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Minimal surfaces with low genus in lens spaces

By *Xingzhe Li* at Ithaca, *Tongrui Wang* at Shanghai and *Xuan Yao* at Princeton

Abstract. Given a Riemannian RP^3 with a bumpy metric or a metric of positive Ricci curvature, we show that there either exist four distinct minimal real projective planes, or there exist one minimal real projective plane together with two distinct minimal 2-spheres. Our proof is based on a variant multiplicity one theorem for the Simon–Smith min-max theory under certain equivariant settings. In particular, we show under the positive Ricci assumption that RP^3 contains at least four distinct minimal real projective planes and four distinct minimal tori. Additionally, the number of minimal tori can be improved to five for a generic positive Ricci metric on RP^3 by the degree method. Moreover, using the same strategy, we show that, in the lens space $L(4m, 2m \pm 1), m \geq 1$, with a bumpy metric or a metric of positive Ricci curvature, there either exist $N(m)$ distinct minimal Klein bottles, or there exist one minimal Klein bottle and three distinct minimal 2-spheres, where $N(1) = 4$, $N(m) = 2$ for $m \geq 2$, and the first case happens under the positive Ricci assumption.

1. Introduction

In 1917, Birkhoff [5] proposed a min-max method and showed the existence of a closed geodesic in any Riemannian 2-sphere S^2. Using multi-parameter families of closed curves to sweep out S^2, Lusternik–Schnirelmann [31] (see also Grayson [18]) further showed that there are at least three closed geodesics in any Riemannian 2-sphere.

In one higher dimension, investigating minimal surfaces in S^3 is also a significant topic in differential geometry. In [28], Simon–Smith set out a min-max process to construct minimal surfaces with controls on topology, which is a variant of the min-max theory established by Almgren [1, 2] and Pitts [39] for minimal hypersurfaces (see also Schoen–Simon [41]). In particular, they proved the existence of an embedded minimal 2-sphere in any Riemannian S^3 (cf. [12]). In analogy with the result of Lusternik–Schnirelmann, S. T. Yau posed the following problem in his famous 1982 Problem Section [52].

The corresponding author is Tongrui Wang.

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Conjecture 1.1 (S. T. Yau [52]). *There are four distinct embedded minimal spheres in any manifold diffeomorphic to* S^3.

Towards this conjecture, White [51] first used degree methods to show the existence of at least two embedded minimal spheres in any Riemannian S^3 with positive Ricci curvature. In particular, if the metric is sufficiently close to the round metric, White [51] showed that there are at least four embedded minimal spheres. Additionally, it was also shown in [51] that the set of bumpy metrics is generic in the Baire sense, where a metric g_M on a given closed manifold M is called *bumpy* if every closed embedded minimal hypersurface is non-degenerate. Using Simon–Smith min-max theory and mean curvature flow, Haslhofer–Ketover [20] applied the catenoid estimates (cf. [27]) to overcome the difficulty of multiplicities and proved the existence of at least two embedded minimal spheres in S^3 with bumpy metrics. Recently, significant progress was made by Wang–Zhou [48], which resolved Yau's conjecture in S^3 with bumpy metrics or positive Ricci curvature via Simon–Smith min-max.

Inspired by the work of Wang–Zhou [48], we set out this paper to investigate minimal surfaces with low genus in lens spaces.

1.1. Minimal projective planes in RP^3. The motivation of Yau's conjecture is closely connected to the topological structure of the space of embedded spheres in S^3 (cf. Hatcher's proof of the Smale conjecture [21, Appendix (14)]). In light of the generalized Smale conjecture for lens spaces (cf. [3,4,26]), one would expect that the space of embedded projective planes in RP^3 can retract onto the space $\mathscr{E}$ of great projective planes (the quotient of great spheres). Note that $\mathscr{E}$ can be parameterized by $\mathbb{RP}^3$.[1)] Therefore, except for the area minimizing projective plane (associated with $H_0(\mathscr{E}, \mathbb{Z}_2)$), we can build three nontrivial (multi-parameter) families of great projective planes sweeping out the ambient manifold RP^3, each of which is associated with the generator of $H_i(\mathscr{E}, \mathbb{Z}_2) \cong \mathbb{Z}_2, i = 1, 2, 3$, respectively (cf. Section 6.1). Analogous to Conjecture 1.1, it is now reasonable to ask whether there exist at least four distinct embedded minimal projective planes in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$?

In this paper, we use equivariant min-max theory and a variant multiplicity one result to investigate the above question. Our first main result is the following dichotomy theorem.

Theorem 1.2. *Given a Riemannian* RP^3 *with a bumpy metric or a metric of positive Ricci curvature, either*

(i) *there exist at least four distinct embedded minimal projective planes, or*

(ii) *there exist one embedded minimal projective plane and two embedded minimal spheres.*

In particular, (i) *is valid when* RP^3 *has positive Ricci curvature.*

In [20], Haslhofer–Ketover proved in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ with positive Ricci curvature that there are at least two embedded minimal projective planes. Our result can now improve the number of minimal projective planes in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ to four under the positive Ricci assumption. Indeed, a minimal sphere Σ in RP^3 will be lifted to two disjoint minimal spheres $\Sigma_+ \sqcup \Sigma_-$ in S^3 so that the antipodal map on S^3 permutes $\Sigma_\pm$. Combining with the *embedded Frankel property* in

1) Throughout the paper, we will use RP^n to denote the ambient manifold or the submanifold, and use $\mathbb{RP}^n$ to denote the parameter space.

manifolds with positive Ricci curvature (i.e. any two closed embedded minimal hypersurfaces must intersect), we conclude Theorem 1.2 (ii) cannot happen if $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ has positive Ricci curvature.

In recent years, we have witnessed the tremendous progress of Almgren–Pitts min-max theory. One peak of the development is the resolution of Yau's conjecture [52, Problem 88] on the existence of infinitely many closed minimal surfaces due to Marques–Neves [34] and Song [43]. Meanwhile, several significant advancements [22,36,44] in the spatial distribution of closed minimal hypersurfaces were made by the multi-parameter min-max theory coupled with the Weyl law for the volume spectrum [29]. Moreover, after the establishment of a min-max theory for prescribed mean curvature hypersurfaces [56, 57], Zhou [54] solved the multiplicity one conjecture in the Almgren–Pitts setting. Combining with the Morse index estimates, Marques–Neves [33, 35] complete their celebrated program on the Morse theory for the area functional (see also [32]). We refer to [55] for a more detailed history.

1.2. Strategy of the proof. Although our proof shares the same spirit as Wang–Zhou's resolution [48], one major difference is that the sweepouts are formed by 1-sided projective planes. However, Wang–Zhou's multiplicity one theorem [48, Theorem B] relies heavily on the min-max theory [48, Theorem 2.4] for prescribing mean curvature functionals $\mathcal{A}^h$, (2.2), while it is inappropriate to define the $\mathcal{A}^h$ functional or prescribed mean curvature functions for 1-sided surfaces. Hence [48, Theorem B] cannot be applied directly.

Let $\pi\colon S^3 \to \mathrm{RP}^3$ be the double cover and g_- the involution map in S^3, i.e.

$$\pi(p) = \pi(g_-(p)) \quad \text{and} \quad g_-(p) \neq p \quad \text{for all } p \in S^3.$$

Heuristically, note that a projective plane RP^2 in RP^3 can be lifted to a sphere $S = \pi^{-1}(\mathrm{RP}^2)$ which separates S^3 into two parts Ω_+, Ω_- so that $\Omega_\pm = g_-(\Omega_\mp)$. Hence if ν is a unit normal along S (with respect to the lifted metric), then $dg_-(\nu) = -\nu$ and the mean curvature H of S with respect to ν is antisymmetric $H(p) = -H(g_-(p))$. This inspired us to lift the sweepouts in RP^3 back into S^3 and consider the $\mathbb{Z}_2 = \{\mathrm{id}, g_-\}$ equivariant min-max theory in S^3 for the prescribed mean curvature functionals $\mathcal{A}^h$ with antisymmetric h (i.e. $h(p) = -h(g_-(p))$).

As one key novelty of this paper, we show the following multiplicity one theorem in a certain equivariant setting analogy to the above, which provides an alternative way to handle the multiplicities in the 1-sided Simon–Smith min-max theory.

Theorem 1.3. *Let (M, g_M) be a closed connected orientable 3-dimensional Riemannian manifold, and let G be a finite group acting freely and effectively as isometries on M so that G admits an index 2 subgroup G_+ with coset $G_- = G \setminus G_+$. Suppose Σ_0 is an orientable G-invariant genus-$\mathfrak{g}_0$ surface, and $\mathscr{X}_{G_\pm}$ consists of all the G-equivariant embedding $\Sigma = \phi(\Sigma_0)$ of Σ_0 into M so that $M \setminus \Sigma = \Omega_+ \sqcup \Omega_-$ and $G_+ \cdot \Omega_\pm = \Omega_\pm$, $G_- \cdot \Omega_\pm = \Omega_\mp$. Then, given any homotopy class of smooth sweepouts formed by surfaces in $\mathscr{X}_{G_\pm}$, the associated min-max varifold can be chosen as a G-varifold induced by a closed embedded G-invariant minimal surface Γ with connected components $\{\Gamma_j\}_{j=1}^J$ and integer multiplicities $\{m_j\}_{j=1}^J$ so that*

(i) *if Γ_j is unstable and 2-sided, then $m_j = 1$;*

(ii) *if Γ_j is 1-sided, then its connected double cover is stable.*

Moreover, the weighted total genus of Γ given by (4.1) is bounded by $\mathfrak{g}_0$.

Remark 1.4. (1) By re-merging the components Γ_{j_i} that lie in the same G-orbit $G \cdot \Gamma_j$, we can assume each Γ_j is G-invariant and Γ_j/G is connected. In applications, one can first take a sweepout of M/G formed by 1-sided surfaces in $\{\Sigma/G \mid \Sigma \in \mathscr{X}_{G_\pm}\}$, and then pull it back to a sweepout of M in $\mathscr{X}_{G_\pm}$. Clearly, the min-max G-surfaces $\{\Gamma_j\}_{j=1}^J$ given by Theorem 1.3 can be reduced to min-max surfaces $\{\Gamma_j/G\}_{j=1}^J$ in M/G with the same multiplicity. One should note that the multiplicity m_j is related to the stability and the orientability of Γ_j by Theorem 1.3, not to Γ_j/G.

(2) For a general compact Lie group G acting by isometries on M^{n+1} with

$$3 \leq \operatorname{codim}(G \cdot x) \leq 7 \quad \text{for all } x \in M,$$

the G-equivariant min-max theory for closed minimal G-hypersurfaces has been established by Ketover [23], Liu [30], and the second author [46, 47] in different settings. However, our scenarios cannot be covered directly by these works since we also need an equivariant min-max for the $\mathcal{A}^h$-functional (2.2) and both h and $\Omega_\pm$ are G_--antisymmetric.

After applying the above theorem to $M = S^3$, $G = \{\mathrm{id}, g_-\}$, $G_+ = \{\mathrm{id}\}$, $G_- = \{g_-\}$, and $\mathscr{X}_{G_\pm} = \{\pi^{-1}(P) \mid P \text{ is an embedded } \mathrm{RP}^2 \subset \mathrm{RP}^3\}$, we obtain some G-invariant unions of disjoint minimal spheres $\{\Gamma_j\}_{j=1}^J$ with multiplicities $\{m_j\}_{j=1}^N$ satisfying (i)–(ii), which alternatively gives a multiplicity result for the min-max outputs $\{\widetilde{\Gamma}_i = \pi(\Gamma_j)\}_{j=1}^J$ in RP^3.

Meanwhile, another difficulty arises as $\{\widetilde{\Gamma}_j\}_{j=1}^J$ may also contain minimal spheres instead of minimal projective planes. Since spheres have no contribution to the total genus, the genus bound does not help to distinguish them. Nevertheless, we can show that (Proposition 5.4) every min-max minimal surface $\bigcup_{j=1}^J \widetilde{\Gamma}_j$ associated with the sweepouts formed by (1-sided) projective planes contains exactly one minimal projective plane, i.e. $\widetilde{\Gamma}_1 \cong \mathrm{RP}^2$ and $\widetilde{\Gamma}_{j\geq 2} \cong S^2$. Hence, if RP^3 has positive Ricci curvature, we see $\bigcup_{j=1}^J \widetilde{\Gamma}_j = \widetilde{\Gamma}_1 \cong \mathrm{RP}^2$ (by the Frankel property) and $\pi^{-1}(\widetilde{\Gamma}_1)$ is unstable in S^3, which implies $m_j = 1$ (Theorem 1.3) and gives four distinct minimal projective planes associated with $H_i(\mathscr{E}, \mathbb{Z}_2)$, $i = 0, 1, 2, 3$.

However, in general, we can neither eliminate minimal spheres in $\{\widetilde{\Gamma}_j\}_{j=1}^J$ nor rule out the existence of minimal projective planes $P \subset \mathrm{RP}^3$ with stable $\pi^{-1}(P) \subset S^3$. Moreover, if we cut RP^3 along a minimal projective plane P with stable $\pi^{-1}(P) \subset S^3$, we would obtain one 3-ball B with boundary $\partial B = \pi^{-1}(P)$. After adding a cylindrical end $\pi^{-1}(P) \times [0, \infty)$ to B and applying Song's technique [43] as in [48, Sections 8.2, 8.4], one can only find minimal spheres (instead of minimal projective planes) confined in $\operatorname{int}(B)$. Therefore, given the above phenomena, it is reasonable to propose the dichotomy theorem (Theorem 1.2) for the coexistence of both minimal projective planes and minimal spheres.

Indeed, we would conjecture that, under a certain metric on RP^3, every minimal real projective plane constructed via min-max could be the same area minimizing one, but with possibly different multiplicities. A possible example is the Riemannian RP^3 so that its double cover S^3 with the pull-back metric looks like a symmetric dumbbell with a long thin neck $S^2 \times [-R, R]$ satisfying that the center $S^2 \times \{0\}$ covers the area minimizing RP^2, and for $t \neq 0$, $S^2 \times \{t\}$ has positive mean curvature pointing towards $S^2 \times \{0\}$.

1.3. Minimal surfaces with Euler characteristic 0 in lens spaces $L(p, q)$. The above idea can also be applied to investigate minimal Klein bottles and minimal tori in lens spaces $L(p, q) \cong S^3/\mathbb{Z}_p$, where $p \geq 2$, $1 \leq q \leq p - 1$, and $\mathbb{Z}_p = \langle \xi_{p,q} \rangle$ acts freely on S^3 by (7.1).

In particular, we have a very similar dichotomy theorem for minimal Klein bottles in $L(4m, 2m \pm 1)$ as shown below. Note that $L(4m, 2m \pm 1)$, $m \geq 1$, are the only lens spaces that contain embedded Klein bottles (cf. [8, Corollary 6.4]).

Theorem 1.5. *Let $m \geq 1$ be an integer, $N(1) = 4$, and $N(m) = 2$ for $m \geq 2$. Then, given a lens space $L(4m, 2m \pm 1)$ with a bumpy metric or a metric of positive Ricci curvature, either*

(i) *there exist $N(m)$ embedded minimal Klein bottles, or*

(ii) *there exist one embedded minimal Klein bottle and three embedded minimal spheres.*

In particular, (i) *is valid when $L(4m, 2m \pm 1)$ has positive Ricci curvature.*

One notices that an embedded Klein bottle K in $L(4m, 2m \pm 1)$ can be lifted to an embedded G-invariant torus T in S^3 so that $S^3 \setminus T$ has two G_+-invariant components $U_+ \sqcup U_-$ with G_- permuting them, where

$$G = \mathbb{Z}_{4m} = \langle \xi_{4m,2m\pm 1} \rangle, \quad G_+ = \mathbb{Z}_{2m} = \langle \xi^2_{4m,2m\pm 1} \rangle \quad \text{and} \quad G_- = G \setminus G_+.$$

Hence, using the idea in Remark 1.4, we can lift the sweepout in $L(4m, 2m \pm 1)$ formed by Klein bottles to a family of $\mathbb{Z}_{4m}$-invariant tori in S^3, and apply Theorem 1.3 to obtain closed embedded $\mathbb{Z}_{4m}$-invariant minimal surfaces $\bigcup_{j=1}^J \Gamma_j$ in S^3 with the total weighted genus bounded by 1. Similarly,

$$\bigcup_{j=1}^{J} \widetilde{\Gamma}_j = \bigcup_{j=1}^{J} \Gamma_j / \mathbb{Z}_{4m}$$

contains exactly one embedded minimal Klein bottle $\widetilde{\Gamma}_1$ (Corollary 7.2). Then $\widetilde{\Gamma}_1$ must have multiplicity one due to the weighted genus bound. The difficulty mainly comes from the possible existence of minimal spheres, which can be handled similarly by considering the coexistence of them.

Moreover, we also establish a new result regarding minimal tori in lens spaces.

Theorem 1.6. *Let $L(p, q)$, $p \geq 2$, be a lens space with a metric of positive Ricci curvature. Then*

(i) *$L(2, 1) \cong \mathrm{RP}^3$ contains at least four embedded minimal tori; in particular, for almost every (in the sense of Baire category) Riemannian metric of positive Ricci curvature on* RP^3*, there are at least five embedded minimal tori;*

(ii) *$L(p, 1)$ and $L(p, p-1)$ with $p > 2$ contains at least three embedded minimal tori;*

(iii) *$L(p, q)$ with $q \notin \{1, p-1\}$ contains at least one embedded minimal torus.*

Remark 1.7. Using the weighted genus bound [25, (2.6)] and the catenoid estimates [27], Ketover concluded under the positive Ricci assumption that there are three minimal tori in $L(p, 1)$, $p \neq 2$ (see [25, Theorem 4.9]), and two minimal Klein bottles in $L(4p, 2p \pm 1)$, $p > 1$ (see [25, Theorem 4.8]). Theorems 1.5 and 1.6 extend these results by some new constructions and the multiplicity one theorem.

Ketover–Liokumovich [26, Theorem 1.4] showed as an equivalent formulation of the Smale conjecture in lens spaces that the space of Heegaard tori in $L(p,q)$ retracts onto the space $\mathscr{E}_{p,q}$ of Clifford tori. Hence the parameterization of $\mathscr{E}_{p,q}$ (cf. [26, Proposition 2.3]) and the Lusternik–Schnirelmann theory support the counts of minimal tori in Theorem 1.6.

It is also noteworthy that the space of Clifford tori in S^3 and RP^3 can both be parameterized by $\mathbb{RP}^2 \times \mathbb{RP}^2$. One would then expect five distinct minimal tori in a Riemannian S^3 and RP^3 (cf. White's conjecture in [50]). However, after White's existence result of one minimal torus in any positive Ricci curved S^3 (see [51, Theorem 6.2]) via the degree method, there has been no substantial progress (to the best of the author's knowledge) in the general construction of more minimal tori in a Riemannian S^3. Nevertheless, our Theorem 1.6 (i) has fulfilled the expectation in RP^3 to a certain extent.

