r^{-n} \mu_{\widetilde{\mathbf{C}}}(B_r) &= \lim_{i \to \infty} r^{-n} \mu^i_\tau(B_r) = \lim_{i \to \infty} \frac{\mu_{\tau\lambda_i^{-2}}(B_{r\lambda_i^{-1}})}{r^n \lambda_i^{-n}} \\
&= \lim_{i \to \infty} \frac{1}{F_i(r\lambda_i^{-1})} \left(F_i(r\lambda_i^{-1}) \frac{\mu_{\tau\lambda_i^{-2}}(B_{r\lambda_i^{-1}})}{r^n \lambda_i^{-n}} + F_i(r\lambda_i^{-1}) - F_i(r\lambda_i^{-1}) \right) \\
&\leq \limsup_{i \to \infty} \frac{1}{F_i(r\lambda_i^{-1})} \left[F_i(\rho) \frac{\mu_{\tau\lambda_i^{-2}}(B_\rho)}{\rho^n} + F_i(\rho) - F_i(r\lambda_i^{-1}) \right] \quad \text{(A.3)} \\
&\leq \limsup_{i \to \infty} \left[\overline{F}(\rho) \frac{\mu_{\tau\lambda_i^{-2}}(B_\rho)}{\rho^n} + \overline{F}(\rho) - 1 \right] \\
&= \overline{F}(\rho) \frac{\mu_{V_0}(B_\rho)}{\rho^n} + \overline{F}(\rho) - 1.
\end{aligned}$$

Thus

$$\frac{\mu_{\widetilde{\mathbf{C}}}(B_r)}{r^n} \leq \overline{F}(\rho) \frac{\mu_{V_0}(B_\rho)}{\rho^n} + \overline{F}(\rho) - 1.$$

Taking $\rho \searrow 0$ reveals

$$\frac{\mu_{\widetilde{\mathbf{C}}}(B_r)}{r^n} \leq \lim_{\rho \searrow 0} \overline{F}(\rho) \frac{\mu_{V_0}(B_\rho)}{\rho^n} + \overline{F}(\rho) - 1 = \lim_{\rho \searrow 0} \frac{\mu_{V_0}(B_\rho)}{\rho^n} = \omega_n \theta_{V_0}(\mathbf{0}) = \frac{\mu_{\mathbf{C}}(B_r)}{r^n}.$$

In summary, we have shown that

$$\mu_{\mathbf{C}} \leq \mu_{\widetilde{\mathbf{C}}} \quad \text{and} \quad \frac{\mu_{\mathbf{C}}(B_r)}{\omega_n r^n} \geq \frac{\mu_{\widetilde{\mathbf{C}}}(B_r)}{\omega_n r^n} \quad \text{for all } 0 < r < \infty.$$

In particular, $\mu_{\mathbf{C}}(B_r) = \mu_{\widetilde{\mathbf{C}}}(B_r)$ for all $0 < r < \infty$. It follows that $\mathbf{C} = \widetilde{\mathbf{C}}$. Indeed, if not, then there exists a bounded subset $A \subseteq \mathbb{R}^N$ such that $\mu_{\mathbf{C}}(A) < \mu_{\widetilde{\mathbf{C}}}(A)$. Taking $R > 0$ large enough so that $B_R \supset A$ would then imply

$$\mu_{\mathbf{C}}(B_R) = \mu_{\mathbf{C}}(B_R \setminus A) + \mu_{\mathbf{C}}(A) < \mu_{\widetilde{\mathbf{C}}}(B_R \setminus A) + \mu_{\widetilde{\mathbf{C}}}(A) = \mu_{\widetilde{\mathbf{C}}}(B_R) = \mu_{\mathbf{C}}(B_R),$$

a contradiction.

Thus $\mathbf{C} = \widetilde{\mathbf{C}}$ and in particular $\theta_{V_0}(\mathbf{0}) = \Theta_\mu(\mathbf{0}, 0)$. This completes the proof. □

As an immediate consequence of Theorem 2.13, we deduce that the uniqueness of tangent *cones* of time-slices is equivalent to the uniqueness of backward tangent *flows*.

Corollary 2.14. *Let $(\mu_t)_{t \in (a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n, \infty]$. Let $(V_t)_{t \in (a,b]}$ be the family of varifolds as in Lemma* 2.7. *Let $(x_0, t_0) \in U \times (a,b]$. Then*

$$\{\mathbf{C} : \mathbf{C} \textit{ is a tangent cone of } V_{t_0} \textit{ at } x_0\} \\ = \{\mathbf{C} : (\mu_{\mathbf{C}})_{t<0} \textit{ is a backward tangent flow of } (\mu_t) \textit{ at } (x_0, t_0)\}.$$

In particular, the tangent cone V_{t_0} at x_0 is unique if and only if the backward tangent flow of (μ_t) at (x_0, t_0) is unique. In other words, $(\eta_{x_0,\lambda_i})_\sharp V_{t_0} \rightharpoonup \mathbf{C}$ for every sequence $\lambda_i \nearrow +\infty$ if and only if $(\mathcal{D}_{x_0,t_0;\lambda_i} \mu)_t \rightharpoonup \mu_{\mathbf{C}}$ (for all $t < 0$) for every sequence $\lambda_i \nearrow +\infty$.

Remark 2.15. We emphasize that Corollary 2.14 applies for backward tangent flows only, not full tangent flows. It may be possible that there exist an integral n-Brakke flow $(\mu_t)_{t \in (a,b)}$ as in Corollary 2.14 and $(x_0, t_0) \in U \times (a,b)$ such that there are a unique backward tangent flow to (μ_t) at (x_0, t_0) but distinct full tangent flows to (μ_t) at (x_0, t_0).

For integral n-Brakke flows $(\mu_t)_{t \in (a,b)}$ as in Corollary 2.14 and $(x_0, t_0) \in U \times (a,b)$, it is possible to adapt the arguments of Lemma 2.10 to show that parabolic rescalings $(\mathcal{D}_{x_0,t_0;\lambda_i} \mu)_t$ subsequentially weakly converge to integral n-Brakke flows $(\widetilde{\mu}_t)_{t \in \mathbb{R}}$ for all $t \in \mathbb{R}$ such that $(\widetilde{\mu})_{t \in \mathbb{R}}$ has generalized mean curvature $H \equiv 0$ and $\widetilde{\mu}_t \leq \widetilde{\mu}_{-1} = \mu_{\mathbf{C}}$ for all $t \in \mathbb{R}$, where $\mathbf{C}$ is a stationary, dilation invariant, integer rectifiable n-varifold in $\mathbb{R}^N$. We remark that if additionally $(\widetilde{\mu}_t)_{t \in \mathbb{R}}$ is unit regular, $\mathbf{C}$ has multiplicity one, and the regular part $\mathbf{C}_{reg}$ of $\mathbf{C}$ is connected, then the constancy theorem implies $\widetilde{\mu}_t = \mu_{\mathbf{C}}$ for all $t \in \mathbb{R}$. In this case, uniqueness of the backward tangent flow implies uniqueness of the full tangent flow.

Remark 2.16. Let $(\mu_t)_{t \in (a,b)}$ be an integral Brakke flow with mean curvature bounds as in Theorem 2.13, and consider the associated flow of varifolds $(V_t)_{t \in (a,b]}$ as in Lemma 2.7.

By combining Theorem 2.13 with White's local regularity theorem [33] (and its generalization to integral n-Brakke flows [27]), it follows that if $t_0 \in (a,b]$ and V_{t_0} also has *unit density* (i.e. $\theta_{V_{t_0}}(x) = 1$ for $\mu_{V_{t_0}}$-a.e. $x \in U$), then the singular set of the flow $\mathcal{M} = (V_t)_{t\in(a,t_0]}$ in the $t = t_0$ time-slice has n-dimensional Hausdorff measure 0, that is, $\mathcal{H}^n(\operatorname{sing}_{t_0}\mathcal{M}) = 0$. This recovers Brakke's main regularity theorem [3, 6.12] in the special case of flows with mean curvature bounds (see also [6, Theorem 5.3] and [14, Theorem 3.2]).

In particular, if V_t has unit density for all $t \in (a,b)$, then $\mathcal{H}^n(\operatorname{sing}_t \mathcal{M}) = 0$ for all $t \in (a,b)$. If additionally $U = \mathbb{R}^N$ and the flow $(\mu_{V_t})_{t\in(a,b)}$ extends to an integral n-Brakke flow $(\tilde{\mu}_t)_{t\in(a,\infty)}$ for all $t \in (a,\infty)$ such that $\tilde{\mu}$ is unit density, is unit regular, and has bounded area ratios, then [18, Theorem 1.1] applies to show that $t \mapsto \mu_{V_t}$ is continuous for all $t \in (a,b)$ in the sense that, for all $f \in C_c^2(\mathbb{R}^N)$ with $f \geq 0$,

$$t \mapsto \mu_{V_t}(f) \text{ is continuous on } (a,b).$$

Combining this fact with [12, Theorem 7.2 (ii)] and Lemma 2.7 gives

$$\mu_t \leq \mu_{V_t} = \lim_{s\searrow t} \mu_{V_s} = \lim_{s\searrow t} \mu_s \leq \mu_t \quad \text{for all } t \in (a,b).$$

Hence $\mu_t = \mu_{V_t}$ for all $t \in (a,b)$ in this case.

We refer the interested reader to [3,6,14,18] for precise definitions.

3. Uniqueness of tangent flows given by regular cones

Definition 3.1. Define the *link* $L(\mathbf{C})$ of a dilation invariant set $\mathbf{C} \subseteq \mathbb{R}^N$ to be

$$L(\mathbf{C}) \doteqdot \mathbf{C} \cap \mathbb{S}^{N-1}.$$

Then $\mathbf{C}$ is said to be a *regular cone* if $L(\mathbf{C})$ is a smooth, (properly) embedded submanifold of $\mathbb{S}^{N-1}$.

If $\mathbf{C} \subseteq \mathbb{R}^N$ is a regular cone, then $\mathbf{C} \setminus \{\mathbf{0}\}$ is a smooth submanifold of $\mathbb{R}^N$ and we write $\mathbf{C}^n$ when $\mathbf{C} \setminus \{\mathbf{0}\}$ has dimension n. Note that a regular cone $\mathbf{C}^n \subseteq \mathbb{R}^N$ naturally gives a dilation invariant integral n-varifold (of multiplicity one) with associated measure $\mathcal{H}^n \llcorner \mathbf{C}$ on $\mathbb{R}^N$.

The main result of this section is the following uniqueness result which restates Theorem 1.2.

Theorem 3.2. *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times (a,b))$. If (μ_t) has a backward tangent flow $\mu_{\mathbf{C}}$ at $(x_0,t_0) \in U \times (a,b]$ given by a regular cone $\mathbf{C}^n$ (with multiplicity one), then $\mu_{\mathbf{C}}$ is the unique backward tangent flow of (μ_t) at (x_0,t_0).*

The proof essentially follows from [22, §7, Theorem 5] applied to the time slice V_{t_0}. However, [22, §7, Theorem 5] requires a regularity assumption [22, equation (7.23)] which we must verify for V_{t_0}. Informally, this regularity assumption states that if V_{t_0} is a small $C^{1,\alpha}$-graph over the regular cone $\mathbf{C}$ in an annulus, then the mean curvature $H_{V_{t_0}}$ of V_{t_0} has interior C^2-bounds.

Since V_{t_0} comes from a Brakke flow, we can prove V_{t_0} satisfies this regularity assumption [22, equation (7.23)] as follows:

(1) show that $C^{1,\alpha}$-graphicality propagates outward in space and backward in time (Lemma 3.7), and

(2) apply interior estimates to improve the $C^{1,\alpha}$ bounds to C^∞ estimates for V_t and apply interior estimates to the evolution equation for $H = H_{V_t}$ to obtain $\|H\|_{C^2} \lesssim \|H\|_{C^0}$ (Lemma 3.8).

The remainder of this section rigorously carries out this argument to prove Theorem 3.2.

Definition 3.3. In what follows, we use $A_{r,R}(x_0)$ to denote the open annulus

$$A_{r,R}(x_0) = B_R(x_0) \setminus \overline{B}_r(x_0)$$

and $A_{r,R} = A_{r,R}(\mathbf{0})$.

For $\mathbf{C}^n \subseteq \mathbb{R}^N$ a regular cone, we slightly abuse notation and write $u\colon \mathbf{C} \cap A_{r,R} \to T^\perp\mathbf{C}$ to mean a function $u\colon \mathbf{C} \cap A_{r,R} \to \mathbb{R}^N$ such that $u(x) \in T_x^\perp\mathbf{C}$ for all $x \in \mathbf{C} \cap A_{r,R}$. For

$$u\colon \mathbf{C} \cap A_{r,R} \to T^\perp\mathbf{C},$$

denote

$$G_{r,R}(u) \doteqdot \left\{ \frac{x + u(x)}{\sqrt{1 + |u(x)|^2/|x|^2}} : x \in \mathbf{C} \cap A_{r,R} \right\} \subseteq \mathbb{R}^N.$$

Note that if $\mathbf{C}$ is a regular cone and $u\colon \mathbf{C} \cap A_{r,R} \to T^\perp\mathbf{C}$ is a C^2-function with $u(x)/|x|$ sufficiently small in C^1, then $G_{r,R}(u)$ is a properly embedded C^2-submanifold.