The main challenges in finding minimal tori lie in two aspects. Firstly, the moduli space $\mathscr{X}$ of embedded tori in S^3 or RP^3 is very complicated, making it difficult to find an element

$$\alpha \in H^1(\mathscr{X} \cup \partial\mathscr{X}, \partial\mathscr{X}; \mathbb{Z}_2) \quad \text{with } \alpha^5 \neq 0 \in H^5(\mathscr{X} \cup \partial\mathscr{X}, \partial\mathscr{X}; \mathbb{Z}_2),$$

while such an α is crucial for applying the Lusternik–Schnirelmann theory. In Section 8.1, we address this by considering part of the boundary $\mathscr{Y} := \{\text{great circles}\} \subset \partial\mathscr{X}$ and identifying an element $\alpha \in H^1(\mathscr{X} \cup \mathscr{Y}, \mathscr{Y}; \mathbb{Z}_2)$ with $\alpha^4 \neq 0$ (Lemma 8.2). This approach allows us to find four minimal tori in RP^3 with positive Ricci curvature and potentially five via White's degree method. Indeed, this construction of α is also valid in S^3. Additionally, another challenge is to distinguish tori from 2-spheres in min-max outputs. In lens space, the positive Ricci assumption is sufficient to exclude all the minimal 2-spheres by the embedded Frankel property, which simplifies the problem considerably. In contrast, this advantage is absent in S^3, where distinguishing between these surfaces remains challenging.

2. Preliminaries

In this section, we collect necessary preliminaries and notation in the equivariant setting.

Let (M^3, g_M) be an oriented connected closed Riemannian 3-dimensional manifold and let G be a finite group acting freely and effectively as isometries on M. By [38], M^3 can be G-equivariantly isometrically embedded into some $\mathbb{R}^L$, namely there is an isometric embedding i from M into $\mathbb{R}^L$ which is equivariant with respect to an orthogonal representation ρ of G on $\mathbb{R}^L$ ($i \circ g = \rho(g) \circ i$ for all $g \in G$). In addition, let G_+ be an index 2 subgroup of G (i.e. $[G : G_+] = 2$), and denote by G_- the coset of G_+ in G (i.e. $G = G_+ \sqcup G_-$). Note that we only specify the ambient manifold M^3 and the free actions of G to S^3 and $\mathbb{Z}_p$ respectively in Sections 6, 7, 8.

To signify the G-invariance, we will add a superscript or a subscript "G" to the usual terminologies. We also use "$G_\pm$" to emphasize G_+-symmetry together with G_--antisymmetry.

- π: the projection $\pi\colon M \mapsto M/G$ given by $p \mapsto [p] := \{g \cdot p \mid g \in G\}$;
- $\mathcal{H}^m$: the m-dimensional Hausdorff measure in $\mathbb{R}^L$;
- $B_r(p)$: Euclidean open ball of radius $r > 0$ centered at p;
- $A_{s,r}(p)$: Euclidean open annulus $B_r(p) \setminus \mathrm{Clos}(B_s(p))$;
- $B^G_r(p) := B_r(G \cdot p) = G \cdot B_r(p)$ and $A^G_{s,r}(p) := A_{s,r}(G \cdot p) = G \cdot A_{s,r}(p)$;

- $h \in C^{\infty}_{G_\pm}(M)$: a smooth mean curvature prescribing function, where for $k \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned}(2.1)\qquad C^k_{G_\pm}(M) := \{f \in C^k(M) \mid f(p) = f(g\cdot p) \text{ for all } p \in M,\ g \in G_+,\\ f(p) = -f(g\cdot p) \text{ for all } p \in M,\ g \in G_-\};\end{aligned}$$

- $\mathcal{C}(M)$, $\mathcal{C}^G(M)$: the space of sets $\Omega \subset M$ with finite perimeter (i.e. Caccioppoli sets, cf. [42, §40]), and the space of $\Omega \in \mathcal{C}(M)$ with $g\cdot\Omega = \Omega$ for all $g \in G$ (i.e. G-invariant Caccioppoli sets);
- $\mathcal{C}^{G_\pm}(M)$: the space of G_+-invariant Caccioppoli sets $\Omega \in \mathcal{C}(M)$ so that G_- permutes Ω and $M \setminus \Omega$ as two (G_+-invariant) Caccioppoli sets, i.e.

$$\begin{aligned}\mathcal{C}^{G_\pm}(M) := \{\Omega \in \mathcal{C}(M) \mid g\cdot\Omega = \Omega \text{ for all } g \in G_+,\\ g\cdot\Omega = M\setminus\Omega \text{ in } \mathcal{C}(M) \text{ for all } g \in G_-\};\end{aligned}$$

- $\mathcal{V}(M)$, $\mathcal{V}^G(M)$: the space of 2-varifolds in M^3, and the space of $V \in \mathcal{V}(M)$ with $g_\# V = V$ for all $g \in G$ (i.e. G-invariant 2-varifolds);
- $\mathfrak{X}(M)$, $\mathfrak{X}^G(M)$: the space of smooth vector fields in M, and the space of $X \in \mathfrak{X}(M)$ with $dgX = X$ for all $g \in G$;
- $\mathrm{Diff}_0(M)$: the connected component of the diffeomorphism group of M containing the identity.

For any $\Omega \in \mathcal{C}(M)$, denote by $\partial\Omega$ the (reduced) boundary of $[[\Omega]]$ as a 3-dimensional mod 2 flat chain, by $|\partial\Omega|$ the induced integral varifold, and by $\nu_{\partial\Omega}$ the outward pointing unit normal of $\partial\Omega$ (see [42, §14]). In addition, denote by $\|\partial\Omega\|$ and $\|V\|$ the induced Radon measure in M associated with $\partial\Omega$ and $V \in \mathcal{V}(M)$.

Remark 2.1. For any $\Omega \in \mathcal{C}^{G_\pm}(M^3)$, the 2-dimensional mod 2 flat chain $\partial\Omega$ and the varifold $|\partial\Omega|$ are both G-invariant. Besides, the outward unit normal $\nu_{\partial\Omega}$ is G_+-invariant so that $dg(\nu_{\partial\Omega}) = -\nu_{\partial\Omega}$ for any $g \in G_-$. As h is G_+-invariant with $h \circ g = -h$ for any $g \in G_-$, we conclude that $h\cdot\nu_{\partial\Omega}$ is G-invariant.

Let $U \subset M$ be an open set. Then we denote by $\mathfrak{Is}(U)$ the set of isotopies of M supported in U. Additionally, after replacing M by U, one can similarly define the localized notions $\mathcal{C}(U)$, $\mathcal{V}(U)$, $\mathfrak{X}(U)$, and $\mathcal{C}^G(U)$, $\mathcal{C}^{G_\pm}(U)$, $\mathcal{V}^G(U)$, $\mathfrak{X}^G(U)$ provided that U is G-invariant.

Note that a map $F\colon M \to M$ is said to be *G-equivariant* if $F(g\cdot p) = g\cdot F(p)$ for all $p \in M$, $g \in G$. We can now introduce the following notation:

- $\mathrm{Diff}^G_0(M)$: the connected component of the G-equivariant diffeomorphism group of M containing the identity;
- $\mathfrak{Is}^G(U)$: the set of G-equivariant isotopies of M supported in an open G-set U.

Furthermore, for any G-invariant subset $A \subset M$ with connected components $\{A_i\}_{i=1}^I$, we say that A is *G-connected* if, for any $i, j \in \{1,\dots,I\}$, there is $g_{i,j} \in G$ with $g_{i,j}\cdot A_j = A_i$. Finally, given a (G-)connected open set $U \subset M$, we say that a set of (G-)connected C^1-embedded (G-)surfaces $\{\Gamma^i\}_{i=1}^l \subset U$ with $\partial\Gamma^i \cap U = \emptyset$ is *ordered*, denoted by $\Gamma^1 \le \cdots \le \Gamma^l$, if, for each i, Γ_i separates U into two (G-)connected components $U^i_\pm$ (i.e. $U \setminus \Gamma_i = U^i_+ \sqcup U^i_-$) so that $\Gamma^1,\dots,\Gamma^{i-1} \subset \mathrm{Clos}(U^i_-)$ and $\Gamma^{i+1},\dots,\Gamma^l \subset \mathrm{Clos}(U^i_+)$.

Since the group actions in this paper are free and effective, we can make the following definition.

Definition 2.2. A G-invariant open set $U \subset M$ is said to be *appropriately small* if each connected component of U is diffeomorphic to $\pi(U) = U/G$.

Namely, an appropriately small open G-set U shall have $\#G$ connected components $\{U_i\}_{i=1}^{\#G}$ with G permuting them. For instance, any open G-set $U \subset B_r^G(p)$ is appropriately small provided $r > 0$ is smaller than the injectivity radius of $G \cdot p$.

2.1. $\mathcal{A}^h$-functional and $\mathcal{VC}^{G_\pm}$-space. For any pairs

$$(V, \Omega) \in \mathcal{V}^G(M) \times \mathcal{C}^{G_\pm}(M) \subset \mathcal{V}(M) \times \mathcal{C}(M)$$

and $h \in C^\infty_{G_\pm}(M)$, define the prescribing mean curvature functional by

$$\mathcal{A}^h(V, \Omega) = \|V\|(M) - \int_\Omega h \, d\mathcal{H}^3. \tag{2.2}$$

Note that $F_\#(V, \Omega) := (F_\# V, F(\Omega))$ is still a pair in $\mathcal{V}^G(M) \times \mathcal{C}^{G_\pm}(M)$ for any G-equivariant diffeomorphism F of M. Hence we have the following definitions generalizing [48, Definition 1.1].

Definition 2.3. A pair $(V, \Omega) \in \mathcal{V}^G(M) \times \mathcal{C}^{G_\pm}(M)$ is said to be $\mathcal{A}^h$-*stationary* (resp. $(G, \mathcal{A}^h)$-*stationary*) in a (resp. G-invariant) open set $U \subset M$ if, for any $X \in \mathfrak{X}(U)$ (resp. $X \in \mathfrak{X}^G(U)$) with generated diffeomorphisms $\{\phi^t\}$,

$$\begin{aligned} \delta \mathcal{A}^h_{V,\Omega}(X) &:= \frac{d}{dt}\Big|_{t=0} \mathcal{A}^h(\phi^t_\#(V, \Omega)) \\ &= \int_{G_2(M)} \operatorname{div}_S X(x) \, dV(x, S) - \int_{\partial\Omega} \langle X, \nu_{\partial\Omega} \rangle h \, d\mu_{\partial\Omega} = 0. \end{aligned} \tag{2.3}$$

In addition, (V, Ω) is said to be $\mathcal{A}^h$-*stable* (resp. $(G, \mathcal{A}^h)$-*stable*) in U if

$$\delta^2 \mathcal{A}^h_{V,\Omega}(X, X) := \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}^h(\phi^t_\#(V, \Omega)) \geq 0$$

for any $X \in \mathfrak{X}(U)$ (resp. $X \in \mathfrak{X}^G(U)$).

In the following lemma, we shall prove that $(G, \mathcal{A}^h)$-stationarity implies $\mathcal{A}^h$-stationarity. Note that the proof also works for general (non-free non-effective) G-actions.

Lemma 2.4. *Given any G-invariant open set $U \subset M$, a pair*

$$(V, \Omega) \in \mathcal{V}^G(M) \times \mathcal{C}^{G_\pm}(M)$$

is $(G, \mathcal{A}^h)$-stationary in U if and only if (V, Ω) is $\mathcal{A}^h$-stationary in U.

Proof. It is sufficient to show that, for any $X \in \mathfrak{X}(U)$, there exists $X_G \in \mathfrak{X}^G(U)$ so that $\delta \mathcal{A}^h_{V,\Omega}(X) = \delta \mathcal{A}^h_{V,\Omega}(X_G)$. Indeed, define $X_G := (\sum_{g \in G} dg^{-1}(X))/\#G$ to be the averaged G-invariant vector field compactly supported in U. Since $\partial\Omega$ and $h\nu_{\partial\Omega}$ are both G-invariant (Remark 2.1), we have $\int_{\partial\Omega}\langle X, h\nu_{\partial\Omega}\rangle\, d\mu_{\partial\Omega} = \int_{\partial\Omega}\langle X_G, h\nu_{\partial\Omega}\rangle\, d\mu_{\partial\Omega}$. In addition, note that the diffeomorphisms generated by X and $dg^{-1}(X)$ are $\{\phi^t\}$ and $\{g^{-1}\circ\phi^t\circ g\}$ respectively. Hence the G-invariance of V implies $\|\phi^t_\# V\|(M) = \|(g^{-1}\circ\phi^t\circ g)_\# V\|(M)$ and $\int \operatorname{div}_S X\, dV = \int \operatorname{div}_S(dg^{-1}(X))\, dV = \int \operatorname{div}_S X_G\, dV$. □

Remark 2.5. In particular, if the open G-set U is appropriately small (Definition 2.2), then we also have the equivalence between the G-stability and stability in U. Indeed, since the stability of (V,Ω) in U is equivalent to the stability in every component U_i of U, $i = 1,\dots,\#G$, it follows from (2.6) and the proof of Lemma 2.18 that

$$\#G\cdot\delta^2\mathcal{A}^h_{V,\Omega}(X,X) = \delta^2\mathcal{A}^h_{V,\Omega}(X_G,X_G),$$

where $X\in\mathfrak{X}(U_i)$ and $X_G = \sum_{g\in G} dg^{-1}(X)\in\mathfrak{X}^G(U)$.

Motivated by Almgren's VZ-space [2], we make the following definition for the weak topological completion of the diagonal pairs

$$\{(|\partial\Omega|,\Omega)\mid \Omega\in\mathcal{C}^{G\pm}(M)\}\subset\mathcal{V}^G(M)\times\mathcal{C}^{G\pm}(M).$$

Definition 2.6. The $\mathcal{VC}$*-space* $\mathcal{VC}(M)$ (resp. $\mathcal{VC}^{G\pm}$*-space* $\mathcal{VC}^{G\pm}(M)$) is the space of all pairs (V,Ω) in $\mathcal{V}(M)\times\mathcal{C}(M)$ (resp. $\mathcal{V}^G(M)\times\mathcal{C}^{G\pm}(M)$) so that $V = \lim_{k\to\infty}|\partial\Omega_k|$ and $\Omega = \lim_{k\to\infty}\Omega_k$ for some sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ in $\mathcal{C}(M)$ (resp. $\mathcal{C}^{G\pm}(M)$).

For any $(V,\Omega),(V',\Omega')\in\mathcal{VC}(M)$, define the $\mathscr{F}$*-distance* between them by

$$\mathscr{F}((V,\Omega),(V',\Omega')) := \mathbf{F}(V,V') + \mathcal{F}(\Omega,\Omega'),$$

where $\mathbf{F}$ is the varifolds $\mathbf{F}$-metric [39, Section 2.1] and $\mathcal{F}$ is the flat metric [42, §31].

Note that $\mathcal{V}^G(M)$ and $\mathcal{C}^{G\pm}(M)$ are closed sub-spaces of $\mathcal{V}(M)$ and $\mathcal{C}(M)$ respectively. One verifies that $\mathcal{VC}^{G\pm}(M)$ is also a closed sub-space of $\mathcal{VC}(M)$. Hence the $\mathscr{F}$-distance can be induced to $\mathcal{VC}^{G\pm}(M)$.

Moreover, combining Lemma 2.4 with [48, Lemmas 1.4–1.7], we have the following results.

Lemma 2.7. *Set $c := \sup_{p\in M}|h(p)|$. Then, for any $C>0$, $(V,\Omega)\in\mathcal{VC}^{G\pm}(M)$, and G-invariant open set $U\subset M$, the following statements are valid:*

(i) $\operatorname{spt}(\partial\Omega)\subset\operatorname{spt}(\|V\|)$ *and* $\|\partial\Omega\|\le\|V\|$;

(ii) *if (V,Ω) is $(G,\mathcal{A}^h)$-stationary in U, then V has c-bounded first variation in U, i.e.*

$$|\delta V(X)| := \left|\int_{G_2(M)}\operatorname{div}_S X(x)\, dV(S,x)\right|\le c\int_M |X|\, d\|V\| \quad \text{for all } X\in\mathfrak{X}(U);$$

(iii) $A^C := \{(V,\Omega)\in\mathcal{VC}^{G\pm}(M)\mid \|V\|(M)\le C\}$ *is a compact metric space with respect to the $\mathscr{F}$-distance;*

(iv) $A^C_0 := \{(V,\Omega)\in A^C\mid (V,\Omega)$ *is* $(G,\mathcal{A}^h)$*-stationary*$\}$ *is a compact subset of A^C with respect to the $\mathscr{F}$-distance.*

Proof. Since $\mathcal{VC}^{G_\pm}(M)$ is a closed sub-space of $\mathcal{VC}(M)$, (i) and (iii) follow directly from [48, Lemmas 1.4, 1.6]. Combining Lemma 2.4 and [48, Lemmas 1.5, 1.7], we conclude (ii) and (iv). □

2.2. $C^{1,1}$ almost embedded $(G_\pm, h)$-surfaces. In this subsection, we consider the $\mathcal{A}^h$-functional for $C^{1,1}$ (almost embedded) $G_\pm$-boundaries.

Definition 2.8. Let $U \subset M$ be an open subset. A $C^{1,1}$ immersed surface $\phi\colon \Sigma \to U$ with $\phi(\partial\Sigma) \cap U = \emptyset$ is said to be a $C^{1,1}$ *almost embedded surface* in U if, at any non-embedded point $p \in \phi(\Sigma)$, there is a neighborhood $W \subset U$ of p so that

- $\Sigma \cap \phi^{-1}(W)$ is a disjoint union of connected components $\bigsqcup_{i=1}^{l} \Gamma^i$;
- $\phi(\Gamma^i) \subset W$ is a $C^{1,1}$ embedding for each $i = 1, \dots, l$;
- for each i, any other component $\phi(\Gamma^j)$ $(j \neq i)$ lies on one side of $\phi(\Gamma^i)$, i.e.

$$\phi(\Gamma^j) \leq \phi(\Gamma^i) \quad \text{or} \quad \phi(\Gamma^i) \leq \phi(\Gamma^j).$$

For simplicity, we will denote $\phi(\Sigma)$ and $\phi(\Gamma^i)$ by Σ and Γ^i respectively in appropriate context. The subset of non-embedded points in Σ, denoted by $\mathcal{S}(\Sigma)$, is called the *touching set*, and $\mathcal{R}(\Sigma) := \Sigma \setminus \mathcal{S}(\Sigma)$ is the *regular set*.

Definition 2.9 ($C^{1,1}$ $G_\pm$-boundary). Let $U \subset M$ be a G-invariant open subset. Then a G-equivariant $C^{1,1}$ almost embedded surface $\phi\colon \Sigma \to U$ is said to be a $C^{1,1}$ *(almost embedded) $G_\pm$-boundary* in U if Σ is oriented and

$$\phi_\#([[\Sigma]]) = \partial\Omega$$

as 2-currents with $\mathbb{Z}_2$-coefficients in U for some $\Omega \in \mathcal{C}^{G_\pm}(U)$.