Definition 3.4. For $\mathbf{C}$ a regular cone, let $\|\cdot\|_{C_*^k}$ and $\|\cdot\|_{C_*^{k,\alpha}}$ $(0 < \alpha \le 1)$ denote the standard C^k and $C^{k,\alpha}$-norms respectively. For example, if $u\colon \mathbf{C} \cap \Omega \to \mathbb{R}^N$, then

$$\|u\|_{C_*^{k,\alpha}(\mathbf{C}\cap\Omega)} = \sum_{j=0}^{k} \sup_{x\in\mathbf{C}\cap\Omega} |\nabla_{\mathbf{C}}^j u|(x) + \sup_{x\neq y\in\mathbf{C}\cap\Omega} \frac{|\nabla_{\mathbf{C}}^k u(x) - \nabla_{\mathbf{C}}^k u(y)|}{|x-y|^\alpha}.$$

For $0 < r < R < \infty$ and $u\colon \mathbf{C} \cap A_{r,R} \to \mathbb{R}^N$, define scale-invariant C^k and $C^{k,\alpha}$-norms by

$$\begin{aligned}\|u\|_{C^k(\mathbf{C}\cap A_{r,R})} &\doteqdot \sum_{j=0}^{k} \sup_{x\in\mathbf{C}\cap A_{r,R}} |x|^{j-1} \sup_{y\in B_{\frac{|x|}{2}}(x)\cap\mathbf{C}\cap A_{r,R}} |\nabla_{\mathbf{C}}^j u|(y) \\ &= \sum_{j=0}^{k} \sup_{x\in\mathbf{C}\cap A_{r,R}} |x|^{j-1} \|\nabla_{\mathbf{C}}^j u\|_{C^0(B_{\frac{|x|}{2}}(x)\cap\mathbf{C}\cap A_{r,R})}\end{aligned}$$

$$\begin{aligned}\|u\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R})} &\doteqdot \|u\|_{C^k(\mathbf{C}\cap A_{r,R})} \\ &\quad + \sup_{x\in\mathbf{C}\cap A_{r,R}} |x|^{k-1+\alpha} \sup_{y\neq z\in B_{\frac{|x|}{2}}(x)\cap\mathbf{C}\cap A_{r,R}} \frac{|\nabla_{\mathbf{C}}^k u(y) - \nabla_{\mathbf{C}}^k u(z)|}{|y-z|^\alpha} \\ &= \|u\|_{C^k(\mathbf{C}\cap A_{r,R})} + \sup_{x\in\mathbf{C}\cap A_{r,R}} |x|^{k-1+\alpha} [\nabla^k u]_{C_*^\alpha(B_{\frac{|x|}{2}}(x)\cap\mathbf{C}\cap A_{r,R})},\end{aligned}$$

where $[\cdot]_{C_*^\alpha}$ denotes the standard C^α semi-norm.

For functions $u\colon \mathbf{C} \cap A_{r,R} \times (a,b) \to \mathbb{R}^N$ that also depend on time $t \in (a,b)$, denote backward parabolic neighborhoods as

$$P_r(x,t) \doteqdot B_r(x) \times (t - r^2, t)$$

and define scale-invariant C^k and $C^{k,\alpha}$-norms by

$$\begin{aligned}
&\|u\|_{C^k(\mathbf{C}\cap A_{r,R}\times(a,b))}\\
&\qquad \doteqdot \sum_{2i+j\le k} \sup_{(x,t)\in\mathbf{C}\cap A_{r,R}\times(a,b)} |x|^{2i+j-1}\|\partial_t^i\nabla_{\mathbf{C}}^j u\|_{C^0_*(P_{\frac{|x|}{2}}(x,t)\cap(\mathbf{C}\cap A_{r,R}\times(a,b)))}\\
&\qquad + \sum_{0<\frac{k-2i-j}{2}<1} \sup_{(x,t)\in\mathbf{C}\cap A_{r,R}\times(a,b)} |x|^{k-1+\alpha}[\partial_t^i\nabla_{\mathbf{C}}^j u]_{t,C_*^{\frac{k-2i-j}{2}}(P_{\frac{|x|}{2}}(x,t)\cap(\mathbf{C}\cap A_{r,R}\times(a,b)))},\\
&\|u\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R}\times(a,b))}\\
&\qquad \doteqdot \sum_{2i+j\le k} \sup_{(x,t)\in\mathbf{C}\cap A_{r,R}\times(a,b)} |x|^{2i+j-1}\|\partial_t^i\nabla_{\mathbf{C}}^j u\|_{C^0_*(P_{\frac{|x|}{2}}(x,t)\cap(\mathbf{C}\cap A_{r,R}\times(a,b)))}\\
&\qquad + \sum_{2i+j=k} \sup_{(x,t)\in\mathbf{C}\cap A_{r,R}\times(a,b)} |x|^{k-1+\alpha}[\partial_t^i\nabla_{\mathbf{C}}^j u]_{x,C_*^{\alpha}(P_{\frac{|x|}{2}}(x,t)\cap(\mathbf{C}\cap A_{r,R}\times(a,b)))}\\
&\qquad + \sum_{0<\frac{k+\alpha-2i-j}{2}<1} \sup_{(x,t)\in\mathbf{C}\cap A_{r,R}\times(a,b)} |x|^{k-1+\alpha}[\partial_t^i\nabla_{\mathbf{C}}^j u]_{t,C_*^{\frac{k+\alpha-2i-j}{2}}(P_{\frac{|x|}{2}}(x,t)\cap(\mathbf{C}\cap A_{r,R}\times(a,b)))}.
\end{aligned}$$

Here, $[\cdot]_{x,C^\alpha_*}$ and $[\cdot]_{t,C^\alpha_*}$ denote the standard C^α semi-norms in the variables x and t respectively. Namely,

$$\begin{aligned}
[u]_{x,C^\alpha_*(\mathbf{C}\times(a,b)\cap\Omega)} &\doteqdot \sup_{(x,t)\neq(x',t)\in\mathbf{C}\times(a,b)\cap\Omega} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha},\\
[u]_{t,C^\alpha_*(\mathbf{C}\times(a,b)\cap\Omega)} &\doteqdot \sup_{(x,t)\neq(x,t')\in\mathbf{C}\times(a,b)\cap\Omega} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha}.
\end{aligned}$$

Since we use the $C^{1,\alpha}$-norm most often, we note explicitly that

$$\begin{aligned}
\|u\|_{C^{1,\alpha}(\mathbf{C}\cap A_{r,R}\times(a,b))} = \sup_{x,t}|x|^{-1} &\sup_{(y,s)\in P_{\frac{|x|}{2}}(x,t)} |u(y,s)| + \sup_{x,t} \sup_{(y,s)\in P_{\frac{|x|}{2}}(x,t)} |\nabla_{\mathbf{C}}u(y,s)|\\
&+ \sup_{x,t}|x|^\alpha \sup_{(y,s)\neq(y',s)\in P_{\frac{|x|}{2}}(x,t)} \frac{|\nabla_{\mathbf{C}}u(y,s)-\nabla_{\mathbf{C}}u(y',s)|}{|y-y'|^\alpha}\\
&+ \sup_{x,t}|x|^\alpha \sup_{(y,s)\neq(y,s')\in P_{\frac{|x|}{2}}(x,t)} \frac{|u(y,s)-u(y,s')|}{|s-s'|^{\frac{1+\alpha}{2}}}
\end{aligned}$$

where also the suprema above are restricted to points in $\mathbf{C}\cap A_{r,R}\times(a,b)$.

Remark 3.5. While the $C^{k,\alpha}$-norms defined above are somewhat non-standard, they have been chosen so that they satisfy the following properties.

(1) (Parabolic scaling invariance) If $u\colon \mathbf{C}\cap A_{r,R}\times(a,b)\to\mathbb{R}^N$, $\lambda>0$, and

$$\tilde{u}\colon \mathbf{C}\cap A_{\lambda r,\lambda R}\times(\lambda^2 a,\lambda^2 b)\to\mathbb{R}^N$$

is given by $\tilde{u}(x,t)=\lambda u(x/\lambda,t/\lambda^2)$, then

$$\|\tilde{u}\|_{C^{k,\alpha}(\mathbf{C}\cap A_{\lambda r,\lambda R}\times(\lambda^2 a,\lambda^2 b))} = \|u\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R}\times(a,b))}.$$

(2) (Time translation invariance) If $u\colon \mathbf{C}\cap A_{r,R}\times(a,b)\to\mathbb{R}^N$, $t_0\in\mathbb{R}$, and

$$\tilde{u}\colon \mathbf{C}\cap A_{r,R}\times(a+t_0,b+t_0)\to\mathbb{R}^N$$

is given by $\widetilde{u}(x,t) = u(x, t - t_0)$, then

$$\|\widetilde{u}\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R}\times(a+t_0,b+t_0))} = \|u\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R}\times(a,b))}.$$

(3) (Equivalent to standard Hölder norms) There exists $C = C(r, R, k, \alpha)$ such that

$$C^{-1}\|u\|_{C^{k,\alpha}_*(\mathbf{C}\cap A_{r,R}\times(a,b))} \leq \|u\|_{C^{k,\alpha}(\mathbf{C}\cap A_{r,R}\times(a,b))} \leq C\|u\|_{C^{k,\alpha}_*(\mathbf{C}\cap A_{r,R}\times(a,b))}.$$

Similar properties hold for the space-time C^k-norms and the spatial C^k and $C^{k,\alpha}$-norms. The proofs of these properties are left as exercises to the reader.

Remark 3.6. The choice of using radius $|x|/2$ balls in the above definitions was somewhat arbitrary. Indeed, it can be shown through a covering argument that, for any $L > 1$, replacing "$|x|/2$" with "$|x|/L$" in the above definitions gives an equivalent norm, that is, the norms differ by a factor of $C = C(k, \alpha, L)$.

Lemma 3.7 (Graphicality propagates out and back). *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^p_{\mathrm{loc}}(U \times (a,b))$ for some $p \in (n, \infty]$. Let $(V_t)_{t\in(a,b]}$ be the associated integral n-Brakke flow from Lemma 2.7. Let $(x_0, t_0) \in U \times (a, b]$ and let $\mathbf{C}^n \subseteq \mathbb{R}^N$ be a regular cone. For any $\epsilon > 0$, there exists $r_0, \delta > 0$ such that the following holds for all $0 < \rho < r_0$: if*

$$(V_{t_0} - x_0) \cap A_{\rho/2,\rho} = G_{\rho/2,\rho}(u)$$

for some $u\colon \mathbf{C} \cap A_{\rho/2,\rho} \to T^\perp C$ with $\|u\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho/2,\rho})} \leq \delta$, then

$$(V_t - x_0) \cap A_{\rho/4,2\rho} = G_{\rho/4,2\rho}(\widetilde{u}(\cdot, t)) \quad (\textit{for all } t \in [t_0 - 4\rho^2, t_0])$$

for some extension $\widetilde{u}\colon (\mathbf{C} \cap A_{\rho/4,2\rho}) \times [t_0 - 4\rho^2, t_0] \to T^\perp\mathbf{C}$ of u with

$$\|\widetilde{u}\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho/4,2\rho}\times[t_0-4\rho^2,t_0])} \leq \epsilon.$$

Proof. By translation, assume without loss of generality that $(x_0, t_0) = (\mathbf{0}, 0)$. Suppose the lemma were false for the sake of contradiction. Then we can take a sequence $r_i = \delta_i \searrow 0$ and obtain $\rho_i \in (0, r_i)$ where the implication fails. That is, $V_0 \cap A_{\rho_i/2,\rho_i} = G_{\rho_i/2,\rho_i}(u_i)$ is a $C^{1,\alpha}$-graph over $\mathbf{C} \cap A_{\rho_i/2,\rho_i}$ with $\|u_i\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho_i/2,\rho_i})} \leq \delta_i$, but (V_t) is not a $C^{1,\alpha}$-graph over $\mathbf{C} \cap A_{\rho_i/4,2\rho_i} \times [t_0 - 4\rho_i^2, t_0]$ with $C^{1,\alpha}$-norm bounded by ϵ in this region.

Parabolically dilate V_t by $\lambda_i \doteqdot \frac{1}{\rho_i} \to +\infty$ to obtain $V^i_t \doteqdot (\eta_{\lambda_i})_\sharp V_{t\lambda_i^{-2}}$ and set $\mu^i_t \doteqdot \mu_{V^i_t}$. After passing to a subsequence, Theorem 2.13 applied to the Brakke flow (μ_{V_t}) implies there exists a stationary, dilation invariant varifold $\mathbf{C}'$ such that

$$V^i_0 \rightharpoonup \mathbf{C}' \quad \text{and} \quad \mu^i_t \rightharpoonup \mu_{\mathbf{C}'} \quad (\text{for all } t < 0)$$

as $i \to \infty$. Since $\mathbf{C}'$ is dilation invariant and

$$\begin{aligned} &V^i_0 \cap A_{1/2,1} = G_{1/2,1}(\lambda_i u_i(\cdot/\lambda_i)) \\ &\qquad \text{with } \|\lambda_i u_i(\cdot/\lambda_i)\|_{C^{1,\alpha}(\mathbf{C}\cap A_{1/2,1})} = \|u_i\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho_i/2,\rho_i})} \leq \delta_i \to 0, \end{aligned}$$

it follows that, in fact, $\mathbf{C}' = \mathbf{C}$. If $\mathbf{C}$ is not stationary, then we contradict $\mathbf{C} = \mathbf{C}'$ and the proof is complete. Thus it suffices to the consider the case where $\mathbf{C}$ is stationary for the remainder of the proof.

Now the stationary flow $(\mu_t^\infty = \mu_{\mathbf{C}})_{t\leq 0}$ given by $\mathbf{C}$ has Gaussian density

$$\Theta_{\mu^\infty}(x,t) = 1 \quad \text{for all } (x,t) \in \overline{A_{1/6,4}} \times [-6,0]$$

since $\mathbf{C}$ has smooth link. The upper semi-continuity of Gaussian density then implies that, for any $\sigma > 0$,

$$\Theta_{\mu^i}(x,t) < 1 + \sigma \quad \text{for all } (x,t) \in \overline{A_{1/6,4}} \times [-6,0],$$

for all $i \gg 1$ sufficiently large.

By White's local regularity theorem [33] (and its generalization to integral Brakke flows in [27]), it follows that, for $i \gg 1$, V_t^i is a smooth mean curvature flow in $A_{1/6,4} \times [-6,0]$ with second fundamental form bounded by a dimensional constant

$$C = C(N) < \infty \quad \text{in } A_{1/5,3} \times [-5,0].$$

Interior regularity for mean curvature flow then implies that the convergence

$$V_t^i \xrightarrow[i\to\infty]{} \mathbf{C}$$

is smooth on $A_{1/4,2} \times [-4,0]$. It follows that, for all $i \gg 1$,

$$V_t^i \cap A_{1/4,2} = G_{1/4,2}(\widetilde{w}_i(\,\cdot\,,t)) \quad (\text{for all } t \in [-4,0])$$

for some $\widetilde{w}_i \colon \mathbf{C} \cap A_{1/4,2} \times [-4,0] \to T^\perp C$ extending $\lambda_i u_i(\,\cdot\,/\lambda_i)$ with

$$\|\widetilde{w}_i\|_{C^{3,\alpha}(C\cap A_{1/4,2}\times[-4,0])} \xrightarrow[i\to\infty]{} 0.$$

Since

$$\|\widetilde{w}_i(\,\cdot\,,0)\|_{C^{1,\alpha}(\mathbf{C}\cap A_{1/2,1})} = \|\lambda_i \widetilde{u}_i(\,\cdot\,/\lambda_i)\|_{C^{1,\alpha}(\mathbf{C}\cap A_{1/2,1})} \leq \delta_i \to 0,$$

and derivatives of $\widetilde{w}_i$ converge to 0 on $A_{1/4,2} \times [-4,0]$, we have that, in fact,

$$\|\widetilde{w}_i\|_{C^{1,\alpha}(\mathbf{C}\cap A_{1/4,2}\times[-4,0])} \leq \epsilon$$

for all $i \gg 1$. Undoing dilations gives that, for $i \gg 1$,

$$V_t \cap A_{\rho_i/4,2\rho_i} = G_{\rho_i/4,2\rho_i}(\widetilde{u}_i(\,\cdot\,,t)) \quad (\text{for all } t \in [-4\rho_i^2,0])$$

for $\widetilde{u}_i(x,t) = \frac{1}{\lambda_i}\widetilde{w}_i(x\lambda_i, t\lambda_i^2)\colon \mathbf{C} \cap A_{\rho_i/4,2\rho_i} \to T^\perp\mathbf{C}$ extending u_i and

$$\|\widetilde{u}_i\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho_i/4,2\rho_i}\times[-4\rho_i^2,0])} \leq \epsilon,$$

which contradicts the choice of the r_i, δ_i, ρ_i. □

Lemma 3.8. *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times (a,b))$, and let $(V_t)_{t\in(a,b]}$ be the associated family of varifolds as in Lemma* 2.7. *Let $(x_0,t_0) \in U \times (a,b]$ and let $\mathbf{C}$ be a regular cone. There exist $\beta, C, r_0 > 0$ such that, for all $0 < \rho < r_0$ the following holds: if*

$$(V_{t_0} - x_0) \cap A_{\rho/2,\rho} = G_{\rho/2,\rho}(u)$$

for some $u\colon \mathbf{C} \cap A_{\rho/2,\rho} \to T^{\perp}\mathbf{C}$ *with* $\|u\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho/2,\rho})} \leq \beta$*, then for any* $0 < \sigma < \rho$,

$$|\nabla_{V_{t_0}} H_{V_{t_0}}| \leq \frac{C\|H\|_{L^\infty L^\infty(A_{\rho/2,\rho}(x_0)\times(t_0-\rho^2,t_0))}}{\sigma} \leq \frac{C\|H\|_{L^\infty L^\infty(B_{r_0}(x_0)\times(t_0-r_0^2,t_0))}}{\sigma} < \infty$$

and

$$|\nabla^2_{V_{t_0}} H_{V_{t_0}}| \leq \frac{C\|H\|_{L^\infty L^\infty(A_{\rho/2,\rho}(x_0)\times(t_0-\rho^2,t_0))}}{\sigma^2} \leq \frac{C\|H\|_{L^\infty L^\infty(B_{r_0}(x_0)\times(t_0-r_0^2,t_0))}}{\sigma^2} < \infty$$

on $V_{t_0} \cap A_{\rho/2+\sigma,\rho-\sigma}(x_0)$.

Proof. Throughout, we assume $0 < r_0 \ll 1$ is small enough so that $\overline{B}_{2r_0}(x_0) \Subset U$ and $(t_0 - 4r_0^2, t_0) \subseteq (a,b)$. By translation, assume without loss of generality that $(x_0,t_0) = (\mathbf{0},0)$. Assume $0 < \rho < r_0$ and

$$V_0 \cap A_{\rho/2,\rho} = G_{\rho/2,\rho}(u)$$

for some $u\colon \mathbf{C} \cap A_{\rho/2,\rho} \to T^{\perp}\mathbf{C}$ with $\|u\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho/2,\rho})} \leq \beta$.

By Lemma 3.7, for any $\epsilon > 0$, we can assume $\beta, r_0 \ll 1$ are sufficiently small (depending on ϵ) so that

$$V_t \cap A_{\rho/4,2\rho} = G_{\rho/4,2\rho}(\widetilde{u}(\cdot,t)) \quad \text{for all } t \in [-4\rho^2, 0],$$

for some extension $\widetilde{u}\colon \mathbf{C} \cap A_{\rho/4,2\rho} \times [-4\rho^2,0] \to T^{\perp}\mathbf{C}$ of u with

$$\|\widetilde{u}\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\rho/4,2\rho}\times[-4\rho^2,0])} \leq \epsilon.$$

Consider the parabolically rescaled flow $W_t \doteqdot (\eta_{1/\rho})_\sharp V_{t\rho^2}$ and note

$$W_t \cap A_{1/4,2} = G_{1/4,2}(\widetilde{w}(\cdot,t)) \quad \text{for all } t \in [-4,0],$$

where

$$\widetilde{w}(x,t) \doteqdot \frac{1}{\rho}\widetilde{u}(x\rho, t\rho^2), \quad \|\widetilde{w}\|_{C^{1,\alpha}(A_{1/4,2}\times[-4,0])} = \|\widetilde{u}\|_{C^{1,\alpha}(A_{\rho/4,2\rho}\times[-4\rho^2,0])} \leq \epsilon.$$

If $\epsilon = \epsilon(N,\mathbf{C})$ is sufficiently small (depending only on N and $\mathbf{C}$), interior estimates (see e.g. [27, 28]) imply that W_t is a smooth mean curvature flow on $A_{1/3,3/2} \times [-3,0]$ with derivative bounds on the second fundamental form $A = A_{W_t}$ of the form

$$\sup_{(x,t)\in A_{1/3,3/2}\times[-3,0]} |\nabla^k_{W_t} A_{W_t}| \leq C_k = C_k(N,\mathbf{C}) \quad (\text{for all } k \in \mathbb{N}). \tag{3.1}$$

In this region $A_{1/3,3/2} \times [-3,0]$ where W_t is a smooth mean curvature flow, the mean curvature $H = H_{W_t}$ satisfies an evolution equation of the form

$$\partial_t H = \Delta H + \nabla A * H + A * \nabla H + A * A * H \tag{3.2}$$

(see [24, Corollary 3.8]). The bounds (3.1) imply (3.2) is a linear parabolic PDE system for H_{W_t} in the domain $A_{1/3,3/2} \times [-3,0]$ with uniform C^k-bounds on the coefficients that depend only on N, $\mathbf{C}$, and k. Interior estimates for parabolic systems (see e.g. [15]) therefore imply that, for some $C = C(N, \mathbf{C})$,

$$\sup_{A_{1/2+\sigma,1-\sigma}} |\nabla^2_{W_0} H_{W_0}| \leq \frac{C}{\sigma^2} \sup_{A_{1/2,1}\times[-1,0]} |H_{W_t}| \quad (\text{for all } 0 < \sigma < 1). \tag{3.3}$$

In terms of V_t, (3.3) becomes

$$\begin{aligned}
\sup_{A_{\rho/2+\sigma,\rho-\sigma}} |\nabla^2_{V_0} H_{V_0}| &= \frac{1}{\rho^3} \sup_{A_{1/2+\sigma/\rho,1-\sigma/\rho}} |\nabla^2_{W_0} H_{W_0}| \\
&\leq \frac{C}{(\sigma/\rho)^2\rho^3} \sup_{A_{1/2,1}\times[-1,0]} |H_{W_t}| \qquad (3.3) \\
&= \frac{C}{\sigma^2\rho} \sup_{A_{1/2,1}\times[-1,0]} |H_{W_t}| \\
&= \frac{C}{\sigma^2} \sup_{A_{\rho/2,\rho}\times[-\rho^2,0]} |H_{V_t}| \\
&\leq \frac{C}{\sigma^2} \|H\|_{L^\infty L^\infty(A_{\rho/2,\rho}\times(-\rho^2,0))} \qquad (\text{Lemma 2.7})
\end{aligned}$$

for all $0 < \sigma < \rho$. Note that, in the last line, H denotes the mean curvature of the Brakke flow $(\mu_t)_{t\in(a,b)}$. An analogous argument applies to estimate $|\nabla_{V_0} H_{V_0}|$. □

The conclusion of Lemma 3.8 is effectively the same statement as [22, (7.23)]. We can now prove Theorem 3.2 by adapting the argument [22, §7] used for tangent cones of stationary varifolds.

Proof of Theorem 3.2. Throughout, we use $(V_t)_{t\in(a,b]}$ to denote the associated family of varifolds given by Lemma 2.7. By translation, assume without loss of generality that $(x_0, t_0) = (\mathbf{0}, 0)$. By Corollary 2.14, it suffices to show $\mathbf{C}$ is the unique tangent cone of V_0 at $\mathbf{0}$.

By Theorem 2.13, there exists some sequence $\lambda_k \nearrow +\infty$ such that

$$V_k \doteqdot (\eta_{\lambda_k})_\sharp V_0 \rightharpoonup \mathbf{C}.$$

and $\theta_{V_k}(\mathbf{0}) = \mu_{\mathbf{C}}(B_1)/\omega_n$ for all k. Note that the mean curvature H_{V_k} of V_k is bounded by

$$\|H_{V_k}\|_{L^\infty(B_2)} = \frac{1}{\lambda_k}\|H_{V_0}\|_{L^\infty(B_{2/\lambda_k})} \leq \frac{1}{\lambda_k}\|H_{V_t}\|_{L^\infty L^\infty(B_\delta\times(-\delta,0])} \xrightarrow[k\to\infty]{} 0$$

so long as $\overline{B_\delta} \times (-\delta, 0) \subseteq U \times (a, b)$ and k is large enough to ensure $2/\lambda_k < \delta$. Since $\mathbf{C}$ is smooth away from $\mathbf{0}$, Allard's regularity theorem [1] (see also [23, Chapter 5]) implies that $V_k \cap A_{1/2,1}$ is smooth for all $k \gg 1$ sufficiently large, and the convergence

$$V_k \cap A_{1/2,1} \xrightarrow[k\to\infty]{} \mathbf{C}$$

is smooth. In particular, for $k \gg 1$, $V_k \cap A_{1/2,1} = G_{1/2,1}(u_k)$ for some $u_k\colon A_{1/2,1} \to T^\perp\mathbf{C}$ with

$$\|u_k\|_{C^{1,\alpha}(A_{1/2,1})} \xrightarrow[k\to\infty]{} 0.$$

Lemma 3.8 implies V_k satisfies the property [22, (7.23)] and therefore [22, §7, Theorem 5] applies. Specifically, [22, §7, Theorem 5] gives that, for $k \gg 1$, $V_k \cap B_1 \setminus \{\mathbf{0}\} = G_{0,1}(\widetilde{u}_k)$ for some extension $\widetilde{u}_k \in C^2(\mathbf{C} \cap B_1 \setminus \{\mathbf{0}\})$ of u_k that satisfies

$$\lim_{\rho \searrow 0} \frac{\widetilde{u}_k(\rho\omega)}{\rho} = \zeta_k(\omega) \quad (\omega \in L(\mathbf{C})),$$

where $\zeta_k \in C^2(L(\mathbf{C}))$ and where the convergence is in the $C^2(L(\mathbf{C}))$-norm. Since the V_k are all dilations of V_0, it follows that

$$\widetilde{u}_k(x) = \frac{\lambda_k}{\lambda_l}\widetilde{u}_l\Big(\frac{\lambda_l}{\lambda_k}x\Big) \quad \text{for all } k, l \gg 1,$$

and thus $\zeta_k \equiv 0$ for all $k \gg 1$. Undoing the dilations reveals that, for a fixed k suitably large,

$$(3.4) \quad V_0 \cap B_{1/\lambda_k} \setminus \{\mathbf{0}\} = G_{0,1/\lambda_k}\Big(\frac{1}{\lambda_k}\widetilde{u}_k(\lambda_k x)\Big) \quad \text{and} \quad \frac{\frac{1}{\lambda_k}\widetilde{u}_k(\lambda_k\rho\omega)}{\rho} \xrightarrow[\rho\searrow 0]{C^2(L(\mathbf{C}))} 0.$$

The uniqueness of the tangent cone now follows. □

As a corollary, we note that Theorem 3.2 implies the flow may be written as a graph over the cone in certain space-time regions near the singularity.