Remark 2.10. Let (Σ, Ω) be a $C^{1,1}$ $G_\pm$-boundary in U. Then both of $\mathcal{R}(\Sigma)$ and $\mathcal{S}(\Sigma)$ are G-invariant. Additionally, it follows from [48, Lemma 1.11] that Σ (as an immersed surface) admits a unit normal ν_Σ so that

- $\nu_\Sigma = \nu_{\partial\Omega}$ along $\operatorname{spt}(\partial\Omega)$ provided $\Omega \neq \emptyset$ or U;
- if Σ decomposes into ordered G-connected sheets $\Gamma^1 \leq \cdots \leq \Gamma^l$ in a G-connected open set $W \subset U$, then ν_Σ must alternate orientations along $\{\Gamma^i\}_{i=1}^{l}$.

For any $C^{1,1}$ $G_\pm$-boundary (Σ, Ω) (in M), define its $\mathcal{A}^h$-functional by

$$\mathcal{A}^h(\Sigma, \Omega) := \mathcal{H}^2(\Sigma) - \int_\Omega h \, d\mathcal{H}^3$$

It then follows from (2.3) and [48, Lemma 1.12] that, for any $X \in \mathfrak{X}(M)$,

$$\begin{aligned} \delta\mathcal{A}^h_{\Sigma,\Omega}(X) &= \int_\Sigma \operatorname{div}_\Sigma X \, d\mathcal{H}^2 - \int_{\partial\Omega} \langle X, \nu_{\partial\Omega}\rangle h \, d\mu_{\partial\Omega} \\ &= \int_\Sigma \operatorname{div}_\Sigma X - \langle X, \nu_\Sigma\rangle h \, d\mathcal{H}^2, \end{aligned} \tag{2.4}$$

where ν_Σ is given in the above remark.

Definition 2.11 ($C^{1,1}$ $(G_\pm, h)$-boundary). A $C^{1,1}$ $G_\pm$-boundary (Σ, Ω) in a G-invariant open set U is said to be *$\mathcal{A}^h$-stationary* (resp. *$(G, \mathcal{A}^h)$-stationary*) in U if, for any $X \in \mathfrak{X}(U)$ (resp. $X \in \mathfrak{X}^G(U)$), $\delta \mathcal{A}^h_{\Sigma,\Omega}(X) = 0$. In particular, we say that (Σ, Ω) is a *$C^{1,1}$ (almost embedded) $(G_\pm, h)$-boundary* in U if it is $(G, \mathcal{A}^h)$-stationary in U.

Similar to Lemma 2.4, we have the following result.

Lemma 2.12. *Given any G-invariant open set $U \subset M$, a $C^{1,1}$ $G_\pm$-boundary (Σ, Ω) is $(G, \mathcal{A}^h)$-stationary in U if and only if (Σ, Ω) is $\mathcal{A}^h$-stationary in U.*

Proof. For any $X \in \mathfrak{X}(U)$, let $X_G := (\sum_{g \in G} dg^{-1}(X))/\#G \in \mathfrak{X}^G(U)$. Since Σ is G-invariant and $\Omega \in \mathcal{C}^{G_\pm}(M)$, we can combine (2.4) with the proof of Lemma 2.4 to conclude $\delta \mathcal{A}^h_{\Sigma,\Omega}(X) = \delta \mathcal{A}^h_{\Sigma,\Omega}(X_G)$ and get the desired result. □

By the above lemma, (Σ, Ω) is a $C^{1,1}$ $(G_\pm, h)$-boundary if and only if it is a $C^{1,1}$ $G_\pm$-boundary (cf. Definition 2.9) and a $C^{1,1}$ h-boundary (in the sense of [48, Definition 1.13]). In particular, combining the first variation formula (2.4) with the standard elliptic regularity theory, we know that the regular set $\mathcal{R}(\Sigma)$ of a $C^{1,1}$ $(G_\pm, h)$-boundary (Σ, Ω) is smoothly embedded of prescribed mean curvature

$$H|_{\mathcal{R}(\Sigma)} = h|_{\mathcal{R}(\Sigma)}$$

with respect to the unit normal $\nu_\Sigma = \nu_{\partial\Omega}$.

2.3. Strong $\mathcal{A}^h$-stationarity. In [48], Wang–Zhou introduced the *strong $\mathcal{A}^h$-stationarity* which plays an important role in the regularity theory of PMC min-max and their multiplicity one theorem.

Definition 2.13 (Strong $\mathcal{A}^h$-stationarity). A $C^{1,1}$ $(G_\pm, h)$-boundary (Σ, Ω) is said to be *strongly $\mathcal{A}^h$-stationary* in an open set U if the following holds.

For any $p \in \mathcal{S}(\Sigma) \cap U$, there exist a small neighborhood $W \subset U$ of p and a decomposition $\bigcup_{i=1}^l \Gamma^i$ of $\Sigma \cap W$ by $l := \Theta^2(\Sigma, p) \geq 2$ connected disks with a natural ordering $\Gamma^1 \leq \cdots \leq \Gamma^l$. Let W^1, W^l be the bottom and top components of $W \setminus \Sigma$. Then, for $i \in \{1, l\}$ and all $X \in \mathfrak{X}(W)$ pointing into W^i along Γ^i,

$$\begin{aligned} \delta \mathcal{A}^h_{\Gamma^i, W^i}(X) &\geq 0 \quad \text{when } W^i \subset \Omega, \\ \delta \mathcal{A}^h_{\Gamma^i, W \setminus W^i}(X) &\geq 0 \quad \text{when } W^i \cap \Omega = \emptyset. \end{aligned} \tag{2.5}$$

Remark 2.14. One can also say (Σ, Ω) is *strongly $(G, \mathcal{A}^h)$-stationary* in U if Σ has a local decomposition by G-connected ordered sheets $\Gamma^1 \leq \cdots \leq \Gamma^l$ in a G-neighborhood $W \subset U$ of $G \cdot p \subset \mathcal{S}(\Sigma)$ so that (2.5) is valid for all G-invariant $X \in \mathfrak{X}^G(W)$ pointing away from all other sheets along Γ^i $(i \in \{1, l\})$. Nevertheless, since the strong $\mathcal{A}^h$-stationarity is a local property, a $C^{1,1}$ $(G_\pm, h)$-boundary (Σ, Ω) is strongly $\mathcal{A}^h$-stationary in U if and only if it is strongly $(G, \mathcal{A}^h)$-stationary in U.

In particular, the results in [48, Section 1.3] remain valid for strongly $\mathcal{A}^h$-stationary (or strongly $(G_\pm, \mathcal{A}^h)$-stationary) $C^{1,1}$ $(G_\pm, h)$-boundaries. The mean curvature H of Σ with respect to ν still satisfies

$$H(p)=\begin{cases}h(p) & \text{when } p\in\mathcal{R}(\Sigma)\cap U,\\ 0 & \text{for } \mathcal{H}^2\text{-a.e. } p\in\mathcal{S}(\Sigma)\cap U.\end{cases}$$

Moreover, for the generalized mean curvature H^l of Γ^l defined with respect to ν_Σ from Remark 2.10, the following holds: given that Ω does not contain the region above the top sheet Γ^l, we know that $H^l=h|_{\Gamma^l}$ in a neighborhood where $h>0$ and that $H^l\geq h|_{\Gamma^l}$ in a neighborhood where $h<0$. We have a corresponding statement for Γ^1 by flipping the order.

2.4. *G*-stability and compactness. Since we only consider G-equivariant deformations of $G_\pm$-boundaries, we will extend Wang–Zhou's compactness theorem for stable h-boundaries to an equivariant version.

Definition 2.15 (G-stable $C^{1,1}$ $(G_\pm,h)$-boundary). Let $U\subset M$ be a G-invariant open set, and let (Σ,Ω) be a $C^{1,1}$ $(G_\pm,h)$-boundary in U. Then (Σ,Ω) is said to be *stable* (resp. G-stable) in U if, for any flow $\{\phi^t\}$ generated by $X\in\mathfrak{X}(U)$ (resp. $X\in\mathfrak{X}^G(U)$),

$$\delta^2\mathcal{A}^h_{\Sigma,\Omega}(X):=\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{A}^h(\phi^t_\#(\Sigma,\Omega))\geq 0.$$

If in addition (Σ,Ω) is strongly $\mathcal{A}^h$-stationary, then this is equivalent to the following stability inequality for all $X\in\mathfrak{X}(U)$ (resp. $X\in\mathfrak{X}^G(U)$):

$$\int_\Sigma|\nabla^\perp X^\perp|^2-\mathrm{Ric}_M(X^\perp,X^\perp)-|A_\Sigma|^2|X^\perp|^2\,d\mathcal{H}^2\geq\int_{\partial\Omega}\langle X^\perp,\nabla h\rangle\langle X,\nu_{\partial\Omega}\rangle\,d\mathcal{H}^2,$$

where $\perp$ denotes the normal part with respect to Σ, and A_Σ denotes the second fundamental form of Σ (as an immersion).

Note that if the touching set $\mathcal{S}(\Sigma)$ is empty (or consider Σ instead of $\phi(\Sigma)$), the first eigenfunction φ_1 of the Jacobi operator $L^h_\Sigma:=\triangle_\Sigma-(\mathrm{Ric}_M(\nu_\Sigma,\nu_\Sigma)+|A_\Sigma|^2+\partial_{\nu_\Sigma}h)$ is G-invariant since $\partial_{\nu_\Sigma}h$ is G-invariant and G-actions are isometries. Hence the (intrinsic) G-stability is equivalent to the (intrinsic) stability provided $\mathcal{S}(\Sigma)=\emptyset$. However, $\mathcal{S}(\Sigma)\neq\emptyset$ in general even if (Σ,Ω) solves the isotopy minimizing problem [48, Theorem 1.25]. Nevertheless, as the group actions we considered here are free and effective, we still have the following lemma.

Lemma 2.16. *Let* (Σ,Ω) *be a* $C^{1,1}$ $(G_\pm,h)$*-boundary in an appropriately small open* G*-subset* U. *Then* (Σ,Ω) *is stable in* U *if and only if* (Σ,Ω) *is* G*-stable in* U.

Proof. The "only if" part is direct. For the "if" part, we only need to show (Σ,Ω) is stable in each connected component $\{U_i\}_{i=1}^k$, $k:=\#G$, of U. Indeed, given $X\in\mathfrak{X}(U_1)$ (with $X\llcorner U_j=0$ for $j\neq 1$), we can define $X_G\in\mathfrak{X}^G(U)$ by $X_G\llcorner g\cdot U_1:=dg(X)$ since U is appropriately small and the G-action is free and effective. Then we notice ∇h and $\nu_{\partial\Omega}$ will keep or change the sign simultaneously under the push forward of $g\in G_+$ or $g\in G_-$. Therefore, $\delta^2\mathcal{A}^h_{\Sigma,\Omega}(X)=\delta^2\mathcal{A}^h_{\Sigma,\Omega}(dg(X))$, and thus $0\leq\delta^2\mathcal{A}^h_{\Sigma,\Omega}(X_G)=k\cdot\delta^2\mathcal{A}^h_{\Sigma,\Omega}(X)$ by the appropriate smallness of U. □

Combining Lemma 2.16 with [48, Proposition 1.24], we conclude the following compactness result.

Proposition 2.17. *Suppose that $U \subset M$ is an appropriately small open G-subset and that $h_j, h \in C^2_{G_\pm}(M)$ (cf. (2.1)) satisfies $\lim_{j\to\infty}\|h_j - h\|_{C^2} = 0$. Let $\{(\Sigma_j, \Omega_j)\}_{j\in\mathbb{N}}$ be a sequence of G-stable $C^{1,1}$ $(G_\pm, h_j)$-boundaries in U so that $\mathcal{H}^2(\Sigma_j) \le \Lambda$ for some $\Lambda > 0$. Then there is a stable $C^{1,1}$ $(G_\pm, h)$-boundary (Σ, Ω) in U so that*

(1) *Σ_j converges to Σ in U as varifolds and also in the sense of $C^{1,\alpha}_{\mathrm{loc}}$ for all $\alpha \in (0, 1)$;*

(2) *Ω_j converge to Ω as currents in $\mathcal{C}^{G_\pm}(U) \subset \mathcal{C}(U)$.*

Moreover,

(i) *if Σ is smooth, then Σ_j converges to Σ in U in the $C^{1,1}_{\mathrm{loc}}$ topology;*

(ii) *if (Σ_j, Ω_j) is strongly $\mathcal{A}^{h_j}$-stationary in U, then (Σ, Ω) is strongly $\mathcal{A}^h$-stationary in U.*

Proof. By Lemma 2.16, we can apply the compactness result [48, Proposition 1.24] and the $C^{1,1}$-regularity result [49, Theorem 1.1] to (Σ_j, Ω_j) and obtain $C^{1,1}$ h-boundary (Σ, Ω) satisfying (1)–(2) and (i)–(ii). Since G acts by isometries, we get the G-invariance of Σ and $\Omega \in \mathcal{C}^{G_\pm}(U)$. □

2.5. G-isotopy minimizing problem. Let

$$0 < \mathbf{r}_0 < \min\big\{\rho_0, \inf_{p\in M} \operatorname{inj}(G\cdot p)\big\}$$

be a sufficiently small constant, where $\rho_0 = \rho_0(M, g_M, \sup|h|) > 0$ is given in [40, §14] and $\operatorname{inj}(G \cdot p)$ is the injectivity radius of the normal exponential map $\exp^\perp_{G\cdot p}$. Note that $B^G_{\mathbf{r}_0}(p)$ is appropriately small for all $p \in M$ and $\inf_{p\in M} \operatorname{inj}(G \cdot p) > 0$ since G acts freely by isometries.

For any open G-subset $U \subset B^G_{\mathbf{r}_0}(p)$, suppose that $\mathcal{R} \in \mathcal{C}^{G_\pm}(U)$ has a smoothly embedded G-invariant boundary $\Sigma := \partial\mathcal{R} \cap U$ in U. Then we say that a pair $(V, \Omega) \in \mathcal{VC}(U)$ (resp. $(V, \Omega) \in \mathcal{VC}^{G_\pm}(U)$) is an *isotopy minimizer* (resp. *G-isotopy minimizer*) of $(\Sigma, \mathcal{R})$ in U if there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ in $\mathfrak{Is}(U)$ (resp. $\mathfrak{Is}^G(U)$) so that

- $\lim_{k\to\infty} \mathcal{A}^h(\phi_k(\Sigma, \mathcal{R})) = \inf\{\mathcal{A}^h(\phi(\Sigma, \mathcal{R})) \mid \phi \in \mathfrak{Is}(U) \text{ (resp. } \phi \in \mathfrak{Is}^G(U))\}$;
- $\lim_{k\to\infty} \mathcal{F}((V, \Omega), \phi_k(\Sigma, \mathcal{R})) = 0$,

where we used the notation $\phi(\Sigma, \mathcal{R}) := (\phi(1, \Sigma), \phi(1, \mathcal{R}))$ for simplicity.

Similar to Lemma 2.16, we also have the following equivalence between isotopy minimizers and G-isotopy minimizers in appropriately small open G-sets by the definitions of $h \in C^\infty_{G_\pm}(U)$ and $\mathcal{C}^{G_\pm}(U)$.

Lemma 2.18. *Let $U \subset M$ be an appropriately small open G-subset and let $(\Sigma, \mathcal{R})$ be given as above. Then a G-isotopy minimizer $(V, \Omega) \in \mathcal{VC}^{G_\pm}(U)$ of $(\Sigma, \mathcal{R})$ in U is also an isotopy minimizer of $(\Sigma, \mathcal{R})$ in U.*

Proof. Denote by $\{U_i\}_{i=1}^{\#G}$ the connected components of U, and fix any component, e.g. U_1. Then, for any $(V', \Omega') \in \mathcal{VC}^{G_\pm}(U)$, we have

$$\mathcal{A}^h(V' \llcorner U_i, \Omega' \llcorner U_i) = \begin{cases} \mathcal{A}^h(V' \llcorner U_1, \Omega' \llcorner U_1) & \text{if } U_i \subset G_+ \cdot U_1, \\ \mathcal{A}^h(V' \llcorner U_1, \Omega' \llcorner U_1) + \int_{U_1} h \, d\mathcal{H}^3 & \text{if } U_i \subset G_- \cdot U_1. \end{cases} \tag{2.6}$$

Therefore, for any G-isotopy minimizing sequence $\{\phi_k(\Sigma, \mathcal{R})\}_{k\in\mathbb{N}}$,

$$\begin{aligned}\mathcal{A}^h(\phi_k(\Sigma,\mathcal{R})) &= \sum_{i=1}^{\#G} \mathcal{A}^h(\phi_k(\Sigma\llcorner U_i, \mathcal{R}\llcorner U_i)) \\ &= (\#G)\cdot \mathcal{A}^h(\phi_k(\Sigma\llcorner U_1, \mathcal{R}\llcorner U_1)) + \frac{\#G}{2}\int_{U_1} h\, d\mathcal{H}^3.\end{aligned}$$

Since any isotopy $\phi \in \mathfrak{Is}(U_1)$ can be extended to a G-isotopy $\phi_G \in \mathfrak{Is}^G(U)$ by taking

$$\phi_G\llcorner U_i = g\circ\phi\circ g^{-1}$$

for $U_i = g\cdot U_1$, it follows from the above equality that $\phi_k(\Sigma\llcorner U_1, \mathcal{R}\llcorner U_1)$ is also an isotopy minimizing sequence of $(\Sigma, \mathcal{R})$ in U_1, which implies $(V\llcorner U_1, \Omega\llcorner U_1)$ is also an isotopy minimizer in U_1. □

By the choice of $\mathbf{r}_0$, any open G-set $U \subset B^G_{\mathbf{r}_0}(p)$ is appropriately small. Hence we have the following regularity theorem for G-isotopy minimizers in an appropriately and sufficiently small open G-set.

Theorem 2.19. *Let $U \subset B^G_{\mathbf{r}_0}(p)$ and $(\Sigma, \mathcal{R})$ be as above. Suppose $(V,\Omega) \in \mathcal{C}^{G_\pm}(U)$ is a G-isotopy minimizer of $(\Sigma,\mathcal{R})$ in U. Then (V,Ω) is a strongly $\mathcal{A}^h$-stationary and stable $C^{1,1}$ $(G_\pm, h)$-boundary in U.*

Proof. This result follows directly from Lemma 2.18 and [48, Theorem 1.25]. □

3. Min-max theory for $C^{1,1}$ $(G_\pm, h)$-boundaries

In this section, we follow the approach in [48] to set up the relative G-equivariant min-max problem for the $\mathcal{A}^h$-functional and prove the main regularity results for $(G, \mathcal{A}^h)$-min-max pairs.

3.1. G-equivariant min-max problem. Fix a G-connected closed surface Σ_0 of genus $\mathfrak{g}_0$. A G-equivariant embedding $\phi\colon \Sigma_0 \to M$ is said to be $G_\pm$-*separating* if

$$M \setminus \phi(\Sigma_0) = \Omega_+ \sqcup \Omega_-,$$

where Ω_+, Ω_- are two nonempty domains sharing a common boundary $\phi(\Sigma_0)$ so that

$$G_+\cdot\Omega_\pm = \Omega_\pm \quad \text{and} \quad G_-\cdot\Omega_\pm = \Omega_\mp.$$

For convenience, write $\Sigma = \phi(\Sigma_0)$ with the orientation induced by the outer normal ν of Ω, where Ω is an arbitrary choice of $\{\Omega_+, \Omega_-\}$. We then denote

$$\mathcal{E}_{G_\pm} := \{(\Sigma,\Omega) \mid \Sigma \text{ is a } G\text{-equivariant } G_\pm\text{-separating embedding of } \Sigma_0 \text{ in } M\}$$

endowed with the oriented smooth topology in the usual sense.