Corollary 3.9. *Let $(\mu_t)_{t\in(a,b)}$ be an integral n-Brakke flow in $U \subseteq \mathbb{R}^N$ with locally uniformly bounded areas and generalized mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times (a,b))$, and let $(V_t)_{t\in(a,b]}$ be the associated integral n-Brakke flow from Lemma 2.7. Assume (μ_t) has a backward tangent flow $\mu_{\mathbf{C}}$ at $(x_0, t_0) \in U \times (a,b]$ given by a regular cone $\mathbf{C}^n$. For any $\epsilon, C > 0$, there exists $r > 0$ such that*

$$(V_t - x_0) \cap A_{\frac{\sqrt{t_0-t}}{C},r} = G_{\frac{\sqrt{t_0-t}}{C},r}(u(\cdot\,,t)) \quad \textit{for all } t \in [t_0 - r^2, t_0],$$

for some function

$$u\colon \Omega \doteqdot \{(x,t) \in \mathbf{C} \times [t_0 - r^2, t_0] : x \in A_{\frac{\sqrt{t_0-t}}{C},r}\} \to T^\perp\mathbf{C} \quad \textit{with } \|u\|_{C^{1,\alpha}(\Omega)} \le \epsilon.$$

Proof. For any $\delta > 0$, the proof of Theorem 3.2, namely (3.4), implies there exists $r > 0$ such that $(V_{t_0} - x_0) \cap B_r \setminus \{\mathbf{0}\}$ can be written as a graph over $\mathbf{C}$ with $C^{1,\alpha}$-norm bounded by δ. Let $\epsilon, C > 0$ be given. If $\delta = \delta(\epsilon, C) \ll 1$ is sufficiently small and r is possibly made smaller, then it follows from Lemma 3.7 that, for any $0 < \rho < r$, $V_t - x_0$ is a graph over $\mathbf{C}$ on the region $A_{\rho/2,\rho} \times [t_0 - C^2\rho^2, t_0]$ with $C^{1,\alpha}$-norm bounded by $\epsilon > 0$. The statement now follows by taking a union over $\rho \in (0, r)$. □

4. Pinching Hardt–Simon minimal surfaces

Throughout this section, we restrict to the case where $(M_t^n)_{t\in[-T,0)}$ is a smooth mean curvature flow of properly embedded hypersurfaces in an open subset $U \subseteq \mathbb{R}^{n+1}$. We denote by $\mathcal{M} = \bigcup_{t\in[-T,0)} M_t \times \{t\} \subseteq \mathbb{R}^{n+1} \times \mathbb{R}$ its space-time track. Fix a regular cone $\mathbf{C}_0^n \subseteq \mathbb{R}^{n+1}$ and let $\mathcal{C} = \{A \cdot \mathbf{C}_0 : A \in O(n+1)\}$ denote all rotations of the cone. We generally use $\mathbf{C}$ to denote a rotation of the cone $\mathbf{C}_0$, that is, $\mathbf{C} \in \mathcal{C}$.

The goal of this section is to prove Theorem 1.3, which says that if $\mathcal{M}$ has bounded H and develops a singularity with backward tangent flow given by an area-minimizing quadratic cone, then there is a type II blow-up limit given by a Hardt–Simon minimal surface (see Section 4.2 for the relevant definitions).

Remark 4.1. The example of the static flow of an area-minimizing quadratic cone shows Theorem 1.3 is false when $\mathcal{M} = (M_t^n)_{t\in(a,b)}$ is replaced with an integral n-Brakke flow $(\mu_t)_{t\in(a,b)}$.

We restrict to the case of smooth mean curvature flows here because the arguments in this section and the proof of Theorem 1.3 make reference to the second fundamental form and rely on a pseudolocality theorem for mean curvature flows (see Lemma 4.4 and Lemma 4.5 below in particular). The pseudolocality theorem fails for general integral n-Brakke flows because Brakke flows may suddenly vanish.

4.1. Flows near a regular cone. We begin with general results for mean curvature flows locally close to a regular cone $\mathbf{C} \in \mathcal{C}$. The approach here was inspired by [17, Section 8].

Definition 4.2. We say $\mathcal{M}$ is *ϵ-close to* $\mathbf{C} \in \mathcal{C}$ if $\mathcal{M}$ is a $C^{1,\alpha}$-graph on

$$\mathbf{C} \cap A_{\epsilon,\epsilon^{-1}} \times [-\epsilon^{-2}, -\epsilon^2]$$

with $C^{1,\alpha}$-norm at most ϵ. In other words,

$$M_t \cap A_{\epsilon,\epsilon^{-1}} = G_{\epsilon,\epsilon^{-1}}(u(\cdot\,,t)) \quad \text{for all } t \in [-\epsilon^{-2}, -\epsilon^2],$$

for some $u\colon \mathbf{C} \cap A_{\epsilon,\epsilon^{-1}} \times [-\epsilon^{-2}, -\epsilon^2] \to T^{\perp}\mathbf{C}$ with

$$\|u\|_{C^{1,\alpha}(\mathbf{C}\cap A_{\epsilon,\epsilon^{-1}}\times[-\epsilon^{-2},-\epsilon^2])} \le \epsilon.$$

We say $\mathcal{M}$ is *ϵ-close to* $\mathbf{C} \in \mathcal{C}$ *at* $X = (x,t)$ if $\mathcal{M} - X$ is ϵ-close to $\mathbf{C}$.

Throughout the remainder of this section, ϵ is always assumed to be less than some small constant $\epsilon_0 = \epsilon_0(n, \mathbf{C}_0)$ that depends only on the regular cone $\mathbf{C}_0$ and implicitly its dimension n.

Definition 4.3. Suppose $\mathcal{M}$ is ϵ-close to $\mathbf{C} \in \mathcal{C}$ at X for some $\epsilon \le \epsilon_0$. Define

$$1 \in [\lambda_*(X), \lambda^*(X)] \subseteq [0,\infty]$$

to be the largest interval such that, for all $\lambda \in [\lambda_*(X), \lambda^*(X)]$, $\mathcal{D}_{\lambda^{-1}}(\mathcal{M} - X)$ is ϵ-close to some $\mathbf{C}' = \mathbf{C}'_\lambda \in \mathcal{C}$ at $(\mathbf{0}, 0)$.

Occasionally, we may write $\lambda_*(X; \mathcal{M}, \epsilon)$ or $\lambda^*(X; \mathcal{M}, \epsilon)$ to emphasize the dependence on $\mathcal{M}$ and ϵ. Observe that $\lambda_*(X)$ and $\lambda^*(X)$ are continuous in the basepoint X.

We note the following consequence of pseudolocality for mean curvature flow in our setting.

Lemma 4.4. *Define constants*

$$0 < c_{\mathbf{C}_0} \doteqdot \frac{1}{2} \sup_{x\in L(\mathbf{C}_0)} |A_{\mathbf{C}_0}|(x) < C_{\mathbf{C}_0} \doteqdot 2 \sup_{x\in L(\mathbf{C}_0)} |A_{\mathbf{C}_0}|(x) < \infty.$$

There exists $C = C(n, \mathbf{C}_0) \gg 1$ and $\epsilon_0 = \epsilon_0(n, \mathbf{C}_0) \ll 1$ such that if $\mathcal{M}$ is ϵ-close to $\mathbf{C} \in \mathcal{C}$ at $X = (x,t)$ for some $\epsilon \le \epsilon_0$, then

$$\frac{c_{\mathbf{C}_0}}{\rho} \le \sup_{|y-x|=\rho} |A_{M_t}|(y) \le \frac{C_{\mathbf{C}_0}}{\rho} \quad \textit{for all } (\epsilon + C)\lambda_*(X) \le \rho \le (\epsilon^{-1} - C)\lambda^*(X). \tag{4.1}$$

Proof. Let $\lambda \in [\lambda_*(X), \lambda^*(X)]$ and consider $\mathcal{M}' \doteqdot \mathcal{D}_{\lambda^{-1}}(\mathcal{M} - X)$. By definition, $\mathcal{M}'$ can be written as a $C^{1,\alpha}$-graph over some $\mathbf{C}' \in \mathcal{C}$ on $A_{\epsilon,\epsilon^{-1}} \times [-\epsilon^{-2}, -\epsilon^2]$, and the $C^{1,\alpha}$-norm is bounded by $\epsilon \leq \epsilon_0$.

Let $\delta = \delta(n, \mathbf{C}_0) \ll 1$ denote some small constant to be determined, which depends only on n and $\mathbf{C}_0$. If $\epsilon_0 \ll 1$ is sufficiently small (depending also on δ) and $C \gg 1$ is sufficiently large (depending also on δ), then pseudolocality for mean curvature flow [13, Theorem 1.5] implies that $\mathcal{M}'$ can be written as a Lipschitz graph over $\mathbf{C}'$ on $A_{\epsilon+C,\epsilon^{-1}-C} \times [-\epsilon^{-2}, 0]$. Namely, if M'_t denotes the time t time-slice of $\mathcal{M}'$, then

$$M'_t \cap A_{\epsilon+C,\epsilon^{-1}-C} = G_{\epsilon+C,\epsilon^{-1}-C}(u(\cdot,t)) \quad \text{for all } t \in [-\epsilon^{-2}, 0],$$

for some $u \colon \mathbf{C}' \cap A_{\epsilon+C,\epsilon^{-1}-C} \times [-\epsilon^{-2}, 0] \to T^\perp \mathbf{C}'$ with

$$\begin{aligned} &\|u(\cdot,t)\|_{C^{0,1}(\mathbf{C}' \cap A_{\epsilon+C,\epsilon^{-1}-C})} \\ &\quad \leq C'\Bigg(\sup_{x \in \mathbf{C}' \cap A_{\epsilon+C,\epsilon^{-1}-C}} \frac{|u(x,t)|}{|x|} + \sup_{x \neq y \in \mathbf{C}' \cap A_{\epsilon+C,\epsilon^{-1}-C}} \frac{|u(x,t) - u(y,t)|}{|x-y|}\Bigg) \leq C'\delta \\ &\quad\quad \text{for all } t \in [-\epsilon^{-2}, 0] \text{ (where } C' \text{ is a universal constant).} \end{aligned}$$

Interior estimates for mean curvature flow [7] then imply u satisfies C^2-estimates on

$$\mathbf{C}' \cap A_{\epsilon+C+1,\epsilon^{-1}-C-1} \times [-\epsilon^{-2} + 1, 0]$$

of the form

$$\|u\|_{C^2(\mathbf{C}' \cap A_{\epsilon+C+1,\epsilon^{-1}-C-1} \times [-\epsilon^{-2}+1,0])} \leq C''\delta$$

for some constant $C'' = C''(n, \mathbf{C}_0)$.

If $\delta \ll 1$ is sufficiently small depending on n and $\mathbf{C}_0$, then this C^2-closeness of M'_0 to the cone $\mathbf{C}'$ implies curvature estimates of the form

$$\frac{c_{\mathbf{C}}}{\rho} \leq \sup_{|y|=\rho} |A_{M'_0}|(y) \leq \frac{C_{\mathbf{C}}}{\rho} \quad \text{for all } \epsilon + C + 1 \leq \rho \leq \epsilon^{-1} - C - 1.$$

Estimate (4.1) now follows from undoing the dilation $\mathcal{D}_{\lambda^{-1}}$ and translation and letting λ vary in $[\lambda_*(X), \lambda^*(X)]$. Note that, since $\delta = \delta(n, \mathbf{C}_0)$, the dependence of ϵ_0 and C on δ can instead be regarded as a dependence on n and $\mathbf{C}_0$. □

Lemma 4.5. *There exists $\epsilon_0 = \epsilon_0(n, \mathbf{C}_0)$ such that the following holds for all $\epsilon \leq \epsilon_0$: if $(M_t^n)_{t \in [-T,0)}$ is smooth in $U \subseteq \mathbb{R}^N$ and its space-time track $\mathcal{M}$ is ϵ-close to $\mathbf{C} \in \mathcal{C}$ at $X = (x,t) \in U \times (-\infty, 0)$, then $\lambda_*(X) > 0$.*

Proof. If not, (4.1) implies

$$|A_{M_t}|(x) = \lim_{\rho \searrow 0} \sup_{|y-x|=\rho} |A_{M_t}|(y) = +\infty,$$

which contradicts the smoothness of the flow at (x,t). □

Remark 4.6. Note that Lemma 4.5 crucially relies on smoothness of the flow and fails when the smooth mean curvature flow (M_t^n) is replaced with an integral n-Brakke flow. Indeed, the static flow of a regular, minimal cone $\mathbf{C}$ has $\lambda_*(\mathbf{0}, -1) = 0$.

Next, we show that $\lambda_*(x,t)$ satisfies a strict inequality in an annulus.

Lemma 4.7. *If $\epsilon_0 = \epsilon_0(n, \mathbf{C}_0) \ll 1$ is sufficiently small (depending on $n, \mathbf{C}_0$), then there exists a constant $C = C(n, \mathbf{C}_0)$ such that the following holds: if $(M_t^n)_{t\in[-T,0)}$ is smooth in $U \subseteq \mathbb{R}^N$ and its space-time track $\mathcal{M}$ is ϵ-close to $\mathbf{C} \in \mathcal{C}$ at $X = (x,t) \in U \times (-\infty, 0)$ for some $\epsilon \leq \epsilon_0$, then*

$$\lambda_*(y,t) > \lambda_*(x,t) \quad \textit{for all } y \in A_{\lambda_*(x,t)[\epsilon+C],\lambda^*(x,t)[\epsilon^{-1}-C]}(x).$$

Proof. Fix $C = C(n, \mathbf{C}_0) \geq 1$ as in Lemma 4.4. Assume $\epsilon_0 = \epsilon_0(n, \mathbf{C}_0) \ll 1$ is small enough so that Lemma 4.4 holds. Let $\epsilon \leq \epsilon_0$. To simplify notation, denote

$$r_0 \doteqdot (\epsilon + C)\lambda_*(x,t), \qquad R_0 \doteqdot (\epsilon^{-1} - C)\lambda^*(x,t),$$
$$r_1 \doteqdot (\epsilon + 100C)\lambda_*(x,t), \quad R_1 \doteqdot (\epsilon^{-1} - 100C)\lambda^*(x,t).$$

Let $y \in A_{r_1,R_1}(x)$. Suppose for the sake of contradiction that $\lambda_*(y,t) \leq \lambda_*(x,t)$. Then

$$(\epsilon + C)\lambda_*(y,t) \leq r_0 \leq (\epsilon^{-1} - C)\lambda^*(y,t)$$

if $\epsilon_0(n, \mathbf{C}_0) \ll 1$. Lemma 4.4 therefore gives curvature estimates of the form

$$\frac{c_{\mathbf{C}_0}}{r_0} \leq \sup_{|y'-y|=r_0} |A_{M_t}|(y'). \tag{4.2}$$

Observe also that the sphere $\partial B(y, r_0)$ is contained in the annulus $A_{r_0,R_0}(x)$ based at x. Indeed, $|y' - y| = r_0$ implies

$$|y' - x| \geq |y - x| - |y' - y| > r_1 - r_0 \geq (1 + 98C)\lambda_*(x,t) > (\epsilon + C)\lambda_*(x,t) = r_0,$$

and an analogous argument applies to show $|y' - y| = r_0$ implies $|y' - x| < R_0$. Therefore, Lemma 4.4 based at x also applies and gives

$$\sup_{|y'-y|=r_0} |A_{M_t}|(y') \leq \sup_{|y'-y|=r_0} \frac{C_{\mathbf{C}_0}}{|y'-x|} \leq \sup_{|y'-y|=r_0} \frac{C_{\mathbf{C}_0}}{|y-x| - |y'-y|}$$
$$< \frac{C_{\mathbf{C}_0}}{r_1 - r_0} < \frac{C_{\mathbf{C}_0}}{4r_0} = \frac{c_{\mathbf{C}_0}}{r_0},$$

which contradicts estimate (4.2). This completes the proof after relabelling $100C$ to C. □

4.2. Finding Hardt–Simon minimal surfaces. Assume additionally throughout this subsection that $\mathcal{M}$ has $H \in L^\infty L^\infty_{\text{loc}}(U \times [-T, 0))$ and that $\mathbf{0} \in U$. Take a sequence $\lambda_i \searrow 0$ and suppose $\mathcal{M}_i \doteqdot \mathcal{D}_{\lambda_i^{-1}} \mathcal{M} \rightharpoonup \mathcal{M}_{\mathbf{C}_0}$, where $\mathcal{M}_{\mathbf{C}_0} = \mathbf{C}_0 \times (-\infty, 0)$ is the flow of the stationary cone $\mathbf{C}_0$.

Since $\mathcal{M}_i \rightharpoonup \mathcal{M}_{\mathbf{C}_0}$, $\mathcal{M}_i$ is ϵ-close to $\mathbf{C}_0$ at $(\mathbf{0}, -1)$ for $i \gg 1$ and

$$\lim_{i\to\infty} \lambda_*(\mathbf{0}, -1; \mathcal{M}_i) = 0.$$

Denote the constant $C = C(n, \mathbf{C}_0)$ from Lemma 4.7 as $\overline{C}$, and obtain $x_i \in \overline{B}_{2\overline{C}}(\mathbf{0})$ such that

$$\lambda_*(x_i, -1; \mathcal{M}_i) = \min_{x \in \overline{B}_{2\overline{C}}(\mathbf{0})} \lambda_*(x, -1; \mathcal{M}_i).$$

Denote $X_i = (x_i, -1)$ and observe

$$\lim_{i\to\infty} \lambda_*(X_i; \mathcal{M}_i) = \lim_{i\to\infty} \lambda_*(\mathbf{0}, -1; \mathcal{M}_i) = 0.$$

Moreover, Lemma 4.7 implies that the minimizer x_i must lie in $\overline{B}_{\lambda_*(\mathbf{0},-1;\mathcal{M}_i)[\epsilon+\overline{C}]}(\mathbf{0})$, which implies $\lim_{i\to\infty} x_i = \mathbf{0}$.

Define

$$\mathcal{M}_i' \doteqdot \mathcal{D}_{\frac{1}{\lambda_*(X_i;\mathcal{M}_i)}}(\mathcal{M}_i - X_i).$$

By construction, for all i, $\mathcal{M}_i'$ is ϵ-close to some $\mathbf{C}_i' \in \mathcal{C}$ at $(\mathbf{0}, 0)$.

Remark 4.8. In what follows, we deviate from the notation of Section 2.2 and write Huisken's monotonic quantity as

$$\Theta((x,t), \mathcal{M}, r) = \int \Phi_{x,t}(\,\cdot\,, t-r^2)\, d\mu_{t-r^2},$$
$$\Theta((x,t), \mathcal{M}) = \lim_{r\searrow 0} \int \Phi_{x,t}(\,\cdot\,, t-r^2)\, d\mu_{t-r^2},$$

where $\mathcal{M} = (\mu_t)_{t\in(a,b)}$ is a Brakke flow in $\mathbb{R}^{n+1}$ and $r > 0$.

Lemma 4.9. *Along some subsequence $i \to \infty$, $\mathcal{M}_i' \rightharpoonup \widehat{\mathcal{M}} \doteqdot (\hat{\mu}_t)_{t\in\mathbb{R}}$, where $\widehat{\mathcal{M}}$ is an eternal integral n-Brakke flow in $\mathbb{R}^{n+1}$ such that*

(1) *(entropy bound) for all $r > 0$ and $(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R}$,*

$$\Theta((x,t), \widehat{\mathcal{M}}, r) \le \Theta((\mathbf{0}, 0), \mathcal{M}_{\mathbf{C}_0}),$$

(2) *for all $t \in \mathbb{R}$, $\hat{\mu}_t = \mu_{\widehat{V}_t}$ for some stationary integral n-varifold $\widehat{V}_t$,*

(3) $\|H\|_{L^\infty L^\infty(\mathbb{R}^{n+1}\times\mathbb{R})} = 0$*, and*

(4) *for all $\lambda \in [1,\infty)$, $\mathcal{D}_{\lambda^{-1}}\widehat{\mathcal{M}}$ is ϵ-close to some $\widehat{\mathbf{C}}_\lambda \in \mathcal{C}$.*

Proof. For any $r > 0$ and $(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R}$,

$$\begin{aligned}
\text{(4.3)}\quad \limsup_{i\to\infty} \Theta((x,t), \mathcal{M}_i', r) &= \limsup_{i\to\infty} \Theta((x,t), \mathcal{D}_{\lambda_*(X_i)^{-1}}(\mathcal{M}_i - X_i), r) \\
&= \limsup_{i\to\infty} \Theta(X_i + (x\lambda_*(X_i), t\lambda_*(X_i)^2), \mathcal{M}_i, r\lambda_*(X_i)) \\
&\le \Theta((\mathbf{0}, -1), \mathcal{M}_{\mathbf{C}_0}, 0)
\end{aligned}$$

by the limiting behavior of $X_i, \mathcal{M}_i, \lambda_*(X_i)$ and the upper semi-continuity of Θ [32]. Huisken's monotonicity formula therefore implies that the $\mathcal{M}_i'$ satisfy local uniform area bounds. Compactness of Brakke flows with local uniform area bounds then allows us to extract a subsequential limit $\widehat{\mathcal{M}}$ of the $\mathcal{M}_i'$. Observe that, since $\lim_{i\to\infty} \lambda_*(X_i) = 0$, the limiting integral n-Brakke flow $\widehat{\mathcal{M}}$ is defined for all $t \in \mathbb{R}$. Additionally, (4.3) shows that, for any $r > 0$ and $(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R}$,

$$\Theta((x,t), \widehat{\mathcal{M}}, r) = \lim_{i\to\infty} \Theta((x,t), \mathcal{M}_i', r) \le \Theta((\mathbf{0}, -1), \mathcal{M}_{\mathbf{C}_0}, 0) = \Theta((\mathbf{0}, 0), \mathcal{M}_{\mathbf{C}_0}, 0).$$

The fact that $\mathcal{M}$ has mean curvature $H \in L^\infty L^\infty_{\mathrm{loc}}(U \times [-T, 0))$ implies that the mean curvature H_i of the rescalings $\mathcal{M}'_i$ has

$$\lim_{i\to\infty} \|H_i\|_{L^\infty L^\infty(K\times[a,b])} = 0 \quad \text{for any } K \times [a,b] \Subset \mathbb{R}^{n+1} \times \mathbb{R}.$$

Thus the limiting Brakke flow $\widehat{\mathcal{M}}$ has $\|H\|_{L^\infty L^\infty(\mathbb{R}^{n+1}\times\mathbb{R})} = 0$.

Let $t_0 \in \mathbb{R}$ and consider the t_0-time-slice $\mathcal{M}'_i(t_0)$ of $\mathcal{M}'_i$. Lemma A.1 applies to give that some subsequence $\mathcal{M}'_i(t_0)$ converges in the weak varifold sense to a stationary integral n-varifold $\widehat{V}_{t_0}$. In particular, the underlying measures converge $\mu_{\mathcal{M}'_i(t_0)} \rightharpoonup \mu_{\widehat{V}_{t_0}}$. On the other hand, the Brakke flow convergence $\mathcal{M}'_i \rightharpoonup \widehat{\mathcal{M}}$ implies $\mu_{\mathcal{M}'_i(t_0)} \rightharpoonup \widehat{\mu}_{t_0}$ and so $\widehat{\mu}_{t_0} = \mu_{\widehat{V}_{t_0}}$.

Let $\lambda \in [1,\infty)$. Observe that $\lim_{i\to\infty} \lambda_*(X_i; \mathcal{M}_i) = 0$ and $\lambda^*(X_*; \mathcal{M}_i) \geq 1$ for all $i \gg 1$. Thus $\lambda\lambda_*(X_i;\mathcal{M}_i) \in [\lambda_*(X_i;\mathcal{M}_i), \lambda^*(X_i;\mathcal{M}_i)]$ for all $i \gg 1$. It follows that

$$\mathcal{D}_{\lambda^{-1}}\mathcal{M}'_i = \mathcal{D}_{\frac{1}{\lambda\lambda_*(X_i;\mathcal{M}_i)}}(\mathcal{M}_i - X_i)$$

is ϵ-close to some $\mathbf{C}'_{\lambda,i} \in \mathcal{C}$ for all $i \gg 1$. After passing to a subsequence and using that $\mathcal{D}_{\lambda^{-1}}\mathcal{M}'_i \rightharpoonup \mathcal{D}_{\lambda^{-1}}\widehat{\mathcal{M}}$, it follows that there exists a limiting cone $\widehat{\mathbf{C}}_\lambda \in \mathcal{C}$ such that $\mathcal{D}_{\lambda^{-1}}\widehat{\mathcal{M}}$ is ϵ-close to $\widehat{\mathbf{C}}_\lambda$. Since $\lambda \in [1,\infty)$ was arbitrary, this completes the proof. □

Definition 4.10. For $p, q \in \mathbb{N}$ with $p + q = n - 1$, define the n-dimensional *quadratic minimal cone* or *generalized Simons cone* $\mathbf{C}^{p,q}$ to be the hypersurface

$$\mathbf{C}^{p,q} \doteqdot \{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2\} \subseteq \mathbb{R}^{n+1}.$$

Remark 4.11. $\mathbf{C}^{p,q}$ is minimal for any p, q. Moreover, $\mathbf{C}^{3,3}$, $\mathbf{C}^{2,4}$ (equivalently $\mathbf{C}^{4,2}$), and $\mathbf{C}^{p,q}$ for any $p + q > 6$ are all area minimizing. In fact, these are the only quadratic minimal cones which are area minimizing (see e.g. [21] and references therein).

Definition 4.12 (Hardt–Simon foliation). Let $\mathbf{C}^{p,q}$ be a quadratic minimal cone which is area minimizing. Denote the two connected components of $\mathbb{R}^{n+1} \setminus \mathbf{C}^{p,q}$ by $E_\pm$. In [9], it is shown that there exist smooth, minimal surfaces $S_\pm \subseteq E_\pm$ which are both asymptotic to $\mathbf{C}^{p,q}$ at infinity and whose dilations $\lambda S_\pm$ ($\lambda > 0$) foliate $E_\pm$, respectively. By dilating, we can assume without loss of generality that both $S_\pm$ are normalized to have $\mathrm{dist}(S_\pm, \mathbf{0}) = 1$.

For any $\lambda \in \mathbb{R}$, define

$$S_\lambda \doteqdot \begin{cases} \lambda S_+ & \text{if } \lambda > 0, \\ \mathbf{C}^{p,q} & \text{if } \lambda = 0, \\ -\lambda S_- & \text{if } \lambda < 0. \end{cases} \tag{4.4}$$

We refer to this family of minimal hypersurfaces as the *Hardt–Simon foliation*.

Theorem 4.13. *Let*

$$\mathbf{C}_0 = \mathbf{C}^{p,q} \subseteq \mathbb{R}^{n+1} \quad (p + q = n - 1)$$

be a generalized Simons cone, and let $\mathcal{C} = \{A \cdot \mathbf{C}_0 : A \in O(n+1)\}$ *denote all rotations of the cone. Let* $0 < \epsilon_0 = \epsilon_0(n, \mathbf{C}_0) \ll 1$ *be sufficiently small so that Lemmas* 4.4–4.7 *hold, and let* $0 < \epsilon \leq \epsilon_0$. *If* $\mathbf{C}_0 = \mathbf{C}^{p,q}$ *is area minimizing, then there exist* $\lambda_0 \neq 0$, $A_0 \in O(n+1)$, *and*

$a_0 \in \mathbb{R}^{n+1}$ such that the eternal Brakke flow $\widehat{\mathcal{M}}$ obtained in Lemma 4.9 is the static flow of the smooth Hardt–Simon minimal surface $M = A_0 \cdot S_{\lambda_0} + a_0$ for all $t \in \mathbb{R}$. Additionally, the scale $\lambda_0 \neq 0$ and center $a_0 \in \mathbb{R}^{n+1}$ are such that, for all $0 < \epsilon' < \epsilon$ and all $\mathbf{C} \in \mathcal{C}$, $\widehat{\mathcal{M}}$ is not ϵ'-close to $\mathbf{C}$.