Let X be a finite-dimensional cubical complex, and let $Z \subset X$ be a subcomplex. Let $\Phi_0\colon X \to \mathcal{E}_{G_\pm}$ be a continuous map. Denote by Π the set of all continuous maps $\Phi\colon X \to \mathcal{E}_{G_\pm}$

which are homotopic to Φ_0 relative to $\Phi_0|_Z\colon Z \to \mathscr{E}_{G_\pm}$. We refer to such a Φ as an (X, Z)-sweepout, or simply a sweepout. Definitions 2.1–2.3 in [48] are directly transferred to our situation as follows.

Definition 3.1. Given (X, Z) and Φ_0 as above, we call Π the (X, Z)-*homotopy class of* Φ_0. The *h-width* of Π is defined by

$$\mathbf{L}^h = \mathbf{L}^h(\Pi) = \inf_{\Phi\in\Pi} \sup_{x\in X} \mathcal{A}^h(\Phi(x)).$$

A sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \Pi$ is called a *minimizing sequence* if

$$\mathbf{L}^h(\Phi_i) := \sup_{x\in X} \mathcal{A}^h(\Phi_i(x)) \to \mathbf{L}^h \quad \text{when } i \to \infty.$$

A subsequence $\{\Phi_{i_j}(x_j) \mid x_j \in X\}_{j\in\mathbb{N}}$ is called a *min-max (sub)sequence* if

$$\mathcal{A}^h(\Phi_{i_j}(x_j)) \to \mathbf{L}^h \quad \text{when } j \to \infty.$$

The *critical set* of a minimizing sequence $\{\Phi_i\}$ is defined by

$$\begin{aligned}\mathbf{C}(\{\Phi_i\}) = \{(V, \Omega) \in \mathcal{VC}^{G_\pm}(M) \mid {} & \text{there exists a min-max subsequence } \{\Phi_{i_j}(x_j)\} \\ & \text{such that } \mathscr{F}(\Phi_{i_j}(x_j), (V, \Omega)) \to 0 \text{ as } j \to \infty\}.\end{aligned}$$

We have the following min-max theorem, and the proof will be given later.

Theorem 3.2 (PMC min-max theorem). *With all notions as above, suppose*

$$\mathbf{L}^h(\Pi) > \max\bigl\{\max_{x\in Z} \mathcal{A}^h(\Phi_0(x)), 0\bigr\}. \tag{3.1}$$

Then there exist a minimizing sequence $\{\Phi_i\} \subset \Pi$ *and a strongly* $\mathcal{A}^h$*-stationary,* $C^{1,1}$ $(G_\pm, h)$*-boundary* (Σ, Ω) *lying in the critical set* $\mathbf{C}(\{\Phi_i\})$ *such that* $\mathcal{A}^h(\Sigma, \Omega) = \mathbf{L}^h(\Pi)$.

3.2. Tightening. We follow the pull-tight process in [48, Section 2.2] with some alterations.

Theorem 3.3 (Pull-tight). *Let* Π *be an* (X, Z)*-homotopy class generated by some continuous* $\Phi_0\colon X \to \mathscr{E}_{G_\pm}$ *relative to* $\Phi_0|_Z$. *For a minimizing sequence* $\{\Phi_i^*\}_{i\in\mathbb{N}} \subset \Pi$ *associated with* $\mathcal{A}^h$, *there exists another minimizing sequence* $\{\Phi_i\}_{i\in\mathbb{N}} \subset \Pi$ *such that* $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi_i^*\})$ *and every element* $(V, \Omega) \in \mathbf{C}(\{\Phi_i\})$ *is either* $(G, \mathcal{A}^h)$*-stationary (and thus* $\mathcal{A}^h$*-stationary by Lemma* 2.4*) or belongs to* $B = \Phi_0(Z) \subset \mathscr{E}_{G_\pm}$.

Proof. Given $C := \mathbf{L}^h + \sup_M |h(p)| \cdot \mathrm{Vol}(M) + 1$, let A^C and A_0^C be defined as in Lemma 2.7. Set $B := \Phi_0(Z)$ and $A_0 = A_0^C \cup B$. For any $\mathcal{X} \in \mathfrak{X}(M)$, we choose

$$\mathcal{X}^G = (\Sigma_{g\in G}\, dg^{-1}(\mathcal{X}))/\#G,$$

which satisfies (Lemma 2.4)

$$\delta A^h_{V,\Omega}(\mathcal{X}) = \delta A^h_{V,\Omega}(\mathcal{X}^G) \quad \text{for every } (V, \Omega) \in \mathcal{VC}^{G_\pm}(M).$$

Then, combining with the constructions in [48, Section 2.2, Steps 1, 2], we obtain a continuous map $\mathcal{X}^G\colon A^C \to \mathfrak{X}^G(M)$ (under the C^1-topology on $\mathfrak{X}^G(M)$) so that $\mathcal{X}^G \llcorner A_0 = 0$ and $\mathcal{X}^G \llcorner (A^C \setminus A_0)$ is continuous under the smooth topology on $\mathfrak{X}^G(M)$. Additionally, by [48, Section 2.2, Step 3], we also have two continuous functions $T, \mathcal{L}\colon (0,\infty) \to (0,\infty)$ with $T(x), \mathcal{L}(x) \to 0$ as $x \to 0$ so that, for each $(V,\Omega) \in A^C$ with $\gamma = \mathscr{F}((V,\Omega), A_0)$, the homotopy map $H\colon I \times A^C \to A^C$ defined by

$$(t,(V,\Omega)) \mapsto (V_{T(\gamma)t}, \Omega_{T(\gamma)t}) := \Phi^G_{V,\Omega}(T(\gamma)t)_\#(V,\Omega)$$

is continuous in the $\mathscr{F}$-metric satisfying

- $H(t,(V,\Omega)) = (V,\Omega)$ if $(V,\Omega) \in A_0^C \cup B$,
- $\mathcal{A}^h(V_1,\Omega_1) - \mathcal{A}^h(V,\Omega) \le -\mathcal{L}(\gamma)$,

where $\{\Phi^G_{V,\Omega}(t,\cdot)\}_{t\ge 0} \subset \mathrm{Diff}^G_0(M)$ is the flow associated with $\mathcal{X}^G(V,\Omega)$.

Without loss of generality, we may assume that $\Phi_i^*(x) \in A^C$ for all $i \in \mathbb{N}$ and $x \in X$. Let $\mathcal{X}_i^G(x) = \mathcal{X}^G(\Phi_i^*(x))$ for $x \in X$, which is continuous under the C^1-topology on $\mathfrak{X}^G(M)$ with $\mathcal{X}_i^G \llcorner Z = 0$. For each $i \in \mathbb{N}$, define $H_i\colon X \to \mathfrak{Is}^G(M)$ by $H_i(x) = H(\cdot, \Phi_i^*(x))$. By smoothing out $\mathcal{X}_i^G$ to some $\widetilde{\mathcal{X}}_i^G\colon X \to \mathfrak{X}^G(M)$, which is continuous under the smooth topology with $\widetilde{\mathcal{X}}_i^G(x) = 0$ for any $x \in Z$ and $\|\mathcal{X}_i^G - \widetilde{\mathcal{X}}_i^G\|_{C^1} \le 1/i$, we define $\widetilde{H}_i\colon X \to \mathfrak{Is}^G(M)$ using $\widetilde{\mathcal{X}}_i^G$ rather than $\mathcal{X}_i^G$. Denoting $\Phi_i(x) = \widetilde{H}_i(1, \Phi_i^*(x))$, we obtain $\Phi_i \in \Pi$ and the same estimate [48, (2.9)], i.e.

$$\mathcal{A}^h(\Phi_i(x)) - \mathcal{A}^h(\Phi_i^*(x)) \le -\mathcal{L}\big(\mathscr{F}(\Phi_i^*(x), A_0)\big) + \frac{C}{i},$$

where $C > 0$ is a universal constant.

Given a min-max sequence $\{\Phi_{i_j}(x_j)\}$, by the above estimate and the fact that $\{\Phi_i^*\}$ is a minimizing sequence, we know that $\{\Phi_{i_j}^*(x_j)\}$ is also a min-max sequence and

$$\mathscr{F}(\Phi_{i_j}^*(x_j), A_0) \to 0 \quad \text{as } j \to \infty.$$

Hence

$$\lim_{j\to\infty} \mathscr{F}(\Phi_{i_j}^*(x_j), \Phi_{i_j}(x_j)) = 0 \quad \text{and} \quad \mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi_i^*\}).$$

Since $\mathscr{F}(\Phi_{i_j}(x_j), A_0) \to 0$ as $j \to \infty$, each element in $\mathbf{C}(\{\Phi_i\})$ is either $(G, \mathcal{A}^h)$-stationary or belongs to $B = \Phi_0(Z) \subset \mathscr{E}_{G\pm}$. □

3.3. Almost minimizing. We now adapt the almost minimizing property to the $(G, \mathcal{A}^h)$-functional using G-equivariant embedded surfaces.

Definition 3.4. Given $\epsilon, \delta > 0$, a G-invariant open set $U \subset M$, and $(\Sigma, \Omega) \in \mathscr{E}_{G\pm}$, we say that (Σ,Ω) *is G-equivariantly $(\mathcal{A}^h, \epsilon, \delta)$-almost minimizing in U* if there does *not* exist any isotopy $\psi \in \mathfrak{Is}^G(U)$ such that

- $\mathcal{A}^h(\psi(t,\Sigma,\Omega)) \le \mathcal{A}^h(\Sigma,\Omega) + \delta$ for all $t \in [0,1]$;
- $\mathcal{A}^h(\psi(1,\Sigma,\Omega)) \le \mathcal{A}^h(\Sigma,\Omega) - \epsilon$.

Definition 3.5 ($(G, \mathcal{A}^h)$-almost minimizing pairs). Given a G-invariant open subset $U \subset M$, a pair $(V,\Omega) \in \mathcal{VC}^{G\pm}(M)$, and a sequence $\{(\Sigma_j, \Omega_j)\}_{j\in\mathbb{N}} \subset \mathscr{E}_{G\pm}$, we say that

(V, Ω) is $(G, \mathcal{A}^h)$*-almost minimizing with respect to* $\{(\Sigma_j, \Omega_j)\}$ *in* U if there exist $\epsilon_j \to 0$ and $\delta_j \to 0$ such that

- $(\Sigma_j, \Omega_j) \to (V, \Omega)$ in the $\mathscr{F}$-metric as $j \to \infty$;
- (Σ_j, Ω_j) is G-equivariantly $(\mathcal{A}^h, \epsilon_j, \delta_j)$-almost minimizing in U.

Sometimes, we also say that (V, Ω) *is* $(G, \mathcal{A}^h)$*-almost minimizing in* U without referring to $\{(\Sigma_j, \Omega_j)\}$.

Moreover, we say that (V_0, Ω_0) is $(G, \mathcal{A}^h)$*-almost minimizing in small* G*-annuli with respect to* $\{(\Sigma_j, \Omega_j)\}$ if, for any $p \in M$, there exists $r_{\mathrm{am}}(G \cdot p) > 0$ such that (V_0, Ω_0) is $(G, \mathcal{A}^h)$-almost minimizing with respect to $\{(\Sigma_j, \Omega_j)\}$ in every $A^G_{s,r}(p) \subset\subset A^G_{0,r_{\mathrm{am}}(G\cdot p)}(p)$.

The lemma below emerges as a consequence of being $(G, \mathcal{A}^h)$-almost minimizing. The proof imitates the argument in [48, Lemma 3.3], which is skipped here.

Lemma 3.6. *Let* $(V, \Omega) \in \mathcal{VC}^{G\pm}(M)$ *be* $(G, \mathcal{A}^h)$*-almost minimizing in a* G*-invariant open set* $U \subset M$*; then*

(i) (V, Ω) *is* $(G, \mathcal{A}^h)$*-stationary in* U*;*

(ii) (V, Ω) *is* $(G, \mathcal{A}^h)$*-stable in* U.

In particular, if U *is appropriately small, then* (V, Ω) *is* $\mathcal{A}^h$*-stationary and* $\mathcal{A}^h$*-stable in* U *by Lemma* 2.4 *and Remark* 2.5.

We also need the following G-equivariant notions generalized from [13, 48].

Definition 3.7. Given $L \in \mathbb{N}$ and $p \in M$, let

$$\mathscr{C}^G = \{A^G_{s_1,r_1}(p), \dots, A^G_{s_L,r_L}(p)\}$$

be a collection of G-annuli centered at $G \cdot p$. We say that $\mathscr{C}^G$ is L*-admissible* if $2r_{j+1} < s_j$ for all $j = 1, \dots, L-1$, and $B^G_{r_1}(p)$ is appropriately small.

In addition, a pair $(V, \Omega) \in \mathcal{VC}^{G\pm}(M)$ is said to be $(G, \mathcal{A}^h)$*-almost minimizing in* $\mathscr{C}^G$ *with respect to a sequence* $\{(\Sigma_j, \Omega_j)\} \subset \mathscr{E}_{G\pm}$ if there exist $\epsilon_j \to 0$ and $\delta_j \to 0$ so that

- $(\Sigma_j, \Omega_j) \to (V, \Omega)$ in the $\mathscr{F}$-metric as $j \to \infty$;
- for each j, (Σ_j, Ω_j) is G-equivariantly $(\mathcal{A}^h, \epsilon_j, \delta_j)$-almost minimizing in at least one G-annulus in $\mathscr{C}^G$.

Comparing with [48, Definition 3.9], our L-admissible collection $\mathscr{C}^G$ of G-annuli requires r_1 is less than the injectivity radius of $G \cdot p$ so that every $\mathrm{An}^G \in \mathscr{C}^G$ is appropriately small.

Now, our goal is to find a $(G, \mathcal{A}^h)$-min-max-pair (V, Ω) which is $(G, \mathcal{A}^h)$-almost minimizing in small G-annuli. Consider the setup in Section 3.1 where the nontriviality condition (3.1) is met. If $\{\Phi_i\}_{i\in\mathbb{N}} \subset \Pi$ is a pull-tight minimizing sequence obtained by Theorem 3.3, then every $(V, \Omega) \in \mathbf{C}(\{\Phi_i\})$ is $\mathcal{A}^h$-stationary.

Theorem 3.8 (Existence of $(G, \mathcal{A}^h)$-almost minimizing pairs). *Let* Π *be an* (X, Z)*-homotopy class generated by some continuous* $\Phi_0 \colon X \to \mathscr{E}_{G\pm}$ *relative to* $\Phi_0|_Z$ *so that* (3.1) *holds. Then there exists a min-max subsequence* $\{(\Sigma_j, \Omega_j) = \Phi_{i_j}(x_j)\}_{j\in\mathbb{N}} \subset \mathscr{E}_{G_\pm}$ *converging to an* $\mathcal{A}^h$*-stationary pair* $(V_0, \Omega_0) \in \mathbf{C}(\{\Phi_i\})$ *so that* (V_0, Ω_0) *is* $(G, \mathcal{A}^h)$*-almost minimizing in every* L*-admissible collection of* G*-annuli with respect to* $\{(\Sigma_j, \Omega_j)\}$*, where* $L = L(m)$ *is an integer depending only on* $m := \dim(X)$.

In addition, up to a subsequence of $\{(\Sigma_j, \Omega_j)\}$*,* (V_0, Ω_0) *is* $(G, \mathcal{A}^h)$*-almost minimizing in small* G*-annuli with respect to* $\{(\Sigma_j, \Omega_j)\}$.

Proof. The proof is essentially the same as that of [13, Appendix] (for the area functional) and [48, Section 3.3] (for the $\mathcal{A}^h$-functional). Namely, if there is no such min-max sequence, then one can apply a combinatorial argument of Almgren–Pitts [39] to find several isotopies supported in many disjoint annuli so that the $\mathcal{A}^h$-functional of a certain sweepout will be pulled down to strictly below $\mathbf{L}^h(\Pi)$ via these isotopies, which contradicts the definition of $\mathbf{L}^h(\Pi)$. Since the proof is combinatorial, the argument would also carry over with our G-equivariant objects. □

An immediate consequence of Lemma 3.6 and Theorem 3.8 is the following.

Corollary 3.9. *The limit* $\mathcal{A}^h$*-stationary pair* $(V_0, \Omega_0) \in \mathbf{C}(\{\Phi_i\})$ *satisfies*

(R) *for every* $L(m)$*-admissible collection* $\mathscr{C}^G$ *of* G*-annuli,* (V_0, Ω_0) *is* $\mathcal{A}^h$*-stable in at least one* G*-annulus in* $\mathscr{C}^G$.

3.4. Regularity of min-max pairs: Part I. In this subsection, we will introduce the notions concerning $(G_\pm, \mathcal{A}^h)$-replacements, and show the regularity for the pairs

$$(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$$

with a certain $(G_\pm, \mathcal{A}^h)$-replacement chain property.

Definition 3.10. Given an open G-subset $U \subset M$ and $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$, a pair $(V^*, \Omega^*) \in \mathcal{VC}^{G_\pm}(M)$ is said to be a $(G_\pm, \mathcal{A}^h)$-*replacement* of (V, Ω) in U if

(i) $(V^*, \Omega^*) = (V, \Omega)$ in $M \setminus \operatorname{Clos}(U)$;

(ii) $\mathcal{A}^h(V^*, \Omega^*) = \mathcal{A}^h(V, \Omega)$;

(iii) (V^*, Ω^*) is a strongly $\mathcal{A}^h$-stationary and stable $C^{1,1}$ $(G_\pm, h)$-boundary in U.

Definition 3.11. As above, (V, Ω) is said to have *(weak) good* $(G_\pm, \mathcal{A}^h)$-*replacement property in* U if, for any $p \in U$, there exists $r_{G\cdot p} > 0$ such that (V, Ω) has an $(G_\pm, \mathcal{A}^h)$-replacement (V^*, Ω^*) in any open G-annulus $\operatorname{An}^G \subset\subset A^G_{0, r_{G\cdot p}}(p)$.

Proposition 3.12 (Classification of tangent cones). *Let* $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ *be* $\mathcal{A}^h$*-stationary in an open* G*-set* U *and have (weak) good* $(G_\pm, \mathcal{A}^h)$*-replacement property in* U*. Then* V *is integer rectifiable in* U*, and for any* $p \in \operatorname{spt}(\|V\|) \cap U$*, every tangent varifold of* V *at* p *is an integer multiple of a plane in* T_pM.