Proof. It follows from the entropy bound Lemma 4.9 (1), compactness of Brakke flows with uniform local area bounds, and Huisken's monotonicity formula that there exists a limiting shrinker $\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}} \rightharpoonup \widehat{\mathcal{M}}_{-\infty}$ along some sequence $\lambda_i \to \infty$. Since $\widehat{\mathcal{M}}$ has $H \equiv 0$ (see Lemma 4.9 (3)), the same argument as in the proof of Lemma 2.10 shows that, for $t < 0$, $\widehat{\mathcal{M}}_{-\infty}$ must be the flow of a stationary cone $\widehat{\mathbf{C}}$ (which is dilation invariant with respect to $\mathbf{0} \in \mathbb{R}^{n+1}$). Because $\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}}$ is ϵ-close to some $\widehat{\mathbf{C}}_{\lambda_i} \in \mathcal{C}$ for all i (Lemma 4.9 (4)), we can pass to a further subsequence (still denoted by i) so that the $\widehat{\mathbf{C}}_{\lambda_i}$ converge to $\widehat{\mathbf{C}}_{-\infty} \in \mathcal{C}$ and $\widehat{\mathcal{M}}_{-\infty} = \mathcal{M}_{\widehat{\mathbf{C}}}$ is ϵ-close to $\widehat{\mathbf{C}}_{-\infty} \in \mathcal{C}$. By the dilation invariance of the cone $\widehat{\mathbf{C}}$, it follows that $\widehat{\mathbf{C}}$ can be written *globally* as a $C^{1,\alpha}$-graph over $\widehat{\mathbf{C}}_{-\infty}$ with $C^{1,\alpha}$-norm less than or equal to ϵ.

Consider the Hardt–Simon foliation (4.4) rotated so that $S_0 = \widehat{\mathbf{C}}_{-\infty}$. If $\epsilon \ll 1$ is sufficiently small depending on $n, \mathbf{C}_0$, then [8, Theorem 3.1] implies that there exist $a \in \mathbb{R}^{n+1}$, $q \in \mathrm{SO}(n+1)$, $\lambda' \in \mathbb{R}$ such that $\widehat{\mathbf{C}}$ is a $C^{1,\beta}$-graph over $a + q(S_{\lambda'})$ and the graphing function u satisfies an improved decay estimate

$$\sup_{x \in B_r(a) \cap (a + q(S_{\lambda'}))} |u(x)| \lesssim r^{1+\beta} \quad \text{for all } r \leq 1/2.$$

Since $\widehat{\mathbf{C}}$ is dilation invariant (with respect to $\mathbf{0}$), this estimate is only possible if $a = \mathbf{0}$, $\lambda' = 0$, and $u \equiv 0$. In other words, $\widehat{\mathbf{C}} = q(S_0) = q(\widehat{\mathbf{C}}_{-\infty}) \in \mathcal{C}$. Thus, for $t < 0$, we have $\widehat{\mathcal{M}}_{-\infty} = \mathcal{M}_{\widehat{\mathbf{C}}}$ and $\widehat{\mathbf{C}} \in \mathcal{C}$.

Because $\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}} \rightharpoonup \widehat{\mathcal{M}}_{-\infty}$ converges as Brakke flows, we can find $t_0 < 0$ and pass to a further subsequence so that the time slices $\widehat{V}_i \doteqdot \lambda_i^{-1} \widehat{V}_{t_0 \lambda_i^2} \rightharpoonup \widehat{\mathbf{C}}$ converge as integral n-varifolds. For $i \gg 1$, [8, Theorem 3.1] applies and gives that, for all $i \gg 1$, there exist $a_i \in \mathbb{R}^{n+1}$, $q_i \in \mathrm{SO}(n+1)$, $\lambda_i' \in \mathbb{R}$ such that $\operatorname{spt} \widehat{V}_i \cap B_{1/2}$ is a $C^{1,\beta}$-graph over $a_i + q_i(S_{\lambda_i'})$. Moreover, after renormalizing the Hardt–Simon foliation (4.4) so that $S_0 = \widehat{\mathbf{C}}$,

$$\lim_{i \to \infty} (|a_i| + |q_i - \mathrm{Id}| + |\lambda_i'|) = 0.$$

Claim 4.14. *$\lambda_i' \neq 0$ for all $i \gg 1$.*

Proof of claim. Consider some index i with $\lambda_i' = 0$. Then the monotonicity formula (A.3) for stationary varifolds implies that

$$\theta_{\widehat{V}_i - a_i}(\mathbf{0}) = \lim_{r \searrow 0} \frac{\mu_{\widehat{V}_i - a_i}(B_r)}{\omega_n r^n} \geq \theta_{q_i(S_0)}(\mathbf{0}) = \theta_{\mathbf{C}_0}(\mathbf{0}) = \Theta(\mathcal{M}_{\mathbf{C}_0}, (\mathbf{0}, 0)).$$

Theorem 2.13 then gives that

$$\begin{aligned}
\theta_{\mathbf{C}_0}(\mathbf{0}) &\leq \theta_{\widehat{V}_i - a_i}(\mathbf{0}) = \theta_{\widehat{V}_i}(a_i) \\
&\leq \Theta_{\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}}}(a_i, t_0) && \text{(Theorem 2.13)} \\
&\leq \Theta_{\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}}}((a_i, t_0), r) && \text{(Huisken's monotonicity formula (2.4), } r > 0) \\
&= \Theta_{\widehat{\mathcal{M}}}((\lambda_i a_i, \lambda_i^2 t_0), \lambda_i r) \\
&\leq \theta_{\mathbf{C}_0}(\mathbf{0}) && \text{(Lemma 4.9 (1)).}
\end{aligned}$$

Thus we have equality throughout and

$$\Theta_{\widehat{\mathcal{M}}}((\lambda_i a_i, \lambda_i^2 t_0), r) = \theta_{\mathbf{C}_0}(\mathbf{0}) \quad \text{for all } r > 0.$$

It follows from Huisken's monotonicity formula (2.4) that $\widehat{\mathcal{M}} - (\lambda_i a_i, \lambda_i^2 t_0)$ is a shrinker (for $t < 0$) and must therefore be equal to the limiting shrinker $\widehat{\mathcal{M}}_{-\infty}$, that is,

$$\widehat{\mathcal{M}} - (\lambda_i a_i, \lambda_i^2 t_0) = \widehat{\mathcal{M}}_{-\infty} = \mathcal{M}_{\widehat{\mathbf{C}}} \quad \text{for } t < 0. \tag{4.5}$$

Write $\widehat{\mathcal{M}} = (\widehat{\mu}_t)_{t\in\mathbb{R}}$, and note that $\widehat{\mu}_t = \mu_{\widehat{\mathbf{C}}+\lambda_i a_i}$ for all $t < \lambda_i^2 t_0$ by (4.5). Moreover, Brakke's inequality and the fact that $\widehat{\mathcal{M}}$ has $H \equiv 0$ (Lemma 4.9 (3)) imply $\widehat{\mu}_{t_1} \geq \widehat{\mu}_{t_2}$ for all $t_1 \leq t_2$. In particular, $\operatorname{spt} \widehat{\mu}_{t_2} \subseteq \operatorname{spt} \widehat{\mu}_{t_1}$ for $t_1 \leq t_2$.

Let $t \in [\lambda_i^2 t_0, -\epsilon^2]$ and recall $\widehat{\mu}_t$ is represented by a stationary integral n-varifold $\widehat{V}_t$ with $H = H_{\widehat{V}_t} = 0$ (Lemma 4.9 (2)). Then

$$\operatorname{spt} \widehat{V}_t = \operatorname{spt} \widehat{\mu}_t \subseteq \operatorname{spt} \widehat{\mu}_{\lambda_i^2 t_0} = \widehat{\mathbf{C}} + \lambda_i a_i.$$

On the other hand, Solomon–White's strong maximum principle [25] applied to the smooth manifold $(\widehat{\mathbf{C}} + \lambda_i a_i) \setminus \{\lambda_i a_i\} \subseteq \mathbb{R}^{n+1} \setminus \{\lambda_i a_i\}$ implies that either

$$(\widehat{\mathbf{C}} + \lambda_i a_i) \setminus \{\lambda_i a_i\} \subseteq \operatorname{spt} \widehat{V}_t \quad \text{or} \quad (\widehat{\mathbf{C}} + \lambda_i a_i) \cap \operatorname{spt} \widehat{V}_t \setminus \{\lambda_i a_i\} = \emptyset.$$

In particular, either $\widehat{\mathbf{C}} + \lambda_i a_i = \operatorname{spt} \widehat{V}_t$ or $\operatorname{spt} \widehat{V}_t \subseteq \{\lambda_i a_i\}$ since $\operatorname{spt} \widehat{V}_t \subseteq \widehat{\mathbf{C}} + \lambda_i a_i$ and $\operatorname{spt} \widehat{V}_t$ is a closed set. However, the second case is impossible since $\widehat{\mathcal{M}}$ is ϵ-close to some $\widehat{\mathbf{C}}_{\lambda=1} \in \mathcal{C}$ by Lemma 4.9 (4). Thus

$$\operatorname{spt} \widehat{V}_t = \widehat{\mathbf{C}} + \lambda_i a_i \quad \text{for all } t \in [\lambda_i^2 t_0, -\epsilon^2].$$

By the entropy bounds (Lemma 4.9 (1)) and the constancy theorem [23, Chapter 8, §4], it follows that, in fact, $\widehat{V}_t = \widehat{\mathbf{C}} + \lambda_i a_i$ for all $t \in [\lambda_i^2 t_0, -\epsilon^2]$. Thus, in combination with (4.5), we have shown

$$\widehat{\mathcal{M}} - (\lambda_i a_i, 0) = \mathcal{M}_{\widehat{\mathbf{C}}} \quad \text{for } t \in [\lambda_i^2 t_0, -\epsilon^2], \tag{4.6}$$

for any index i such that $\lambda_i' = 0$.

Suppose for the sake of contradiction that there exist arbitrarily large indices i with $\lambda_i' = 0$. Then there exists i_0 such that $\lambda_{i_0}' = 0$ and $\lambda_{i_0}^2 t_0 \leq -\epsilon^{-2} - 1$. Hence equality (4.6) holds for $t \in [-\epsilon^{-2} - 1, -\epsilon^2]$ and

$$\begin{aligned} &\mathcal{D}_{\lambda_*(X_j;\mathcal{M}_j)^{-1}}(\mathcal{M}_j - X_j - (\lambda_*(X_j)\lambda_{i_0} a_{i_0}, 0)) \\ &= \mathcal{M}_j' - (\lambda_{i_0} a_{i_0}, 0) \\ &\rightharpoonup \widehat{\mathcal{M}} - (\lambda_{i_0} a_{i_0}, 0) \quad (\text{as } j \to \infty) \\ &= \mathcal{M}_{\widehat{\mathbf{C}}} \qquad\qquad (\text{for } t \in [-\epsilon^{-2} - 1, -\epsilon^2]). \end{aligned}$$

In particular, White's local regularity theorem [33] implies $\mathcal{M}_j' - (\lambda_{i_0} a_{i_0}, 0)$ converges to $\mathcal{M}_{\widehat{\mathbf{C}}}$ in $C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \times [-\epsilon^{-2} - 1, -\epsilon^2])$ as $j \to \infty$. Using smoothness of the flows $\mathcal{M}_j'$, it then follows that, for some large enough j, $\mathcal{D}_{\lambda^{-1}}(\mathcal{M}_j' - (\lambda_{i_0} a_{i_0}, 0))$ is ϵ-close to $\widehat{\mathbf{C}}$ at $(\mathbf{0}, 0)$ for all λ in a neighborhood of 1. Equivalently, $\mathcal{D}_{\lambda^{-1}}(\mathcal{M}_j - X_j - (\lambda_*(X_j)\lambda_{i_0} a_{i_0}, 0))$ is ϵ-close to $\widehat{\mathbf{C}} \in \mathcal{C}$ at $(\mathbf{0}, 0)$ for all λ in a neighborhood of $\lambda_*(X_j)$. This however contradicts the definitions of X_j and λ_*, and this contradiction completes the proof of the claim. □

With Claim 4.14 in hand, $\widehat{V}_i \cap B_{1/2}$ is a $C^{1,\beta}$-graph over a *smooth* minimal surface $a_i + q_i(S_{\lambda_i'})$ ($\lambda_i' \neq 0$) for all $i \gg 1$. Interior regularity for stationary varifolds then implies that $\widehat{V}_i \cap B_{1/4}$ is smooth for all $i \gg 1$, and thus so is $\operatorname{spt} \widehat{\mu}_{t_0 \lambda_i^2} \cap B_{\lambda_i/4}$.