Proof. The proof can be taken almost verbatim from [12, Lemma 6.4] and [40, Lemma 20.2]. □

Definition 3.13 ($(G_\pm, \mathcal{A}^h)$-replacement chain property). Let $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ and let $U \subset M$ be an open G-set. Then (V, Ω) is said to have the $(G_\pm, \mathcal{A}^h)$-*replacement chain property in* U if the following statement holds. For any sequence $B_1^G, \dots, B_k^G \subset\subset U$ of open G-subsets, there exist a sequence

$$(V, \Omega) = (V_0, \Omega_0), (V_1, \Omega_1), \dots, (V_k, \Omega_k) \subset \mathcal{VC}^{G_\pm}(M)$$

satisfying that

(i) (V_j, Ω_j) is an $(G_\pm, \mathcal{A}^h)$-replacement of (V_{j-1}, Ω_{j-1}) in B_j^G for $j = 1, \dots, k$;

(ii) (V_j, Ω_j) is $\mathcal{A}^h$-stationary and stable in U;

(iii) for another sequence
$$B_1^G, \dots, B_k^G, \widetilde{B}_{k+1}^G, \dots, \widetilde{B}_\ell^G \subset\subset U$$
of open G-subsets, the sequence of $(G_\pm, \mathcal{A}^h)$-replacements $(\widetilde{V}_j, \widetilde{\Omega}_j)$ can be chosen so that $(\widetilde{V}_j, \widetilde{\Omega}_j) = (V_j, \Omega_j)$ for $j = 1, \dots, k$.

Note that the $(G_\pm, \mathcal{A}^h)$-replacement chain property of $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ implies that (V, Ω) is $\mathcal{A}^h$-stationary and stable in U, and has the (weak) good $(G_\pm, \mathcal{A}^h)$-replacement property in U, which further indicates the rectifiability of V by Proposition 3.12. Additionally, if (V^*, Ω^*) is a $(G_\pm, \mathcal{A}^h)$-replacement of (V, Ω) in $B^G \subset U$, then (V^*, Ω^*) not only is an $\mathcal{A}^h$-replacement of (V, Ω) in $B^G \subset U$ in the sense of [48, Definition 3.4] but also has certain symmetries, i.e. $(V^*, \Omega^*) \in \mathcal{VC}^{G_\pm}(M) \subset \mathcal{VC}(M)$. Hence the above definitions concerning $(G_\pm, \mathcal{A}^h)$-replacements are stronger than [48, Definitions 3.4, 3.5, 3.6]. We then have the following regularity theorem by [48, Theorem 4.4].

Theorem 3.14 (First regularity). *Let* $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ *satisfy the* $(G_\pm, \mathcal{A}^h)$*-replacement chain property in a given open* G*-set* $U \subset M$*. Then* (V, Ω) *is induced by a strongly* $\mathcal{A}^h$*-stationary and stable* $C^{1,1}$ $(G_\pm, h)$*-boundary in* U.

Proof. By the above definitions, $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ also has the replacement chain property in U in the sense of [48, Definition 3.6]. Hence the desired regularity result follows from [48, Theorem 4.4] and the fact that $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$. □

3.5. Regularity of min-max pairs: Part II. In this subsection, we prove the regularity of the $(G, \mathcal{A}^h)$-almost minimizing pairs $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ in an appropriately small open G-set U by constructing $(G_\pm, \mathcal{A}^h)$-replacement chains. Throughout this subsection, we always assume that

$U \subset M$ is an *appropriately small* (cf. Definition 2.2) open G-set

with connected components $\{U_i\}_{i=1}^{\#G}$.

To begin with, consider a constrained $\mathcal{A}^h$-minimizing problem. For any G-equivariant $G_\pm$-separating embedded surface $(\Sigma, \Omega) \in \mathcal{E}_{G_\pm}$ and $\delta > 0$, take

$$\mathfrak{Is}_\delta^{G,h}(U) := \{\psi \in \mathfrak{Is}^G(U) \mid \mathcal{A}^h(\psi(t, (\Sigma, \Omega))) \le \mathcal{A}^h(\Sigma, \Omega) + \delta\},$$
$$\mathfrak{Is}_\delta^{h}(U_i) := \{\psi \in \mathfrak{Is}(U_i) \mid \mathcal{A}^h(\psi(t, (\Sigma, \Omega))) \le \mathcal{A}^h(\Sigma, \Omega) + \delta\}.$$

Then a sequence $\{(\Sigma_k, \Omega_k)\}_{k\in\mathbb{N}} \subset \mathscr{E}_{G_\pm}$ is said to be *minimizing in problem* $(\Sigma, \Omega, \mathfrak{Is}_\delta^{G,h}(U))$ if there exists a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathfrak{Is}_\delta^{G,h}(U)$ with $(\Sigma_k, \Omega_k) = \psi_k(1, (\Sigma, \Omega))$ so that

$$\mathcal{A}^h(\Sigma, \Omega) \geq \mathcal{A}^h(\Sigma_k, \Omega_k) \to m_\delta^G := \inf\{\mathcal{A}^h(\psi(1, (\Sigma, \Omega))) \mid \psi \in \mathfrak{Is}_\delta^{G,h}(U)\} \text{ as } k \to \infty$$

Similarly, if $\{\widehat{\psi}_k\}_{k\in\mathbb{N}} \subset \mathfrak{Is}_\delta^h(U_i)$ with $(\widehat{\Sigma}_k, \widehat{\Omega}_k) = \widehat{\psi}_k(1, (\Sigma, \Omega))$ so that

$$\mathcal{A}^h(\Sigma, \Omega) \geq \mathcal{A}^h(\widehat{\Sigma}_k, \widehat{\Omega}_k) \to m_\delta := \inf\{\mathcal{A}^h(\widehat{\psi}(1, (\Sigma, \Omega))) \mid \widehat{\psi} \in \mathfrak{Is}_\delta^h(U_i)\} \text{ as } k \to \infty,$$

then we say that $\{(\widehat{\Sigma}_k, \widehat{\Omega}_k)\}_{k\in\mathbb{N}}$ is *minimizing in problem* $(\Sigma, \Omega, \mathfrak{Is}_\delta^h(U_i))$.

Using the arguments in Lemma 2.18, we also have the following result indicating the equivalence between minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}_\delta^{G,h}(U))$ and minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}_{\delta/\#G}^h(U_i))$.

Lemma 3.15. *Given*

$$i \in \{1, \dots, \#G\}, \quad \{\psi_k\}_{k\in\mathbb{N}} \subset \mathfrak{Is}_\delta^{G,h}(U) \quad \textit{and} \quad (\Sigma_k, \Omega_k) = \psi_k(1, (\Sigma, \Omega)),$$

define

$$\widehat{\psi}_k := \begin{cases} \psi_k & \textit{in } U_i, \\ \mathrm{id} & \textit{in } M \setminus U_i, \end{cases} \qquad (\widehat{\Sigma}_k, \widehat{\Omega}_k) := \widehat{\psi}_k(1, (\Sigma, \Omega)) = \begin{cases} (\Sigma_k, \Omega_k) & \textit{in } U_i, \\ (\Sigma, \Omega) & \textit{in } M \setminus U_i. \end{cases}$$

Then (Σ_k, Ω_k) *is minimizing in problem* $(\Sigma, \Omega, \mathfrak{Is}_\delta^{G,h}(U))$ *if and only if* $(\widehat{\Sigma}_k, \widehat{\Omega}_k)$ *is minimizing in problem* $(\Sigma, \Omega, \mathfrak{Is}_{\delta/\#G}^h(U_i))$.

Proof. By (2.6) and the above definitions, we have

$$\begin{aligned} \mathcal{A}^h\big(\psi_k(t, (\Sigma, \Omega))\big) &= \#G \cdot \mathcal{A}^h\big(\widehat{\psi}_k(t, (\Sigma \cap U_i, \Omega \cap U_i))\big) \\ &\quad + \frac{\#G}{2} \int_{U_i} h + \mathcal{A}^h(\Sigma \setminus U, \Omega \setminus U) \\ &= \#G \cdot \mathcal{A}^h\big(\widehat{\psi}_k(t, (\Sigma, \Omega))\big) + C_0, \end{aligned} \tag{3.2}$$

where $C_0 = C_0(U, \Sigma, \Omega, h, \#G)$ is a constant (independent of ψ_k). In particular,

$$\begin{aligned} \mathcal{A}^h(\Sigma, \Omega) &= \#G \cdot \mathcal{A}^h(\Sigma \cap U_i, \Omega \cap U_i) + \frac{\#G}{2} \int_{U_i} h + \mathcal{A}^h(\Sigma \setminus U, \Omega \setminus U) \\ &= \#G \cdot \mathcal{A}^h(\Sigma, \Omega) + C_0. \end{aligned}$$

Thus $\{\widehat{\psi}_k\} \subset \mathfrak{Is}_{\delta/\#G}^h(U_i)$, and every $\widehat{\phi} \in \mathfrak{Is}_{\delta/\#G}^h(U_i)$ can be recovered to $\phi \in \mathfrak{Is}_\delta^{G,h}(U)$ by taking

$$\phi \llcorner U_j = g \circ \widehat{\phi} \circ g^{-1} \quad \text{for } U_j = g \cdot U_i.$$

Moreover, the above formulae also imply that $m_\delta^G = \#G \cdot m_{\delta/\#G} + C_0$. Therefore,

$$\mathcal{A}^h(\Sigma, \Omega) \geq \mathcal{A}^h(\Sigma_k, \Omega_k) \to m_\delta^G$$

if and only if $\mathcal{A}^h(\Sigma, \Omega) \geq \mathcal{A}^h(\widehat{\Sigma}_k, \widehat{\Omega}_k) \to m_{\delta/\#G}$ by the above equalities again. □

In the following lemma, we show that any isotopy in a small enough open G-set which does not increase $\mathcal{A}^h(\Sigma_k, \Omega_k)$ can also be replaced by an isotopy in $\mathfrak{Is}_\delta^{G,h}(U)$.

Lemma 3.16. *Let $\{(\Sigma_k, \Omega_k)\}_{k\in\mathbb{N}}$ be minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}^{G,h}_{\delta}(U))$. Given any G-subset $U' \subset\subset U$, there exist $\rho_0 > 0$, $k_0 \gg 1$ so that, for any $k \geq k_0$ and $B^G_{2\rho}(x) \subset U'$ with $\rho < \rho_0$, if $\varphi \in \mathfrak{Is}^G(B^G_\rho(x))$ with*

$$\mathcal{A}^h\big(\varphi(1, (\Sigma_k, \Omega_k))\big) \leq \mathcal{A}^h(\Sigma_k, \Omega_k),$$

then there is $\Phi \in \mathfrak{Is}^G(B^G_{2\rho}(x))$ with

$$\Phi(1, \cdot) = \varphi(1, \cdot) \quad \text{and} \quad \mathcal{A}^h\big(\Phi(t, (\Sigma_k, \Omega_k))\big) \leq \mathcal{A}^h(\Sigma_k, \Omega_k) + \delta \quad \text{for all } t \in [0, 1].$$

Moreover, ρ_0 depends on $\mathcal{H}^2(\Sigma)$, $\|h\|_{L^\infty}$, U', M, δ, but does not depend on $\{(\Sigma_k, \Omega_k)\}_{k\in\mathbb{N}}$.

Proof. Let $\{\widehat{\psi}_k\}$ and $\{(\widehat{\Sigma}_k, \widehat{\Omega}_k)\}$ be given as in Lemma 3.15 with respect to some fixed i. Hence $\{(\widehat{\Sigma}_k, \widehat{\Omega}_k)\}$ is minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}^h_{\delta/\#G}(U_i))$ so that [48, Lemma 4.5] is applicable in $U'_i := U' \cap U_i$, which gives us the desired $\rho_0 > 0$ and $k_0 \gg 1$.

Indeed, take any $\varphi \in \mathfrak{Is}^G(B^G_\rho(x))$ so that

$$\mathcal{A}^h(\varphi(1, (\Sigma_k, \Omega_k))) \leq \mathcal{A}^h(\Sigma_k, \Omega_k),$$

where $k \geq k_0$, $\rho < \rho_0$, and $x \in U_i$ with $B^G_{2\rho}(x) \subset U'$. Let $\widehat{\varphi} = \varphi$ in $B_\rho(x) = U_i \cap B^G_\rho(x)$ and $\widehat{\varphi} = \mathrm{id}$ outside $B_\rho(x)$. Then $\widehat{\varphi} \in \mathfrak{Is}(B_\rho(x))$ and $\mathcal{A}^h(\widehat{\varphi}(1, (\Sigma_k, \Omega_k))) \leq \mathcal{A}^h(\Sigma_k, \Omega_k)$ by (3.2). Thus, by [48, Lemma 4.5], we have an isotopy $\widehat{\Phi} \in \mathfrak{Is}(B_{2\rho}(x))$ with $\widehat{\Phi}(1, \cdot) = \widehat{\varphi}(1, \cdot)$ and

$$\mathcal{A}^h\big(\widehat{\Phi}(t, (\Sigma_k, \Omega_k))\big) \leq \mathcal{A}^h(\Sigma_k, \Omega_k) + \delta/\#G \quad \text{for all } t \in [0, 1].$$

After recovering $\widehat{\Phi}$ to $\Phi \in \mathfrak{Is}^G(B^G_{2\rho}(x))$ by taking $\Phi \llcorner U_j = g \circ \widehat{\Phi} \circ g^{-1}$ for $U_j = g \cdot U_i$, we can use (3.2) again to show that Φ is the desired isotopy. □

Combining Lemma 3.16, 2.18 and Proposition 2.17, we have the following regularity result for the minimizers in problem $(\Sigma, \Omega, \mathfrak{Is}^{G,h}_\delta(U))$.

Proposition 3.17. *Suppose $(\Sigma, \Omega) \in \mathcal{E}_{G_\pm}$ is G-equivariantly $(\mathcal{A}^h, \epsilon, \delta)$-almost minimizing in an appropriately small open G-set U. Let $\{\psi_k\} \subset \mathfrak{Is}^{G,h}_\delta(U)$ so that*

$$\{(\Sigma_k, \Omega_k) = \psi_k(1, (\Sigma, \Omega))\}$$

is minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}^{G,h}_\delta(U))$. Then, up to a subsequence, (Σ_k, Ω_k) converges to some $(\widehat{V}, \widehat{\Omega}) \in \mathcal{VC}^{G_\pm}(M)$ so that

(i) *$\mathcal{A}^h(\Sigma, \Omega) - \epsilon \leq \mathcal{A}^h(\widehat{V}, \widehat{\Omega}) \leq \mathcal{A}^h(\Sigma, \Omega)$;*

(ii) *$(\widehat{V}, \widehat{\Omega}) \llcorner U$ is a strongly $\mathcal{A}^h$-stationary and stable $C^{1,1}$ $(G_\pm, h)$-boundary in U.*

Proof. Statement (i) follows directly from Definition 3.4. Similar to the proof of Lemma 3.6 (cf. [48, Lemma 3.3]), one can show that $(\widehat{V}, \widehat{\Omega}) \llcorner U$ is strongly $(G, \mathcal{A}^h)$-stationary and G-stable in U because $\{(\Sigma_k, \Omega_k)\}$ is minimizing in problem $(\Sigma, \Omega, \mathfrak{Is}^{G,h}_\delta(U))$. Then, since U is appropriately small, we have $(\widehat{V}, \widehat{\Omega}) \llcorner U$ is strongly $\mathcal{A}^h$-stationary and stable in U by Lemma 2.4 and Remark 2.5

Next, for any $p \in U$, take $r_0 \in (0, \min\{r_1, \rho_0\})$, where $r_1 = \mathrm{dist}_M(p, \partial U)/4$ and ρ_0 is given by Lemma 3.15 for $U' = B^G_{r_1}(p)$. We claim that $(\widehat{V}, \widehat{\Omega})$ has the $(G_\pm, \mathcal{A}^h)$-replacement chain property in $B^G_{r_0}(p)$, which indicates the regularity of $(\widehat{V}, \widehat{\Omega}) \llcorner U$ by Theorem 3.14 and the arbitrariness of $p \in U$. Indeed, since U is appropriately small, the proof of [48, Proposition 4.6] would carry over with our G-equivariant objects by using Theorem 2.19, Proposition 2.17 and Lemma 3.16 in place of [48, Theorem 1.25, Proposition 1.24, Lemma 4.5] respectively. □

Now, we can show the regularity for $(G, \mathcal{A}^h)$-almost minimizing pairs.

Theorem 3.18 (Regularity of $(G_\pm, \mathcal{A}^h)$-almost minimizing pairs). *Given any appropriately small open G-set $U \subset M$, let $(V, \Omega) \in \mathcal{VC}^{G_\pm}(M)$ be $(G_\pm, \mathcal{A}^h)$-almost minimizing with respect to $\{(\Sigma_j, \Omega_j)\}_{j \in \mathbb{N}} \subset \mathscr{E}_{G_\pm}$ in U. Then $(V, \Omega) \llcorner U$ is induced by a strongly $\mathcal{A}^h$-stationary and stable $C^{1,1}$ $(G_\pm, h)$-boundary.*

Proof. Since U is appropriately small, it follows from Lemma 3.6 that $(V, \Omega) \llcorner U$ is strongly $\mathcal{A}^h$-stationary and stable.

Fix any open G-subsets
$$B^G_1 \subset\subset U' \subset\subset U.$$
Note that (Σ_j, Ω_j) is G-equivariantly $(\mathcal{A}^h, \epsilon_j, \delta_j)$-almost minimizing in U for some $\epsilon_j \to 0$ and $\delta_j \to 0$ as $j \to \infty$. Hence it follows that, for each $j \in \mathbb{N}$, we can take a minimizing sequence $\{(\Sigma^1_{j,l}, \Omega^1_{j,l})\}_{l \in \mathbb{N}}$ for problem $(\Sigma_j, \Omega_j, \mathfrak{Is}^{G,h}_{\delta_j}(B^G_1))$, and apply Proposition 3.17 to see
$$\lim_{l \to \infty} (\Sigma^1_{j,l}, \Omega^1_{j,l}) = (V^1_j, \Omega^1_j) \in \mathcal{VC}^{G_\pm}(M)$$
(up to a subsequence) so that

(1) $\mathcal{A}^h(\Sigma_j, \Omega_j) - \epsilon_j \le \mathcal{A}^h(V^1_j, \Omega^1_j) \le \mathcal{A}^h(\Sigma_j, \Omega_j)$;

(2) $(V^1_j, \Omega^1_j) = (\Sigma_j, \Omega_j)$ in $M \setminus \operatorname{Clos}(B^G_1)$;

(3) (V^1_j, Ω^1_j) is a strongly $\mathcal{A}^h$-stationary stable $C^{1,1}$ $(G_\pm, h)$-boundary in B^G_1.

Up to a subsequence, (V^1_j, Ω^1_j) converges to $(V^1, \Omega^1) \in \mathcal{VC}^{G_\pm}(M)$. Combining Proposition 2.17 with (1)–(3), we see (V^1, Ω^1) is a $(G_\pm, \mathcal{A}^h)$-replacement of (V, Ω) in B^G_1. Moreover, $(\Sigma^1_{j,l}, \Omega^1_{j,l})$ is also G-equivariantly $(\mathcal{A}^h, \epsilon_j, \delta_j)$-almost minimizing in U since
$$(\Sigma^1_{j,l}, \Omega^1_{j,l}) = \psi^1_l(1, (\Sigma_j, \Omega_j)) \quad \text{for some } \psi^1_l \in \mathfrak{Is}^{G,h}_{\delta_j}(B^G_1)$$
and $\mathcal{A}^h(\Sigma^1_{j,l}, \Omega^1_{j,l}) \le \mathcal{A}^h(\Sigma_j, \Omega_j)$.

Next, for another open G-set $B^G_2 \subset\subset U'$, the above arguments can be applied in B^G_2 to a diagonal sequence $\{(\Sigma^1_{j,l(j)}, \Omega^1_{j,l(j)})\}_{j \in \mathbb{N}}$ that converges to (V^1, Ω^1). Then we obtain a $(G_\pm, \mathcal{A}^h)$-replacement
$$(V^2, \Omega^2) = \lim (\Sigma^2_{j,l(j)}, \Omega^2_{j,l(j)})$$
of (V^1, Ω^1) in B^G_2 so that each $(\Sigma^2_{j,l(j)}, \Omega^2_{j,l(j)})$ is G-equivariantly $(\mathcal{A}^h, \epsilon_j, \delta_j)$-almost minimizing in U. Hence (V, Ω) has $(G_\pm, \mathcal{A}^h)$-replacement chain property in U' by repeating this procedure, which indicates the regularity of (V, Ω) by Theorem 3.14. □

3.6. Proof of Theorem 3.2. By Theorem 3.8, there exists an $\mathcal{A}^h$-stationary pair
$$(V_0, \Omega_0) \in \mathbf{C}(\{\Phi_i\})$$

that is $(G, \mathcal{A}^h)$-almost minimizing in small G-annuli with respect to a min-max subsequence $\{(\Sigma_j, \Omega_j) = \Phi_{i_j}(x_j)\}_{j\in\mathbb{N}} \subset \mathscr{E}_{G_\pm}$. Let $\{B^G_{r_i}(p_i)\}_{i=1}^{\mathfrak{m}}$ be a finite set of appropriately small open G-sets covering M with radius

$$r_i = \frac{1}{2}\min\{r_{\mathrm{am}}(G\cdot p_i), \mathrm{inj}(G\cdot p_i)\},$$

where $r_{\mathrm{am}}(G\cdot p_i)$ is given by Definition 3.5. After applying Theorem 3.18 to (V_0, Ω_0) in any open G-set $U \subset\subset A^G_{0,r_i}(p_i)$, we see $(V_0, \Omega_0) = (\Sigma_0, \Omega_0)$ is a strongly $\mathcal{A}^h$-stationary $C^{1,1}$ $(G_\pm, h)$-boundary in $M \setminus \{G\cdot p_1, \dots, G\cdot p_{\mathfrak{m}}\}$, also $\mathcal{A}^h$-stable in any $U \subset\subset A^G_{0,r_i}(p_i)$. Finally, using Propositions 2.17, 3.12 in place of [48, Propositions 1.24, 4.1], the arguments in [48, Section 4.4] can be taken almost verbatim to show the regularity of (V_0, Ω_0) extends across each $G\cdot p_i$.

4. Compactness for min-max $(G_\pm, h)$-boundaries

4.1. Strong convergence and weighted genus bound. Given an $h \in C^\infty_{G_\pm}(M)$ and a sequence of positive numbers $\epsilon_k \to 0$ as $k \to \infty$, we denote $\mathcal{A}^{\epsilon_k h}$ simply by $\mathcal{A}^k$. With notation from Section 3.1, consider the equivariant min-max problem associated with Π for each $\mathcal{A}^k$, $k \in \mathbb{N}$. By assuming that the nontriviality condition (3.1) is met for all k, we apply Theorem 3.8 to the $\mathcal{A}^k$-functional for each k. This yields a $(G, \mathcal{A}^k)$-min-max pair

$$(V_k, \Omega_k) \in \mathcal{VC}^{G_\pm}(M)$$

and an associated min-max sequence $\{(\Sigma_{k,j}, \Omega_{k,j})\}_{j\in\mathbb{N}} \subset \mathscr{E}_{G_\pm}$ such that (V_k, Ω_k) is $\mathcal{A}^k$-stationary and $(G, \mathcal{A}^k)$-almost minimizing in small G-annuli with respect to $\{(\Sigma_{k,j}, \Omega_{k,j})\}$. By Theorem 3.2, (V_k, Ω_k) is a strongly $\mathcal{A}^k$-stationary, $C^{1,1}$ $(G_\pm, \epsilon_k h)$-boundary (Σ_k, Ω_k) with $\mathcal{A}^k(\Sigma_k, \Omega_k) = \mathbf{L}^{\epsilon_k h}(\Pi)$.

In this part, our goal is to show the smooth regularity of a subsequential varifold limit V_∞ of $\{\Sigma_k\}$ and upgrade the convergence to $C^{1,1}_{\mathrm{loc}}$. Moreover, for specially chosen h, we prove the weighted genus bound (see (4.1)).

To begin with, by Corollary 3.9, we note that V_∞ satisfies

(R′) for every $L(m)$-admissible collection $\mathscr{C}^G$ of G-annuli,
V_∞ is stable (for area) in at least one G-annulus in $\mathscr{C}^G$.

Arguing similarly to the proof of [48, Proposition 5.1], we have the following proposition, which is crucial for the removable singularity step.

Proposition 4.1. *There exists a subsequence of* $\{(\Sigma_k, \Omega_k)\}_{k\in\mathbb{N}}$ *such that*

(S) *given any* $p \in M$, *there exists* $r_{G\cdot p} > 0$ *such that,*
for each $A^G_{s,r}(p)$ *with* $0 < s < r < r_{G\cdot p}$,
(Σ_k, Ω_k) *is* $\mathcal{A}^k$*-stable in* $A^G_{s,r}(p)$ *for all sufficiently large* k.

Theorem 4.2. $\mathrm{spt}\|V_\infty\|$ *is a closed embedded* G*-invariant minimal surface* Σ_∞. *Moreover, there exists a finite set of points* $\mathcal{Y} \subset M$ *such that, up to a subsequence,* $\{\Sigma_k\}_{k\in\mathbb{N}}$ *converges in* $C^{1,1}_{\mathrm{loc}}$ *to* Σ_∞ *in any compact subset of* $M \setminus \mathcal{Y}$.

Proof. The theorem is readily verified by combining Proposition 2.17, property (S), and the standard removable singularity theorem (see [41]). □

Although V_∞ may not be G-equivariantly $\mathcal{A}^0$-almost minimizing in small G-annuli, we may choose a special $h \in C^\infty_{G_\pm}(M)$ so that the limit minimal surface still has the total genus less than $\mathfrak{g}_0$, the genus of elements in $\mathcal{E}_{G_\pm}$.

Theorem 4.3 (Genus bound). *Let (M, g_M) be a closed orientable 3-dimensional Riemannian manifold and let G be a finite group acting freely and effectively as isometries on M so that G admits an index 2 subgroup G_+ with coset $G_- = G \setminus G_+$. Consider V_∞ as above. Suppose that there are finitely many pairwise disjoint appropriately small open G-balls*

$$B^G_1, \ldots, B^G_\alpha \subset M$$

such that

(1) $\pi(\operatorname{spt}\|V_\infty\| \cap B^G_j)$ *is an embedded disk for* $j = 1, \ldots, \alpha$*;*

(2) $h \equiv 0$ *in a small neighborhood of* $\operatorname{spt}\|V_\infty\| \setminus \bigcup_j B^G_j$.

Assume that $V_\infty = \sum_{i=1}^N m_i[\Gamma_i]$, where $\{\Gamma_i\}_{i=1}^N$ is a pairwise disjoint collection of connected, closed, embedded, minimal surfaces. Denote by $I_O \subset \{1, \ldots, N\}$ (resp. I_U) the collection of i such that Γ_i is orientable (resp. non-orientable). Then we have

$$\sum_{i\in I_O} m_i \cdot \mathfrak{g}(\Gamma_i) + \frac{1}{2}\sum_{i\in I_U} m_i \cdot (\mathfrak{g}(\Gamma_i) - 1) \le \mathfrak{g}_0, \tag{4.1}$$

where $\mathfrak{g}_0$ and $\mathfrak{g}(\Gamma_i)$ are the genus of Σ_0 and Γ_i respectively.

Proof. Let $\{\gamma_i\}_{i=1}^k$ be a collection of simple closed curves contained in $\bigcup_{i=1}^N \Gamma_i$. By assumption (1), we can perturb $\{\gamma_i\}_{i=1}^k$ in the same isotopy class so that $\bigcup_i \gamma_i$ does not intersect $\bigcup_j B^G_j$. Hence, by assumption (2), $\epsilon_k h \equiv 0$ in a neighborhood of $\bigcup_i \gamma_i$. Additionally, one notices that [48, Proposition 5.3] can be easily generalized to our G-equivariant setting. Hence, by possibly perturbing $\{\gamma_i\}$ and shrinking $r_{G\cdot p} > 0$, we can assume that $(\Sigma_{k,j(k)}, \Omega_{k,j(k)})$ is G-equivariantly $(\mathcal{A}^0, \epsilon_k, \delta_k)$-almost minimizing in $B^G_{r_{G\cdot p}}(p)$ for any $p \in \bigcup_i \gamma_i$. To prove the curve lifting lemma, we follow the strategy in [24, Proposition 2.2] (see also [14]). For simplicity, we only consider the setting of two appropriately small open G-balls B^G_1 and B^G_2 intersecting along a curve γ. Suppose further that the G-equivariantly $\mathcal{A}^0$-almost minimizing property holds in $B^G_1 \cup B^G_2$. By taking successive $(G_\pm, \mathcal{A}^0)$-replacements of $(\Sigma_{k,j(k)}, \Omega_{k,j(k)})$ on B^G_1 and B^G_2, we obtain a new G-invariant surface W_k (arising topologically from $\Sigma_{k,j(k)}$ after finitely many G-equivariant surgeries), which has the same limit as $\Sigma_{k,j(k)}$. Combining Schoen's estimates, a no-folding property, and an integrated Gauss–Bonnet argument, we show the graphical smooth convergence of W_k away from finitely many points. Hence we can lift γ with the correct multiplicity by perturbing it slightly to avoid those singularities. □

4.2. Existence of supersolution. For a sequence of strongly $\mathcal{A}^{\epsilon_k h}$-stationary $C^{1,1}$ $\epsilon_k h$-boundaries $\{(\Sigma_k, \Omega_k)\}$ converging as varifolds to a closed 2-sided minimal surface Σ, Wang–Zhou [48] proved in the non-equivariant setting that Σ admits a nonnegative weak supersolution to a variant of the Jacobi equation provided that $h \llcorner \Sigma$ changes sign and the convergence is $C^{1,1}_{\mathrm{loc}}$ away from a finite set $\mathcal{Y}$ with multiplicity $m \geq 2$. Noting G is finite and $h \in C^\infty_{G_\pm}(M)$ must change sign unless $h \equiv 0$, we immediately have the following generalization.

Proposition 4.4. *Let $\{(\Sigma_k, \Omega_k)\}_{k\in\mathbb{N}}$ be a sequence of strongly $\mathcal{A}^{\epsilon_k h}$-stationary $C^{1,1}$ $(G_\pm, \epsilon_k h)$-boundaries with $\lim_{k\to\infty}\epsilon_k = 0$ so that Σ_k converges as varifolds to a closed embedded 2-sided minimal G-surface Σ with multiplicity $m \geq 2$. Suppose the convergence is also $C^{1,1}_{\mathrm{loc}}$ away from a finite G-set $\mathcal{Y}$. Then Σ admits a nonnegative G-invariant function $\varphi \in W^{1,2}(\Sigma)$ satisfying $\|\varphi\|_{L^2(\Sigma)} = 1$ and*

$$\int_\Sigma \langle \nabla\varphi, \nabla f\rangle - \big(\mathrm{Ric}(\nu,\nu) + |A^\Sigma|^2\big)\varphi f \, d\mathcal{H}^2 \geq \int_\Sigma 2chf \, d\mathcal{H}^2 \quad \text{for all } f \in C^1(\Sigma) \text{ and } f \geq 0, \tag{4.2}$$

where $c \geq 0$ is a constant so that $c = 0$ if $m \geq 3$ is odd.

Proof. As we explained before, [48, Proposition 6.4] can be applied directly to get a nonnegative function $\varphi \in W^{1,2}(\Sigma)$ satisfying (4.2). To show φ is G-invariant, we recall the constructions in [48, Section 6]. Take any open G-set $\mathcal{U} \subset\subset \Sigma \setminus \mathcal{Y}$ and a unit normal ν of Σ. Then, for sufficiently large k, Σ_k admits a decomposition by ordered m-sheets $\Gamma^1_k \leq \cdots \leq \Gamma^m_k$ inside a thickened G-neighborhood $\mathcal{U}_\delta = \mathcal{U} \times (-\delta, \delta)$, and each sheet Γ^ι_k $(1 \leq \iota \leq m)$ is a normal graph of some function $u^\iota_k \in C^{1,1}(\mathcal{U})$ such that $u^1_k \leq \cdots \leq u^m_k$, and $u^\iota_k \to 0$ in $C^{1,1}(\mathcal{U})$ as $k \to \infty$. Although a single sheet Γ^i_k may not be G-invariant, we notice $\Gamma^m_k \cup \Gamma^1_k$ is G-invariant, and thus $|\varphi_k| = \varphi_k := u^m_k - u^1_k$ is G-invariant. Since $\varphi \llcorner \mathcal{U}$ is the $C^{1,\alpha}$ limit of $\varphi_k/\|\varphi_k\|_{L^2(\mathcal{U})}$, we know that φ is G-invariant. □

5. Multiplicity one for generalized Simon–Smith min-max theory

In this section, we will generalize two multiplicity one theorems in [48] to the equivariant setting, namely the relative min-max in the space of oriented G-equivariant $G_\pm$-separating surfaces in Section 5.1 and the classical min-max in the space of unoriented G-equivariant $G_\pm$-separating surfaces in Section 5.2.

5.1. Multiplicity one for relative equivariant min-max. To start with, we have the following compactness theorem, which admits a proof similar to [48].

Theorem 5.1. *Let $L \in \mathbb{N}$ and let $C > 0$ be a constant. Consider $\{\Sigma_k\}$ a sequence of closed embedded G-invariant minimal surfaces satisfying $\sup_k \mathcal{H}^2(\Sigma_k) \leq C$ and property* (R′)*. Then Σ_k converges subsequentially as varifolds to a closed embedded G-invariant minimal surface Σ possibly with integer multiplicities. Moreover, Σ is degenerate if $\Sigma_k \neq \Sigma$ for infinitely many k.*

Recall that the space $\mathscr{E}_{G_\pm}$ of G-equivariant $G_\pm$-separating surfaces of genus $\mathfrak{g}_0$ is defined in Section 3. By choosing an appropriate prescribing function $h \in C^\infty_{G_\pm}(M)$, we obtain the first multiplicity one type result as follows.

Theorem 5.2. *Let (M, g_M) be a closed connected orientable 3-dimensional Riemannian manifold, and let G be a finite group acting freely and effectively as isometries on M so that G admits an index 2 subgroup G_+ with coset $G_- = G \setminus G_+$. Suppose that X is a finite-dimensional cubical complex with $Z \subset X$ a subcomplex. Consider $\Phi_0 \colon X \to \mathcal{E}_{G_\pm}$ a continuous map and Π the (X, Z)-homotopy class of Φ_0. Assume that*

$$\mathbf{L}(\Pi) > \max_{x \in Z} \mathcal{H}^2(\Phi_0(x)).$$

Then there exists a closed embedded G-invariant minimal surface Γ with connected components $\{\Gamma_j\}_{j=1}^J$ and integer multiplicities $\{m_j\}_{j=1}^J$ so that

(i) *if Γ_j is unstable and 2-sided, then $m_j = 1$;*

(ii) *if Γ_j is 1-sided, then its connected double cover is stable.*

Moreover, the weighted total genus of Γ (4.1) is bounded by $\mathfrak{g}_0$.

Proof. It is sufficient to verify the theorem when g_M is G-bumpy, i.e. any finite cover of a closed embedded G-invariant minimal hypersurface in (M, g_M) is non-degenerate. Given a constant $C > 0$ (e.g. $C := \mathbf{L}(\Pi) + 1$), set $\mathcal{M}(C)$ the space of all closed embedded G-invariant minimal surfaces Γ satisfying $\mathcal{H}^2(\Gamma) \le C$ and property (R′). By Theorem 5.1, it follows that $\mathcal{M}(C) = \{S_1, \dots, S_\alpha\}$ is a finite set. Take $p_1, \dots, p_\alpha$ in M so that $p_i \in S_j$ if and only if $j = i$. Let $r > 0$ be small enough such that

- $B_r^G(p_1), \dots, B_r^G(p_\alpha)$ are pairwise disjoint appropriately small open G-balls;
- $B_r^G(p_i)$ intersects S_j if and only if $j = i$;
- $\pi(B_r^G(p_i) \cap S_i)$ is an embedded disk for all $i = 1, \dots, \alpha$.

Now, choose $h \in C^\infty_{G_\pm}(M)$ with $h(M) \subset [-1, 1]$ satisfying that, for all $i = 1, \dots, \alpha$,

(1) $h = 0$ outside $\bigcup_i B_r^G(p_i)$;

(2) $h > 0$ in some component of $B_{r/2}^G(p_i)$ (then $h < 0$ in another component of $B_{r/2}^G(p_i)$);

(3) if S_i is 2-sided, then $\int_{S_i} h\phi_i \, d\mathcal{H}^2 = 0$, where ϕ_i is the first eigenfunction of the Jacobi operator on S_i;

(4) if S_i is 1-sided, then $\int_{\widetilde{S}_i} h\phi_i \, d\mathcal{H}^2 = 0$, where ϕ_i is the first eigenfunction of the Jacobi operator on $\widetilde{S}_i$ and $\widetilde{S}_i$ is the connected double cover of S_i.

Choose $\epsilon_k \to 0$ so that the nontriviality condition (3.1) with h replaced by $\epsilon_k h$ is met for sufficiently large k. By combining Theorem 3.2, Theorem 4.2 and arguing similarly to [48], we obtain a closed embedded G-invariant minimal surface $\Gamma_\infty = \bigcup_{j=1}^J \Gamma_j$, where each Γ_j is G-invariant and belongs to $\mathcal{M}(C)$. Assuming $\Gamma_j = S_{i_j}$ for $i_j \in \{1, \dots, \alpha\}$, the sign of h must change on each Γ_j. If Γ_j is 2-sided with $m_j \ge 2$, it follows from Proposition 4.4 (nonnegativity and nontriviality of φ), the positivity of ϕ_{i_j} and our choice of h that the first eigenvalue satisfies $\lambda_1(\Gamma_j) \ge 0$. Hence any connected component Γ_j is stable if Γ_j is 2-sided with $m_j \ge 2$, which shows the first item. The same argument applied to the double cover of 1-sided G-invariant $\Gamma' \subset \Gamma_\infty$ proves the second item.

By the choice of h, we have $h = 0$ outside $B_r^G(p_i)$. Since $\pi(B_r^G(p_i) \cap \Gamma_j)$ is a disk, it follows from Theorem 4.3 that Γ admits the genus bound. □

5.2. Multiplicity one for classical equivariant min-max. In this part, we show an equivariant version of a multiplicity one type result in analogy to [48, Theorem 7.3]. This relies on a version of Simon–Smith min-max theory for unoriented G-equivariant $G_\pm$-separating surfaces.