Let $t \leq -\epsilon^2$ and $R > \epsilon^2$. Let $i \gg 1$ be sufficiently large such that $\operatorname{spt} \widehat{\mu}_{t_0\lambda_i^2} \cap B_{\lambda_i/4}$ is smooth, $t_0\lambda_i^2 < t$, and $\lambda_i/4 > R$. Since $\widehat{\mu}_t \leq \widehat{\mu}_{t_0\lambda_i^2}$, it follows that

$$\operatorname{spt} \widehat{V}_t \cap B_R = \operatorname{spt} \widehat{\mu}_t \cap B_R \subseteq \operatorname{spt} \widehat{\mu}_{t_0\lambda_i^2} \cap B_R.$$

Because $\operatorname{spt} \widehat{\mu}_{t_0\lambda_i^2} \cap B_R$ is smooth, the strong maximum principle [25] applies and implies that either

$$\operatorname{spt} \widehat{V}_t \cap B_R = \operatorname{spt} \widehat{\mu}_{t_0\lambda_i^2} \cap B_R \quad \text{or} \quad \operatorname{spt} \widehat{V}_t \cap B_R = \emptyset.$$

However, the second case is impossible, since it would imply (together with $\widehat{\mu}_{-\epsilon^2} \leq \widehat{\mu}_t$) that $\operatorname{spt} \widehat{\mu}_{-\epsilon^2} \cap B_R = \emptyset$, which contradicts that $\widehat{\mathcal{M}}$ is ϵ-close to some $\widehat{\mathbf{C}}_{\lambda=1} \in \mathcal{C}$ (Lemma 4.9 (4)). Hence $\operatorname{spt} \widehat{V}_t \cap B_R = \operatorname{spt} \widehat{\mu}_{t_0\lambda_i^2} \cap B_R$. By letting i, R, and t vary, it follows that there exists a smooth manifold M such that $M = \operatorname{spt} \widehat{\mu}_t = \operatorname{spt} \widehat{V}_t$ for all $t \leq -\epsilon^2$.

Additionally, M is minimal ($H_M \equiv 0$) and

$$\lambda_i^{-1} M \rightharpoonup \widehat{\mathbf{C}} \in \mathcal{C} \quad \text{(as varifolds)}$$

since $\mathcal{D}_{\lambda_i^{-1}} \widehat{\mathcal{M}} \rightharpoonup \mathcal{M}_{\widehat{\mathbf{C}}}$ as Brakke flows. The proof of [8, Corollary 3.7] holds in this setting and implies that M is necessarily a smooth Hardt–Simon minimal surface, that is, $M = q(S_\lambda) + a$ for some $q \in \mathrm{SO}(n+1)$, $\lambda \neq 0$, and $a \in \mathbb{R}^{n+1}$. It then follows from the constancy theorem [23, Chapter 8, §4] and the entropy bounds for $\widehat{\mathcal{M}}$ (Lemma 4.9 (1)) that $\widehat{V}_t = M = q(S_\lambda) + a$ for all $t \leq -\epsilon^2$. In summary, $\widehat{\mathcal{M}}$ is the stationary flow of a smooth Hardt–Simon minimal surface $M = q(S_\lambda) + a$ for all $t \leq -\epsilon^2$.

Using the strong maximum principle [25] and the fact that $\widehat{\mu}_{t_1} \geq \widehat{\mu}_{t_2}$ for $t_1 \leq t_2$, it can be shown that there exists $T \in [-\epsilon^2, +\infty]$ such that the flow $\widehat{\mathcal{M}} = (\mu_{\widehat{V}_t})_{t \in \mathbb{R}}$ has

$$\widehat{V}_t = \begin{cases} M & \text{for } t < T, \\ \emptyset & \text{for } t > T. \end{cases}$$

Since $\widehat{\mathcal{M}}$ is a limit of smooth flows $\mathcal{M}_i'$, $\widehat{\mathcal{M}}$ is unit-regular [20, Theorem 4.2] and therefore $T = +\infty$. Thus $\widehat{\mathcal{M}} = (\mu_M)_{t \in \mathbb{R}}$ is the stationary flow of the smooth Hardt–Simon minimal surface $M = q(S_\lambda) + a$. In particular, White's local regularity theorem [33] implies $\mathcal{M}_i'$ converges to $\widehat{\mathcal{M}}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times \mathbb{R})$ as $i \to \infty$.

Finally, suppose for the sake of contradiction that there exist $0 < \epsilon' < \epsilon$ and $\mathbf{C} \in \mathcal{C}$ such that $\widehat{\mathcal{M}}$ is ϵ'-close to $\mathbf{C}$. Since $\mathcal{M}_i'$ converges to $\widehat{\mathcal{M}}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times \mathbb{R})$, one can take $\epsilon'' \in (\epsilon', \epsilon)$ and deduce that $\mathcal{M}_i'$ is ϵ''-close to $\mathbf{C}$ for $i \gg 1$ sufficiently large. Since $\mathcal{M}_i'$ is smooth and $\epsilon'' < \epsilon$, $\mathcal{D}_{\lambda^{-1}} \mathcal{M}_i' = \mathcal{D}_{\lambda^{-1}\lambda_*(X_i;\mathcal{M}_i)^{-1}}(\mathcal{M}_i - X_i)$ must then be ϵ-close to $\mathbf{C}$ for all λ in a neighborhood of 1. This, however, contradicts the definition of $\lambda_*(X_i; \mathcal{M}_i)$. □

Theorem 4.13 and the results of this section complete the proof of Theorem 1.3.

A. Varifolds with generalized mean curvature $H \in L^p_{\mathrm{loc}}$

In this appendix, we collect some standard results for varifolds with generalized mean curvature $H \in L^p_{\mathrm{loc}}$ that are cited throughout the article. For example, the compactness statement (Lemma A.1) and monotonicity formula (Proposition A.3) are given here.

Lemma A.1 (Compactness). *Let $2 \le n < N$, let $U \subseteq \mathbb{R}^N$ be open, and $p \in (1,\infty]$. The collection of integer rectifiable n-varifolds in U with locally uniformly bounded area and locally uniformly bounded generalized mean curvature $H \in L^p_{\mathrm{loc}}(U)$ is weakly compact. In other words, if V_i is a sequence of integral n-varifolds in U such that*

(1) *the V_i have locally uniformly bounded area, i.e.* $\sup_i \mu_{V_i}(K) < \infty$ *(for all $K \Subset U$), and*

(2) *the V_i have generalized mean curvature $H_i \in L^p_{\mathrm{loc}}(U)$ with uniform $L^p_{\mathrm{loc}}(U)$ bounds, i.e.* $\sup_i \|H_i\|_{L^p(K,d\mu_{V_i})} < \infty$ *(for all $K \Subset U$),*

then there exists a subsequence V_{i_j} and an integral n-varifold V_∞ in U such that $V_{i_j} \rightharpoonup V_\infty$ weakly as varifolds. Moreover, V_∞ has $\mu_{V_\infty}(K) \le \limsup_i \mu_{V_i}(K)$ (for all $K \Subset U$) and generalized mean curvature $H_\infty \in L^p_{\mathrm{loc}}(U)$ with

$$\|H_\infty\|_{L^p(K,d\mu_{V_\infty})} \le \limsup_i \|H_i\|_{L^p(K,d\mu_{V_i})} \quad \textit{(for all } K \Subset U\textit{)}.$$

Proof. Allard's compactness theorem [1] gives compactness under the weaker assumption where (2) is replaced by the bound

$$\sup_i |\delta V_i(X)| \le C_K \|X\|_{C^0(K)} \quad \text{for all } X \in C^1_c(K,\mathbb{R}^N),\ K \Subset U.$$

Therefore, there exists a subsequential limit $V_{i_j} \rightharpoonup V_\infty$ such that V_∞ is an integral n-varifold with locally finite area and the weaker property that

$$|\delta V_\infty(X)| \le C_K \|X\|_{C^0} \quad \text{for all } X \in C^1_c(K,\mathbb{R}^N),\ K \Subset U.$$

However, convergence as varifolds $V_{i_j} \rightharpoonup V_\infty$ implies that, for any $K \subseteq U$ compact and $X \in C^1_c(K,\mathbb{R}^N)$,

$$\begin{aligned} \text{(A.1)}\quad & |\delta V_\infty(X)| \\ &= \lim_{j\to\infty} |\delta V_{i_j}(X)| = \lim_{j\to\infty} \left| \int H_{i_j} \cdot X \, d\mu_{V_{i_j}} \right| \\ &\le \begin{cases} \left(\limsup_{i\to\infty} \|H_i\|_{L^p_{\mathrm{loc}}(K,d\mu_{V_i})}\right) \left(\int |X|^{\frac{p}{p-1}} \, d\mu_{V_\infty} \right)^{\frac{p-1}{p}} & \text{if } p \in (1,\infty), \\ \left(\limsup_{i\to\infty} \|H_i\|_{L^\infty_{\mathrm{loc}}(K,d\mu_{V_i})}\right) \int |X| \, d\mu_{V_\infty} & \text{if } p = \infty. \end{cases} \end{aligned}$$

The Radon–Nikodym theorem then implies that $\delta V_\infty = -H_\infty \in L^p_{\mathrm{loc}}(U, d\mu_{V_\infty})$. Then (A.1) also implies $\|H_\infty\|_{L^p(K,d\mu_{V_\infty})} \le \limsup_i \|H_i\|_{L^p(K,d\mu_{V_i})}$. □

Lemma A.2. *Let $2 \le n < N$, let $U \subseteq \mathbb{R}^N$ be open, and $p \in (1,\infty]$. Let $(V_i)_{i \in \mathbb{N}\cup\{\infty\}}$ be a collection of integer rectifiable n-varifolds with generalized mean curvature $H_i \in L^p_{\mathrm{loc}}(U)$. Assume*

$$\sup_i \mu_{V_i}(K) + \sup_i \|H_i\|_{L^p(K,d\mu_{V_i})} < \infty \quad \textit{(for all } K \Subset U\textit{)}.$$

If

$$\text{(A.2)}\qquad \int f \, d\mu_{V_i} \xrightarrow{i\to\infty} \int f \, d\mu_{V_\infty} \quad \textit{for all } f \in C^\infty_c(\mathbb{R}^N) \textit{ with } f \ge 0,$$

then $V_i \rightharpoonup V_\infty$ as varifolds as $i \to \infty$.

Proof. We first show $d\mu_{V_i} \rightharpoonup d\mu_{V_\infty}$. Let $f \in C_c^0(U)$. There exist $\delta > 0$ and a compact set $K \subseteq U$ such that

$$\operatorname{supp} f \subseteq B_\delta(\operatorname{supp} f) \subseteq K \subseteq U,$$

where $B_\delta(\operatorname{supp} f)$ denotes the radius δ neighborhood of $\operatorname{supp} f$. Denote

$$C_K \doteqdot \sup_i \mu_{V_i}(K) < \infty.$$

Let $\epsilon > 0$. Split $f = f_+ - f_-$ into positive and negative parts. Convolving f_+, f_- with a suitable mollifiers, we can find $\widetilde{f}_+, \widetilde{f}_- \in C_c^\infty(U)$ such that

$$\widetilde{f}_\pm \geq 0, \quad \operatorname{supp} \widetilde{f}_\pm \subseteq K, \quad \text{and} \quad \|\widetilde{f}_\pm - f_\pm\|_{C^0} < \frac{1}{6C_K}\epsilon.$$

Then

$$\begin{aligned}
&\left|\int f\, d\mu_{V_i} - \int f\, d\mu_{V_\infty}\right| \\
&\quad \leq \left|\int f_+\, d\mu_{V_i} - \int f_+\, d\mu_{V_\infty}\right| + \left|\int f_-\, d\mu_{V_i} - \int f_-\, d\mu_{V_\infty}\right| \\
&\quad \leq \left|\int f_+ - \widetilde{f}_+\, d\mu_{V_i}\right| + \left|\int \widetilde{f}_+\, d\mu_{V_i} - \int \widetilde{f}_+\, d\mu_{V_\infty}\right| + \left|\int f_+ - \widetilde{f}_+\, d\mu_{V_\infty}\right| \\
&\qquad + \left|\int f_- - \widetilde{f}_-\, d\mu_{V_i}\right| + \left|\int \widetilde{f}_-\, d\mu_{V_i} - \int \widetilde{f}_-\, d\mu_{V_\infty}\right| + \left|\int f_- - \widetilde{f}_-\, d\mu_{V_\infty}\right| \\
&\quad \leq 4\|f - \widetilde{f}\|_{C^0} C_K + \left|\int \widetilde{f}_+\, d\mu_{V_i} - \int \widetilde{f}_+\, d\mu_{V_\infty}\right| + \left|\int \widetilde{f}_-\, d\mu_{V_i} - \int \widetilde{f}_-\, d\mu_{V_\infty}\right| \\
&\quad < \frac{2}{3}\epsilon + \left|\int \widetilde{f}_+\, d\mu_{V_i} - \int \widetilde{f}_+\, d\mu_{V_\infty}\right| + \left|\int \widetilde{f}_-\, d\mu_{V_i} - \int \widetilde{f}_-\, d\mu_{V_\infty}\right| \\
&\quad < \epsilon \quad (\text{for all } i \gg 1),
\end{aligned}$$

where the last inequality follows by (A.2). This completes the proof that $d\mu_{V_i} \rightharpoonup d\mu_{V_\infty}$.