Consider (M, g_M) and G as in Theorem 5.2. Fix Σ_0 a G-connected closed surface of genus $\mathfrak{g}_0$. We equip

$$\mathscr{X}_{G_\pm}(\Sigma_0) := \{\phi(\Sigma_0) \mid \phi\colon \Sigma_0 \to M \text{ is a } G\text{-equivariant } G_\pm\text{-separating embedding}\}$$

with the unoriented smooth topology. Let X be a finite-dimensional cubical complex. Given a fixed continuous map $\Phi_0\colon X \to \mathscr{X}_{G_\pm}$, we denote by Π the collection of all continuous $\Phi\colon X \to \mathscr{X}_{G_\pm}$ which is homotopic to Φ_0. Such a Φ is called a *sweepout by* Σ_0, or simply a sweepout. Define

$$\mathbf{L}(\Pi) := \inf_{\Phi \in \Pi} \sup_{x \in X} \mathscr{H}^2(\Phi(x)).$$

In our setting, the space $\mathscr{E}_{G_\pm}$ defined in Section 3 double covers $\mathscr{X}_{G_\pm}$. Let $\bar{\lambda} \in H^1(\mathscr{X}_{G_\pm}; \mathbb{Z}_2)$ be the generator dual to the nontrivial element of $\pi_1(\mathscr{X}_{G_\pm})$ corresponding to the projection $\pi\colon \mathscr{E}_{G_\pm} \to \mathscr{X}_{G_\pm}$. Note that a loop ϕ in $\mathscr{X}_{G_\pm}$ forms a sweepout in the sense of Almgren–Pitts if and only if $\bar{\lambda}[\phi] \neq 0$.

Theorem 5.3. *Consider the above setup and let* Π *be a homotopy class of sweepouts by* Σ_0 *with* $\mathbf{L}(\Pi) > 0$*. Then there exists a closed embedded* G*-invariant minimal surface* Γ *with* G*-connected components* $\{\Gamma_j\}_{j=1}^J$ *and integer multiplicities* $\{m_j\}_{j=1}^J$ *so that*

$$\mathbf{L}(\Pi) = \mathscr{H}^2(\Gamma) = \sum_{j=1}^J m_j \mathscr{H}^2(\Gamma_j)$$

and

(i) *if* Γ_j *is unstable and* 2*-sided, then* $m_j = 1$*;*

(ii) *if* Γ_j *is* 1*-sided, then its connected double cover is stable.*

Moreover, the genus bound (4.1) *holds for* Γ *with* $\mathfrak{g}_0 = \mathfrak{g}(\Sigma_0)$.

Proof. We follow the proof of [48, Theorem 7.3] (also see [54, Theorem 5.2]) with some alterations. Similar to before, we only need to check the theorem for G-bumpy metric g_M. The existence of a closed embedded G-invariant minimal surface Σ satisfying $\mathscr{H}^2(\Sigma) = \mathbf{L}(\Pi)$ and property (R′) is guaranteed by Theorem 3.3, Theorem 3.8 and Theorem 3.2.

Let $\mathcal{S}$ be the collection of all stationary 2-varifolds with mass lying in

$$[\mathbf{L}(\Pi) - 1, \mathbf{L}(\Pi) + 1],$$

whose support is a closed embedded G-invariant minimal surface satisfying (R′). Note that $\mathcal{S}$ is a finite set by bumpiness. Given a small $\bar{\epsilon} > 0$, set

$$Z_i = \{x \in X \mid \mathbf{F}(\Phi_i(x), \mathcal{S}) \geq \bar{\epsilon}\} \quad \text{and} \quad Y_i = \overline{X \setminus Z_i}.$$

Since each Y_i is topologically trivial, by adapting a continuous version of Pitts' combinatorial argument to $\{\Phi_i\}$, we can find another minimizing sequence, still denoted by $\{\Phi_i\}$, such that $\mathbf{L}(\{\Phi_i|_{Z_i}\}) < \mathbf{L}(\Pi)$ for sufficiently large i. Lifting the maps $\Phi_i\colon Y_i \to \mathscr{X}_{G_\pm}$ to their double covers $\widetilde{\Phi}_i : Y_i \to \mathscr{E}_{G_\pm}$, we have, for i large enough,

$$\sup_{x \in \partial Y_i} \mathcal{H}^2(\widetilde{\Phi}_i(x)) \leq \sup_{x \in Z_i} \mathcal{H}^2(\widetilde{\Phi}_i(x)) < \mathbf{L}(\Pi).$$

Denote $\widetilde{\Pi}_i$ the $(Y_i, \partial Y_i)$-homotopy class associated with $\widetilde{\Phi}_i|_{Y_i}$ in $\mathscr{E}_{G_\pm}$. By employing a contradiction argument as in [54, Lemma 5.8], we obtain

$$\mathbf{L}(\widetilde{\Pi}_i) \geq \mathbf{L}(\Pi) > \sup_{x \in \partial Y_i} \mathcal{H}^2(\widetilde{\Phi}_i(x)).$$

Then the proof is completed by applying Theorem 5.2 to $\widetilde{\Pi}_i$ and letting $i \to \infty$. □

Proposition 5.4. *In the above theorem, there exists a subset $\mathcal{J} \subset \{1, \ldots, J\}$ so that $\Gamma' = \bigcup_{j \in \mathcal{J}} \Gamma_j$ is a $G_\pm$-separating G-surface.*

Proof. Extracting a diagonal subsequence, one easily obtains a sequence $\Omega_k \in \mathcal{C}^{G_\pm}(M)$ with $\partial\Omega_k \in \mathscr{X}_{G_\pm}$ so that $|\partial\Omega_k| \to V_\infty = \sum_{j=1}^{J} m_j |\Gamma_j| \in \mathcal{V}^G(M)$ as $k \to \infty$ in the varifolds sense, where V_∞ is the min-max varifold. It also follows from the compactness theorem that Ω_k converges (up to a subsequence) to some $\Omega_\infty \in \mathcal{C}^{G_\pm}(M)$. Hence $\|\partial\Omega_\infty\| \leq \|V_\infty\|$ and $\partial\Omega_\infty$ is a G-invariant integral current supported in $\bigcup_{j=1}^{J} \Gamma_j$. As an elementary fact, $\partial\Omega_\infty$ is also a G-invariant integral n-cycle in $\bigcup_{j=1}^{J} \Gamma_j$ (cf. [53, Appendix 8]), which implies $\partial\Omega_\infty = \sum_{j=1}^{J} k_j [[\Gamma_j]]$ for some $k_j \in \{0, 1\}$ by the constancy theorem [42, Theorem 26.27]. Since $\mathrm{Vol}(\Omega_\infty) = \mathrm{Vol}(M)/2$, we see that $\emptyset \neq \Gamma' := \cup\{\Gamma_j \mid k_j = 1\}$ is the smooth boundary of $\Omega_\infty \in \mathcal{C}^{G_\pm}(M)$, which is $G_\pm$-separating. □

6. Minimal $\mathbf{RP}^2$ in $\mathbf{RP}^3$

In this section, let $M := S^3 \subset \mathbb{R}^4$ be the unit 3-sphere, and $G := \mathbb{Z}_2$ acts on M by the identity map $G_+ = \{[0]\}$ and the antipodal map $G_- = \{[1]\}$. Consider the space

$$\mathscr{X}_{G_\pm} := \{\phi(S^2) \mid \phi : S^2 \to S^3 \text{ a } G\text{-equivariant } G_\pm\text{-separating smooth embedding}\}$$

endowed with the unoriented smooth topology, where $G = \mathbb{Z}_2$ is given as above.

6.1. Sweepouts formed by real projective planes. We now describe three classes of sweepouts that detect three nontrivial cohomology classes in $H^k(\mathscr{X}_{G_\pm}; \mathbb{Z}_2)$, $k \in \{1, 2, 3\}$.

To begin with, use (a_1, a_2, a_3, a_4) and $[a_1, a_2, a_3, a_4]$ with $\sum_{i=1}^{4} a_i^2 = 1$ to denote the points in $\mathbb{S}^3$ and $\mathbb{RP}^3$ respectively. Define then

$$\widetilde{\mathcal{G}}((a_1, a_2, a_3, a_4)) := \partial(\{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 < 0\} \cap \mathbb{S}^3),$$
$$\mathcal{G}([a_1, a_2, a_3, a_4]) := \{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0\} \cap \mathbb{S}^3,$$

where $x_1, \ldots, x_4$ are the coordinate functions in $\mathbb{R}^4$. Note that $\widetilde{\mathcal{G}}(\mathbb{S}^3) \cong \mathbb{S}^3$ is the space of oriented great spheres and is a double cover of $\mathcal{G}(\mathbb{RP}^3) \cong \mathbb{RP}^3$, the space of unoriented great spheres. Since $\mathcal{G}(\mathbb{RP}^3) \subset \mathscr{X}_{G_\pm}$, we can define our three maps

$$\Phi_1 : \mathbb{RP}^1 \to \mathscr{X}_{G_\pm}, \quad [a_1, a_2] \to \mathcal{G}([a_1, a_2, 0, 0]),$$
$$\Phi_2 : \mathbb{RP}^2 \to \mathscr{X}_{G_\pm}, \quad [a_1, a_2, a_3] \to \mathcal{G}([a_1, a_2, a_3, 0]),$$
$$\Phi_3 : \mathbb{RP}^3 \to \mathscr{X}_{G_\pm}, \quad [a_1, a_2, a_3, a_4] \to \mathcal{G}([a_1, a_2, a_3, a_4]).$$

One notices that, for any $k \in \{1,2,3\}$, Φ_k is a k-sweepout of $M = S^3$ in the sense of Almgren–Pitts. Indeed, consider the closed curve

$$\gamma\colon S^1 = [0,2\pi]/\{0 \sim 2\pi\} \to \operatorname{dmn}(\Phi_k) = \mathbb{RP}^k$$

given by $\gamma(e^{i\theta}) = [\cos(\theta/2), \sin(\theta/2), 0, 0]$, which is a generator of $\pi_1(\mathbb{RP}^k)$. Then the curve $\phi_k := \Phi_k \circ \gamma$ in $\mathcal{G}(\mathbb{RP}^3)$ can be lifted to the curve

$$\widetilde{\phi}_k\colon \theta \in [0,2\pi] \mapsto \widetilde{\mathcal{G}}\big((\cos(\theta/2), \sin(\theta/2), 0, 0)\big)$$

in the double cover $\widetilde{\mathcal{G}}(\mathbb{S}^3)$ satisfying that $\widetilde{\phi}_k(0)$ is $\widetilde{\phi}_k(2\pi)$ with the opposite orientation. Hence ϕ_k is a sweepout in the sense of Almgren–Pitts. Combining with the fact that the generator $\lambda \in H^1(\mathbb{RP}^k;\mathbb{Z}_2)$ satisfies $\lambda(\gamma) = 1$ and $\lambda^k \neq 0$, we conclude Φ_k is a k-sweepout in the sense of Almgren–Pitts (cf. [34, Definition 4.1]).

Next, denote by $\iota\colon \mathscr{X}_{G_\pm} \to \mathcal{Z}_2(S^3;\mathbb{Z}_2)$ the natural inclusion map into the space of mod-2 integral 2-cycles, and by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_2(S^3;\mathbb{Z}_2);\mathbb{Z}_2)$. Then the above result indicates $(\iota \circ \Phi_k)^*(\bar{\lambda}^k) \neq 0 \in H^k(\mathbb{RP}^k;\mathbb{Z}_2)$ (cf. [34, Definition 4.1]). In particular, we have

$$\alpha := \iota^*(\bar{\lambda}) \in H^1(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$$

satisfies $\alpha^k \neq 0 \in H^k(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$ for each $k \in \{1,2,3\}$.

Finally, define $\mathscr{P}_k$, $k \in \{1,2,3\}$, to be the collection of continuous maps Φ from any cubical complex X to $\mathscr{X}_{G_\pm}$ that detects $\alpha^k \in H^k(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$, i.e.

$$\mathscr{P}_k := \{\Phi\colon X \to \mathscr{X}_{G_\pm} \mid \Phi^*(\alpha^k) \neq 0 \in H^k(X;\mathbb{Z}_2)\}.$$

Clearly, $\Phi_k \in \mathscr{P}_k$ for all $k = 1,2,3$. Recall that

$$\mathbf{L}(\mathscr{P}_k) := \inf_{\Phi \in \mathscr{P}_k} \sup_{x \in \operatorname{dmn}(\Phi)} \mathscr{H}^2(\Phi(x)).$$

6.2. Proof of Theorem 1.2. We have the following direct corollary by applying Theorem 5.3 and Proposition 5.4 to any homotopy class in $\mathscr{P}_k$.

Corollary 6.1. *Suppose $\Pi \subset \mathscr{P}_k$, $k \in \{1,2,3\}$, is a homotopy class of sweepouts in $\mathscr{X}_{G_\pm}$. Then $\mathbf{L}(\Pi) > 0$ and the G-connected min-max minimal G-surfaces $\{\Gamma_j\}_{j=1}^J$ associated to Π given by Theorem 5.3 satisfy that*

(i) *there is one G-component, Γ_1, that is a $G_\pm$-separating minimal 2-sphere, i.e. $\Gamma_1 \in \mathscr{X}_{G_\pm}$;*

(ii) *every other G-component Γ_j, $j \geq 2$, is a G-invariant disjoint union of two minimal 2-spheres.*

Proof. Firstly, $\mathbf{L}(\Pi) > 0$ since $\Pi \subset \mathscr{P}_k$ are k-sweepouts in the sense of Almgren–Pitts. By Theorem 5.3, $\{\Gamma_j\}_{j=1}^J$ satisfies the genus bound (4.1), which implies each Γ_j is either a $G_\pm$-separating minimal sphere in $\mathscr{X}_{G_\pm}$ or a G-invariant disjoint union of two minimal spheres. Combining Proposition 5.4 with the fact that every two elements in $\mathscr{X}_{G_\pm}$ have nonempty intersections, there is exactly one element of $\{\Gamma_j\}_{j=1}^J$ that is in $\mathscr{X}_{G_\pm}$. □

Fix any Riemannian metric g_{RP^3} on RP^3. By [6, Proposition 2.3], there exists an embedded area minimizing RP^2 denoted as Σ_0, i.e.

$$\mathscr{H}^2(\Sigma_0) = \inf\{\mathscr{H}^2(\Sigma) \mid \Sigma \text{ is an embedded } \mathrm{RP}^2 \subset \mathrm{RP}^3\}.$$

Let $M = S^3$ with the pull-back metric g_M, $G = \mathbb{Z}_2$, and $\mathscr{X}_{G\pm}$ be defined as above with respect to $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$. Then Σ is an embedded minimal projective plane in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ if and only if $\pi^{-1}(\Sigma) \in \mathscr{X}_{G\pm}$ is a minimal 2-sphere in (M, g_M).

Theorem 6.2. *Using the above notation, suppose $\mathscr{X}_{G\pm}$ contains only finitely many elements that are minimal in $(M = S^3, g_M)$. Then*

$$0 < 2M_0 < \mathbf{L}(\mathscr{P}_1) < \mathbf{L}(\mathscr{P}_2) < \mathbf{L}(\mathscr{P}_3),$$

where $M_0 := \mathscr{H}^2(\Gamma_0)$.

Proof. The fact that $0 < \mathbf{L}(\mathscr{P}_k) < \mathbf{L}(\mathscr{P}_{k+1})$ for $k = 1, 2$ follows from the standard Lusternik–Schnirelmann theory (see [48, Lemma 8.3] or [34, Theorem 6.1]). Indeed, one can take $\mathcal{S}$ to be the collection of G-invariant integral varifolds with mass bounded by $\mathbf{L}(\mathscr{P}_{k+1})$ whose support is an element in $\mathscr{X}_{G\pm}$. Then, using $\alpha \in H^1(\mathscr{X}_{G\pm}; \mathbb{Z}_2)$ and $\mathscr{X}_{G\pm}$ in place of $\bar{\lambda} \in H^1(\mathcal{Z}_2(S^3; \mathbb{Z}_2); \mathbb{Z}_2)$ and $\mathcal{Z}_2(S^3; \mathbb{Z}_2)$, the proof of [34, Theorem 6.1] would carry over, leading to a contradiction to Corollary 6.1 (i). Our goal is now to prove $2M_0 < \mathbf{L}(\mathscr{P}_1)$.

Suppose by contradiction that $2M_0 = \mathbf{L}(\mathscr{P}_1)$. By the finiteness assumption, the union

$$\mathcal{S} := \bigcup \{\Gamma \in \mathscr{X}_{G\pm} \mid \mathscr{H}^2(\Gamma) = 2M_0\}$$

is a closed set. Fix a $\mathbb{Z}_2$-invariant open ball $B \subset\subset S^3 \setminus B_r(\mathcal{S})$. We have the following claim.

Claim 1. *For any $\delta > 0$, there is $\Phi_\delta \in \mathscr{P}_1$ so that* $\sup_{x \in \mathrm{dmn}(\Phi_\delta)} \mathscr{H}^2(\Phi_\delta(x) \cap B) < \delta$.

Proof of Claim 1. Suppose by contradiction that, for some $\delta_0 > 0$, every $\Phi \in \mathscr{P}_1$ has $x \in \mathrm{dmn}(\Phi)$ with $\mathscr{H}^2(\Phi(x) \cap B) \geq \delta_0$. Let $\{\Phi_i\}$ be a minimizing sequence in $\mathscr{P}_1$, i.e.

$$\lim_{i \to \infty} \sup_{x \in \mathrm{dmn}(\Phi_i)} \mathscr{H}^2(\Phi_i(x)) = \mathbf{L}(\mathscr{P}_1) = 2M_0.$$

For each i, we may pick some $x_i \in \mathrm{dmn}(\Phi_i)$ so that $\mathscr{H}^2(\Phi_i(x_i) \cap B) \geq \delta_0$ by assumptions. Based on the fact that

$$2M_0 \leq \mathscr{H}^2(\Phi_i(x_i)) \leq \sup_{x \in \mathrm{dmn}(\Phi_i)} \mathscr{H}^2(\Phi_i(x)) \to 2M_0 \quad \text{as } i \to \infty,$$

we have $\Phi_i(x_i)/\mathbb{Z}_2 \subset \mathrm{RP}^3$ converges (up to a subsequence) as varifolds to an area minimizing projective plane (cf. [6, Proposition 2.3]), and thus the varifolds limit V_∞ of $\Phi_i(x_i)$ satisfies $\mathrm{spt}\|V_\infty\| \subset \mathcal{S}$. This forces

$$\lim_{i \to \infty} \mathscr{H}^2(\Phi_i(x_i) \cap B) = 0,$$

which contradicts the choice of x_i. □

Let $B_r(q) \subset B$ be a small ball with $0 < r < r_0$ and $\delta \in (0, \alpha_0 r^2)$, where $\alpha_0, r_0 > 0$ are given by [34, Proposition 8.2] with respect to $\mathcal{Z}_2(S^3; \mathbb{Z}_2)$. Consider Φ_δ in Claim 1 with respect to this given δ. Then, by definitions, $\iota \circ \Phi_\delta \colon \mathrm{dmn}(\Phi_\delta) \to \mathcal{Z}_2(S^3; \mathbb{Z}_2)$ is a 1-sweepout in the sense of Almgren–Pitts. However, since $\mathbf{M}(\iota \circ \Phi_\delta(x) \llcorner B_r(q)) \leq \mathbf{M}(\iota \circ \Phi_\delta(x) \llcorner B) < \delta < \alpha_0 r^2$ for all $x \in \mathrm{dmn}(\Phi_\delta)$, we obtain a contradiction to [34, Proposition 8.2]. □

At the end of this section, we prove Theorem 1.2.