Next, we prove the varifold convergence $V_i \rightharpoonup V_\infty$. Suppose for the sake of contradiction that $V_i \not\rightharpoonup V_\infty$. Then there exist $f \in C_c^0(G(n, U))$, $\epsilon > 0$, and a subsequence V_{i_j} such that

$$\left|\int f\, dV_{i_j} - \int f\, dV_\infty\right| > \epsilon \quad \text{for all } j.$$

Since the V_{i_j} have locally uniformly bounded areas and locally uniformly bounded generalized mean curvatures $H_{i_j} \in L^p_{\mathrm{loc}}(U)$, Lemma A.1 implies there exist an integer rectifiable n-varifold V'_∞ and a subsequence still denoted V_{i_j} such that $V_{i_j} \rightharpoonup V'_\infty$. In particular,

$$d\mu_{V_\infty} = \lim_{j\to\infty} d\mu_{V_{i_j}} = d\mu_{V'_\infty}.$$

Because V_∞, V'_∞ are integer rectifiable n-varifolds with $d\mu_{V_\infty} = d\mu_{V'_\infty}$, $V_\infty = V'_\infty$. We then have a contradiction that $V_{i_j} \not\rightharpoonup V_\infty = V'_\infty$ and $V_{i_j} \rightharpoonup V'_\infty = V_\infty$. This contradiction proves that, in fact, $V_i \rightharpoonup V_\infty$. □

Proposition A.3 (Monotonicity formula). *Let $2 \leq n < N$, let $U \subseteq \mathbb{R}^N$ be open, and $p \in (n, \infty]$. Let V be an integer rectifiable n-varifold in U with generalized mean curvature*

$H \in L^p_{\mathrm{loc}}(U)$. Let $x_0 \in U$ and $\overline{B_R(x_0)} \subseteq U$. Then, for any $0 < \sigma < \rho < R$,

$$\text{(A.3)}\qquad \Big(e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}}\frac{\mu_V(B_\rho(x_0))}{\rho^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}} - 1\Big) - \Big(e^{\frac{\|H\|}{1-\frac{n}{p}}\sigma^{1-\frac{n}{p}}}\frac{\mu_V(B_\sigma(x_0))}{\sigma^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\sigma^{1-\frac{n}{p}}} - 1\Big) \geq \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}}\, d\mu_V \geq 0,$$

where $\|H\| = \|H\|_{L^p(B_R(x_0), d\mu_V)}$. In particular,

$$e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}}\frac{\mu_V(B_\rho(x_0))}{\rho^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}} - 1 \text{ is non-decreasing in } \rho$$

and

$$\theta_V(x_0) \doteqdot \lim_{\rho \searrow 0} \frac{\mu_V(B_\rho(x_0))}{\rho^n} \text{ exists.}$$

Proof. The proof follows [23, Chapter 4, §4]. We provide the proof for $p \in (n, \infty)$ and let the reader make the necessary adjustments to the proof in the case of $p = \infty$.

For any $0 < \rho < R/(1+\epsilon)$,

$$\text{(A.4)}\qquad \frac{d}{d\rho}(\rho^{-n} I(\rho)) = \rho^{-n}\frac{d}{d\rho}J(\rho) - \rho^{-n}\int \rho^{-1}(x-x_0)\cdot H\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\, d\mu_V(x),$$

where $\varphi_\epsilon \colon [0,\infty) \to \mathbb{R}_{\geq 0}$ is a smooth, non-increasing function with $\varphi_\epsilon(s) \equiv 1$ for $s \in [0,1]$ and $\operatorname{supp}\varphi_\epsilon \subseteq [0, 1+\epsilon]$,

$$x_0 \in U \quad \text{and} \quad \overline{B_R(x_0)} \subseteq U,$$
$$I(\rho) \doteqdot \int \varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\, d\mu_V(x) \geq 0,$$
$$J(\rho) \doteqdot \int \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\, d\mu_V(x) \geq 0.$$

The integral on the right-hand side of (A.4) can be estimated by

$$\begin{aligned}
&\Big|\rho^{-n}\int \frac{x-x_0}{\rho}\cdot H\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\, d\mu_V(x)\Big| \\
&\quad \leq (1+\epsilon)\rho^{-n}\|H\|_{L^p(B_{(1+\epsilon)\rho}(x_0), d\mu_V)}(I(\rho))^{1-\frac{1}{p}} \\
&\quad \leq (1+\epsilon)\rho^{-n/p}\|H\|_{L^p(B_R(x_0), d\mu_V)}(\rho^{-n}I(\rho))^{1-\frac{1}{p}} \qquad \Big(0 < \rho < \frac{R}{1+\epsilon}\Big) \\
&\quad \leq (1+\epsilon)\rho^{-n/p}\|H\|_{L^p(B_R(x_0), d\mu_V)}(1+\rho^{-n}I(\rho)),
\end{aligned}$$

where in the last step we used $a \geq 0 \Rightarrow a^{1-1/p} \leq 1 + a$. Inserting this estimate into (A.4) yields

$$\text{(A.5)}\qquad \frac{d}{d\rho}(1+\rho^{-n}I(\rho)) \geq \rho^{-n}\frac{d}{d\rho}J(\rho) - (1+\epsilon)\rho^{-n/p}\|H\|_{L^p(B_R(x_0), d\mu_V)}(1+\rho^{-n}I(\rho)) \quad \text{for all } 0 < \rho < R/(1+\epsilon).$$

To simplify the notation, we write $\|H\| = \|H\|_{L^p(B_R(x_0),d\mu_V)}$ for the remainder of the proof. Multiplying (A.5) by the integrating factor

$$F_\epsilon(\rho) \doteqdot e^{\int_0^\rho (1+\epsilon)\|H\|\tilde{\rho}^{-n/p}\,d\tilde{\rho}} = e^{\frac{(1+\epsilon)\|H\|}{1-n/p}\rho^{1-n/p}} \geq 1$$

and using the fact that $\frac{d}{d\rho}J \geq 0$ yields

$$\frac{d}{d\rho}\big(F_\epsilon(\rho)\rho^{-n}I(\rho) + F_\epsilon(\rho) - 1\big) \geq F_\epsilon(\rho)\rho^{-n}\frac{d}{d\rho}J \geq \rho^{-n}\frac{d}{d\rho}J.$$

Integrating from σ to ρ then gives

$$\Big(F_\epsilon(\rho)\frac{I_\epsilon(\rho)}{\rho^n} + F_\epsilon(\rho) - 1\Big) - \Big(F_\epsilon(\sigma)\frac{I_\epsilon(\sigma)}{\sigma^n} + F_\epsilon(\sigma) - 1\Big) \geq \int_\sigma^\rho \tilde{\rho}^{-n}\frac{d}{d\tilde{\rho}}J\,d\tilde{\rho}.$$

Using that $\frac{d}{d\rho}(\varphi_\epsilon(|x-x_0|/\rho))$ is supported on the region $\rho \leq |x-x_0| \leq (1+\epsilon)\rho$, it follows that

$$\begin{aligned}
\int_\sigma^\rho \tilde{\rho}^{-n}\frac{d}{d\tilde{\rho}}J\,d\tilde{\rho} &= \int_\sigma^\rho \int \frac{|(x-x_0)^\perp|^2}{|x-x_0|^2}\tilde{\rho}^{-n}\frac{d}{d\tilde{\rho}}\Big(\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\Big)\,d\mu_V\,d\tilde{\rho} \\
&\geq \int_\sigma^\rho \int \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}}\frac{d}{d\tilde{\rho}}\Big(\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big)\Big)\,d\mu_V\,d\tilde{\rho} \\
&= \int \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}}\Big(\varphi_\epsilon\Big(\frac{|x-x_0|}{\rho}\Big) - \varphi_\epsilon\Big(\frac{|x-x_0|}{\sigma}\Big)\Big)\,d\mu_V \\
&\geq \int_{B_\rho(x_0)\setminus B_{(1+\epsilon)\sigma}(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}}\,d\mu_V.
\end{aligned}$$

Letting $\epsilon \searrow 0$ finally yields

$$\begin{aligned}
&\Big(e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}}\frac{\mu_V(B_\rho(x_0))}{\rho^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}} - 1\Big) \\
&\qquad - \Big(e^{\frac{\|H\|}{1-\frac{n}{p}}\sigma^{1-\frac{n}{p}}}\frac{\mu_V(B_\sigma(x_0))}{\sigma^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\sigma^{1-\frac{n}{p}}} - 1\Big) \\
&\geq \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{n+2}}\,d\mu_V \geq 0.
\end{aligned}$$

In particular, we have monotonicity of

$$\rho \mapsto e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}}\frac{\mu_V(B_\rho(x_0))}{\rho^n} + e^{\frac{\|H\|}{1-\frac{n}{p}}\rho^{1-\frac{n}{p}}} - 1.$$

Hence its limit as $\rho \searrow 0$ exists and thus $\theta_V(x_0)$ is well-defined. □

Acknowledgement. I would like to thank Felix Schulze for many helpful conversations, particularly regarding the results in [17]. I appreciate the anonymous referee's comments that improved the presentation of this paper.

References

[1] *W. K. Allard*, On the first variation of a varifold, Ann. of Math. (2) **95** (1972), 417–491.

[2] *S. Angenent*, *P. Daskalopoulos* and *N. Sesum*, Type II smoothing in mean curvature flow, preprint 2021, `https://arxiv.org/abs/2108.08725`.

[3] *K. A. Brakke*, The motion of a surface by its mean curvature, Math. Notes **20**, Princeton University, Princeton 1978.

[4] *O. Chodosh* and *F. Schulze*, Uniqueness of asymptotically conical tangent flows, Duke Math. J. **170** (2021), no. 16, 3601–3657.

[5] *T. H. Colding* and *W. P. Minicozzi, II*, Uniqueness of blowups and Ł ojasiewicz inequalities, Ann. of Math. (2) **182** (2015), no. 1, 221–285.

[6] *K. Ecker*, Regularity theory for mean curvature flow, Progr. Nonlinear Differential Equations Appl. **57**, Birkhäuser, Boston 2004.

[7] *K. Ecker* and *G. Huisken*, Interior estimates for hypersurfaces moving by mean curvature, Invent. Math. **105** (1991), no. 3, 547–569.

[8] *N. Edelen* and *L. Spolaor*, Regularity of minimal surfaces near quadratic cones, Ann. of Math. (2) **198** (2023), no. 3, 1013–1046.

[9] *R. Hardt* and *L. Simon*, Area minimizing hypersurfaces with isolated singularities, J. reine angew. Math. **362** (1985), 102–129.

[10] *G. Huisken*, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. **20** (1984), no. 1, 237–266.

[11] *T. Ilmanen*, Singularities of mean curvature flow of surfaces, unpublished.

[12] *T. Ilmanen*, Elliptic regularization and partial regularity for motion by mean curvature, Mem. Amer. Math. Soc. **108** (1994), no. 520, 1–90.

[13] *T. Ilmanen*, *A. Neves* and *F. Schulze*, On short time existence for the planar network flow, J. Differential Geom. **111** (2019), no. 1, 39–89.

[14] *K. Kasai* and *Y. Tonegawa*, A general regularity theory for weak mean curvature flow, Calc. Var. Partial Differential Equations **50** (2014), no. 1–2, 1–68.

[15] *O. A. Ladyzenskaja*, *V. A. Solonnikov* and *N. N. Ural'ceva*, Linear and quasi-linear equations of parabolic type, American Mathematical Society, Providence 1988.

[16] *T.-K. Lee* and *X. Zhao*, Uniqueness of conical singularities for mean curvature flows, J. Funct. Anal. **286** (2024), no. 1, Paper No. 110200.

[17] *J. D. Lotay*, *F. Schulze* and *G. Székelyhidi*, Neck pinches along the Lagrangian mean curvature flow of surfaces, preprint 2022, `https://arxiv.org/abs/2208.11054`.

[18] *A. Payne*, Mass drop and multiplicity in mean curvature flow, preprint 2020, `https://arxiv.org/abs/2009.14163`.

[19] *F. Schulze*, Uniqueness of compact tangent flows in mean curvature flow, J. reine angew. Math. **690** (2014), 163–172.

[20] *F. Schulze* and *B. White*, A local regularity theorem for mean curvature flow with triple edges, J. reine angew. Math. **758** (2020), 281–305.

[21] *P. Simoes*, On a class of minimal cones in $\mathbf{R}^n$, Bull. Amer. Math. Soc. **80** (1974), 488–489.

[22] *L. Simon*, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems, Ann. of Math. (2) **118** (1983), no. 3, 525–571.

[23] *L. Simon*, Introduction to geometric measure theory, Tsinghua Lectures 2014.

[24] *K. Smoczyk*, Mean curvature flow in higher codimension: Introduction and survey, in: Global Differential Geometry, Springer Proc. Math. **17**, Springer, Heidelberg (2012), 231–274.

[25] *B. Solomon* and *B. White*, A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals, Indiana Univ. Math. J. **38** (1989), no. 3, 683–691.

[26] *M. Stolarski*, Existence of mean curvature flow singularities with bounded mean curvature, Duke Math. J. **172** (2023), no. 7, 1235–1292.

[27] *S. Stuvard* and *Y. Tonegawa*, End-time regularity theorem for Brakke flows, Math. Ann. **390** (2024), no. 3, 3317–3353.

[28] *Y. Tonegawa*, A second derivative Hölder estimate for weak mean curvature flow, Adv. Calc. Var. **7** (2014), no. 1, 91–138.

[29] *Y. Tonegawa*, Brakke's mean curvature flow, SpringerBriefs in Math., Springer, Singapore 2019.

[30] *J. J. L. Velázquez*, Curvature blow-up in perturbations of minimal cones evolving by mean curvature flow, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **21** (1994), no. 4, 595–628.

[31] *B. White*, Partial regularity of mean-convex hypersurfaces flowing by mean curvature, Int. Math. Res. Not. IMRN **1994** (1994), no. 4, 185–192.
[32] *B. White*, Stratification of minimal surfaces, mean curvature flows, and harmonic maps, J. reine angew. Math. **488** (1997), 1–35.
[33] *B. White*, A local regularity theorem for mean curvature flow, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.
[34] *B. White*, Mean curvature flow with boundary, Ars Inven. Anal. **2021** (2021), Paper No. 4.
[35] *J. J. Zhu*, Ł ojasiewicz inequalities, uniqueness and rigidity for cylindrical self-shrinkers, Camb. J. Math. **13** (2025), no. 1, 173–224.

Maxwell Stolarski, Warwick Mathematics Institute, University of Warwick,
Coventry, CV4 7AL, United Kingdom
https://orcid.org/0009-0002-1544-3827
e-mail: max.stolarski@warwick.ac.uk

Eingegangen 13. April 2024, in revidierter Fassung 2. Oktober 2025