Proof of Theorem 1.2. Firstly, we have a minimizing RP^2 embedded in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ denoted as Σ_0 (cf. [6, Proposition 2.3]).

Case I: g_{RP^3} *is a metric with positive Ricci curvature.* In this case, g_M is a $\mathbb{Z}_2$-invariant metric on $M = S^3$ with positive Ricci curvature. Hence

(6.1) every $\mathbb{Z}_2$-connected minimal 2-sphere in (M, g_M) is $(\mathbb{Z}_2)_\pm$-separating,

i.e. there is no minimal 2-sphere in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$. Otherwise, there will be a pair of minimal $S^2 \subset M$, denoted as $\Gamma = \{\Gamma_+, \Gamma_-\}$, such that $[1] \cdot \Gamma_\pm = \Gamma_\mp$ and $\Gamma_+ \cap \Gamma_- = \emptyset$, where $[1] \in \mathbb{Z}_2$ acting on $M = S^3$ by the antipodal map. This violates the embedded Frankel property [15].

Next, without loss of generality, we assume

(6.2) $(M = S^3, g_M)$ contains finitely many minimal 2-spheres in $\mathscr{X}_{\mathbb{Z}_{2,\pm}}$,

whose quotients in RP^3 are minimal projective planes. By Theorem 6.2 and (6.2), we have

$$0 < 2M_0 < \mathbf{L}(\mathscr{P}_1) < \mathbf{L}(\mathscr{P}_2) < \mathbf{L}(\mathscr{P}_3). \tag{6.3}$$

Additionally, although each $\mathscr{P}_k$, $k \in \{1, 2, 3\}$, may contain many different homotopy classes, it follows from the finiteness assumptions (6.2) and Corollary 6.1 that the min-max values of the homotopy classes in $\mathscr{P}_k$ must be stabilized to $\mathbf{L}(\mathscr{P}_k)$. Moreover, for each $k \in \{1, 2, 3\}$, $\mathbf{L}(\mathscr{P}_k)$ is realized by the area of an embedded connected minimal 2-sphere $\Gamma_k \in \mathscr{X}_{G_\pm}$ with integer multiplicity $m_k \in \mathbb{Z}_+$, i.e. $\mathbf{L}(\mathscr{P}_k) = m_k \mathcal{H}^2(\Gamma_k)$.

Since manifolds with positive Ricci curvature contain no 2-sided stable minimal hypersurface, we see Γ_i must be unstable, and thus $m_i = 1$ by Theorem 5.3. We can then conclude that $\Sigma_0, \pi(\Gamma_1), \pi(\Gamma_2), \pi(\Gamma_3)$ are distinct embedded minimal real projective planes in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$.

Case II: g_{RP^3} *is a bumpy metric.* In this case, every embedded $\mathbb{Z}_2$-invariant minimal surface in $(M = S^3, g_M)$ is non-degenerate. Without loss of generality, we also assume that

(6.4) $(M = S^3, g_M)$ contains finitely many $\mathbb{Z}_2$-invariant minimal 2-spheres,

i.e. $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ has finitely many embedded minimal RP^2 and finitely many embedded minimal S^2. In particular, (6.3) is still valid by Theorem 6.2.

For each $k \in \{1, 2, 3\}$, it follows from (6.4) and Theorem 5.3 that the min-max values of homotopy classes in $\mathscr{P}_k$ must be stabilized to $\mathbf{L}(\mathscr{P}_k)$, and there are disjoint embedded $\mathbb{Z}_2$-invariant $\mathbb{Z}_2$-connected minimal 2-spheres $\{\Gamma_{k,j}\}_{j=1}^{J_k}$ and integer multiplicities $\{m_{k,j}\}_{j=1}^{J_k}$ so that

$$\mathbf{L}(\mathscr{P}_k) = \sum_{j=1}^{J_k} m_{k,j} \mathcal{H}^2(\Gamma_{k,j}). \tag{6.5}$$

Also, by Corollary 6.1, we can assume $\Gamma_{k,1} \in \mathscr{X}_{G_\pm}$, and $\Gamma_{k,j}$ with $j \geq 2$ is a $\mathbb{Z}_2$-invariant disjoint union of two minimal 2-spheres. By Theorem 5.2, $\Gamma_{k,1}$ is either unstable with multiplicity one or stable.

Sub-case II (a): $\Gamma_{k,1}$ *is unstable with multiplicity one for all* $k \in \{1,2,3\}$. Suppose Theorem 1.2 (i) fails, i.e. the number of distinct minimal embedded real projective planes in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ is strictly less than 4. Clearly, this assumption and (6.3), (6.5) indicate there must be some $k_0 \in \{1,2,3\}$ with $J_{k_0} \geq 2$. Thus $\Gamma_{k_0,2}$ is a $\mathbb{Z}_2$-invariant union of two minimal spheres $\Gamma_\pm$ with $[1] \cdot \Gamma_\pm = \Gamma_\mp$ and $\Gamma_\pm$ lies in the different components of $M \setminus \Gamma_{k_0,1}$.

If Γ_+ is stable, and thus strictly stable due to the bumpiness of the metric, then by [48, Proposition 8.8] (using Song's strategy [43]), we can find another two minimal spheres in (M, g_M) lying in a fundamental domain of RP^3 (i.e. in a component of $M \setminus \Gamma_{k_0,1}$ containing Γ_+). Together, we obtain three minimal 2-spheres and one area minimizing real projective plane Σ_0 in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$.

If Γ_+ is unstable, then since $\Gamma_{k_0,1}$ is also assumed to be unstable, there exists an (isotopic area minimizing) stable minimal sphere $\widetilde{\Gamma}_+$ lying between Γ_+ and $\Gamma_{k_0,1}$. Using [48, Proposition 8.8], we obtain at least one more minimal sphere in a fundamental domain of RP^3 (i.e. in the component of $M \setminus \widetilde{\Gamma}_+$ containing Γ_+) which is different from $\Gamma_+, \widetilde{\Gamma}_+$. Therefore, we still have one area minimizing minimal real projective plane Σ_0 and three minimal spheres in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$.

Sub-case II (b): $\Gamma_{k,1}$ *is stable for some* $k \in \{1,2,3\}$. Since $\Gamma_{k,1}$ is stable (and thus strictly stable due to the bumpiness) for some $k \in \{1,2,3\}$, we can apply [48, Proposition 8.8] again to find another two distinct minimal spheres in the fundamental domain of $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$ (i.e. in a component of $M \setminus \Gamma_{k,1}$). In conclusion, we can find one area minimizing real projective plane Σ_0 and two distinct minimal spheres in $(\mathrm{RP}^3, g_{\mathrm{RP}^3})$. □

7. Minimal Klein bottles in $L(4m, 2m \pm 1)$

Consider the unit 3-sphere $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. For any two coprime integers $p \geq 1$ and $q \in [1,p)$, we have the following cyclic $\mathbb{Z}_p$-action on $\mathbb{S}^3$ generated by $\xi_{p,q}$:

$$[1] \cdot (z,w) = \xi_{p,q}(z,w) := (e^{2\pi i/p} \cdot z, e^{2\pi q i/p} \cdot w). \tag{7.1}$$

Then the lens space $L(p,q)$ is defined to be $\mathbb{S}^3/\mathbb{Z}_p$.

By [8, Corollary 6.4], the Klein bottle embeds into $L(4m, 2m \pm 1)$ only. Hence, in this section, we always set

$$M := S^3, \quad G := \mathbb{Z}_p = \langle \xi_{p,q} \rangle, \quad G_+ = \mathbb{Z}_{p/2} := \langle \xi_{p,q}^2 \rangle, \quad G_- := G \setminus G_+, \tag{7.2}$$

where $p = 4m$, $q = 2m \pm 1$ and $m \geq 1$. In addition, consider the space

$$\mathscr{X}_{G_\pm} = \{\phi(T^2) \mid \phi\colon T^2 \to S^3 \text{ a } G\text{-equivariant } G_\pm\text{-separating smooth embedding}\},$$

endowed with the smooth unoriented topology. One can verify that, for any G-surface $\Sigma \subset M$, Σ/G is a Klein bottle in $L(4m, 2m \pm 1)$ if and only if $\Sigma \in \mathscr{X}_{G_\pm}$.

7.1. Sweepouts formed by Klein bottles.

We now describe the sweepouts that detect the nontrivial cohomology classes of $H^k(\mathscr{X}_{G_\pm}; \mathbb{Z}_2)$, where $k = 3$ for $m = 1$, and $k = 1$ for $m > 1$. Firstly, we mention the following results given by Ketover [25, Proposition 4.2].

Proposition 7.1 ([25]). *Let $L(p,q)$ denote the lens space endowed with the round metric, $p \geq 2$. Then $L(p,q)$ admits an embedded Klein bottle if and only if $p = 4m$ and $q = 2m \pm 1$ for $m \geq 1$. If $m > 1$, then $L(4m, 2m \pm 1)$ admits an $\mathbb{S}^1$-family of minimal Klein bottles. If $m = 1$, then $L(4,1)$ admits an $\mathbb{S}^1 \times \mathbb{RP}^2$-family of minimal Klein bottles.*

For the sake of completeness, we provide a relatively detailed explanation following the constructions in [25, Section 4] (see also [45]). Firstly, one can identify $\mathbb{S}^3$ and $\mathbb{S}^2$ with the group of unit quaternions and pure unit quaternions (without real part) respectively, i.e.

$$\mathbb{S}^3 := \{a + bi + cj + dk \mid |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}, \quad \mathbb{S}^2 := \{bi + cj + dk \in \mathbb{S}^3\}.$$

Note that any orientation preserving isometry $f \in \mathrm{Isom}_+(\mathbb{S}^3) = SO(4)$ can be represented by $f(q) = q_1 q q_2^{-1}$ for some $q_1, q_2 \in \mathbb{S}^3$. Hence an oriented 2-plane in $\mathbb{R}^4$ spanned by two orthonormal vectors $u, v \in \mathbb{S}^3$ can be written as $\langle u, v\rangle = (q_1, q_2) \cdot \langle 1, i\rangle := \langle q_1 q_2^{-1}, q_1 i q_2^{-1}\rangle$ for some $q_1, q_2 \in \mathbb{S}^3$. Denote by $\widetilde{G}_2(\mathbb{R}^4)$ the Grassmannian manifold consisting of all oriented 2-dimensional subspaces of $\mathbb{R}^4$. Then the space of oriented geodesics in round $\mathbb{S}^3$ is homeomorphic to $\widetilde{G}_2(\mathbb{R}^4) \cong \mathbb{S}^2 \times \mathbb{S}^2$, where the homeomorphism is specified by the map

$$P \colon \widetilde{G}_2(\mathbb{R}^4) \to \mathbb{S}^2 \times \mathbb{S}^2, \quad P((q_1, q_2) \cdot \langle 1, i\rangle) = (q_1 i q_1^{-1}, q_2 i q_2^{-1}).$$

Note that $(a,b), (-a,-b) \in \mathbb{S}^2 \times \mathbb{S}^2$ correspond to the same geodesic with opposite orientations.

Next, for $a \in \mathbb{S}^2$, $B \subset \mathbb{S}^2$, denote

$$\tau(a, B) := \{x \in \mathbb{S}^3 \mid x \in P^{-1}(a,b) \cap \mathbb{S}^3,\ b \in B\}.$$

This is known to be a Clifford torus if $a \in \mathbb{S}^2$ and B is a great circle of $\mathbb{S}^2$ (cf. [25, (4.16)]). Given $b \in \mathbb{S}^2$, denote by

$$E(b) := \partial(\{x \in \mathbb{S}^2 \mid b_1 x_1 + b_2 x_2 + b_3 x_3 < 0\}) \subset \mathbb{S}^2$$

an oriented equator. Since $\tau(a, E(b)) = \tau(-a, E(-b))$ is $\tau(a, E(-b)) = \tau(-a, E(b))$ with the opposite orientation, we see the space of unoriented Clifford tori is homeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$. In addition, by [45, Section 5.1], we know that $\xi_{p,q}$ can be represented by $\xi_{p,q}(x) = e^{i\pi(q+1)/p} \cdot x \cdot e^{-i\pi(q-1)/p}$ using quaternions. Moreover, given $a \in \mathbb{S}^2$ and a great circle $E \subset \mathbb{S}^2$,

$$\xi_{p,q}(\tau(a, E)) = \tau(\widehat{\eta}^1_{p,q}(a), \widehat{\eta}^2_{p,q}(E)), \tag{7.3}$$

where $\widehat{\eta}^1_{p,q}, \widehat{\eta}^2_{p,q}$ is the rotation (in the $\{j,k\}$-plane) on $\mathbb{S}^2$ by the angle of $2\pi(q+1)/p$ and $2\pi(q-1)/p$ respectively with fixed points $\{\pm i\}$.

Case A: *$p = 4$ and $q = 1$ (or $q = 3$).* In this case, one verifies that $\xi_{4,1}$ is orientation preserving, and $\xi_{4,1}(\tau(\cos(\theta) j + \sin(\theta) k, E(b))) = \tau(-\cos(\theta) j - \sin(\theta) k, E(b))$ is $\tau(\cos(\theta) j + \sin(\theta) k, E(b))$ with the opposite orientation for any $\theta \in [0, 2\pi]$, $b \in \mathbb{S}^2$, which implies the support of $\tau(\cos(\theta) j + \sin(\theta) k, E(b))$ is an element in $\mathscr{X}_{G_\pm}$. Hence we have a family of Klein bottles in $L(4,1) \cong L(4,3)$ parameterized by $\mathscr{G} \colon \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathscr{X}_{G_\pm}$,

$$\mathscr{G}([a_1, a_2], [b_1, b_2, b_3]) := \mathrm{spt}\big(\tau(a_1 j + a_2 k, E(b_1 i + b_2 j + b_3 k))\big).$$

We can now define three maps into $\mathscr{X}_{G_\pm}$ as follows:

$$\begin{aligned} \Phi_1\colon \mathbb{RP}^1 &\to \mathscr{X}_{G_\pm}, & [b_1,b_2] &\mapsto \mathscr{G}([1,0],[b_1,b_2,0]),\\ \Phi_2\colon \mathbb{RP}^2 &\to \mathscr{X}_{G_\pm}, & [b_1,b_2,b_3] &\mapsto \mathscr{G}([1,0],[b_1,b_2,b_3]),\\ \Phi_3\colon \mathbb{RP}^1\times\mathbb{RP}^2 &\to \mathscr{X}_{G_\pm}, & ([a_1,a_2],[b_1,b_2,b_3]) &\mapsto \mathscr{G}([a_1,a_2],[b_1,b_2,b_3]). \end{aligned}$$

One notices that Φ_k, $k\in\{1,2,3\}$, is a k-sweepout in the sense of Almgren–Pitts (cf. [34, Definition 4.1]). Indeed, denote by α_l the generator of $H^1(\mathbb{RP}^l;\mathbb{Z}_2)$, $l\in\{1,2\}$, by $\bar\lambda$ the generator of $H^1(\mathcal{Z}_2(M;\mathbb{Z}_2);\mathbb{Z}_2)$ and by $\iota\colon \mathscr{X}_{G_\pm}\to\mathcal{Z}_2(M;\mathbb{Z}_2)$ the natural inclusion. Then the closed curves $\gamma_1(e^{i\theta}) = ([\cos(\theta/2),\sin(\theta/2)],[1,0,0])$ and $\gamma_2(e^{i\theta}) = ([1,0],[\cos(\theta/2),\sin(\theta/2),0])$ in $\mathbb{RP}^1\times\mathbb{RP}^2$ satisfy that

- $\iota\circ\mathscr{G}\circ\gamma_1\colon S^1\to\mathcal{Z}_2(M;\mathbb{Z}_2)$, $\iota\circ\mathscr{G}\circ\gamma_2\colon S^1\to\mathcal{Z}_2(M;\mathbb{Z}_2)$ are sweepouts in the sense of Almgren–Pitts, since $\tau(-1j,E(1j))$ and $\tau(1j,E(-1j))$ respectively are $\tau(1j,E(1j))$ with the opposite orientation;
- $\alpha_1\oplus\alpha_2(\gamma_1) = \alpha_1\oplus\alpha_2(\gamma_2) = 1$, $\alpha_1\oplus\alpha_2(\gamma_1+\gamma_2) = 0$,

where $0\neq\alpha_1\oplus\alpha_2\in H^1(\mathbb{RP}^1\times\mathbb{RP}^2;\mathbb{Z}_2)$. Combining the first bullet with the fact that

$$\alpha_2^2\neq 0\in H^2(\mathbb{RP}^2;\mathbb{Z}_2),$$

we conclude Φ_1 and Φ_2 are 1-sweepout and 2-sweepout in the sense of Almgren–Pitts respectively. Moreover, we also have $\mathscr{G}^*\iota^*(\bar\lambda) = \alpha_1\oplus\alpha_2\in H^1(\mathbb{RP}^1\times\mathbb{RP}^2;\mathbb{Z}_2)$ by the above two bullets. Together with $(\alpha_1\oplus\alpha_2)^3\neq 0\in H^3(\mathbb{RP}^1\times\mathbb{RP}^2;\mathbb{Z}_2)$, we conclude that Φ_3 is a 3-sweepout in the sense of Almgren–Pitts.

Furthermore, since $0\neq(\alpha_1\oplus\alpha_2)^3 = \mathscr{G}^*\iota^*(\bar\lambda^3)\in H^3(\mathbb{RP}^1\times\mathbb{RP}^2;\mathbb{Z}_2)$, we know that

$$\alpha := \iota^*(\bar\lambda)\in H^1(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$$

satisfies $\alpha^k\neq 0\in H^k(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$ for every $k\in\{1,2,3\}$. Thus we can define $\mathscr{P}_k$, $k\in\{1,2,3\}$, to be the collection of continuous maps Φ from any cubical complex X to $\mathscr{X}_{G_\pm}$ that detects $\alpha^k\in H^k(\mathscr{X}_{G_\pm};\mathbb{Z}_2)$, i.e.

$$\mathscr{P}_k := \{\Phi\colon X\to\mathscr{X}_{G_\pm} \mid \Phi^*(\alpha^k)\neq 0\in H^k(X;\mathbb{Z}_2)\}. \tag{7.4}$$

Clearly, the above $\Phi_k\in\mathscr{P}_k$ for all $k=1,2,3$.

Case B: $p=4m$ *and* $q=2m\pm1$ *with* $m\geq 2$. In this case, one verifies that

$$\xi_{p,q}\big(\tau(\cos(\theta)j+\sin(\theta)k, E(1j))\big)$$

is $\tau(\cos(\theta)j+\sin(\theta)k,E(1j))$ with the opposite orientation for any $\theta\in[0,2\pi]$. Hence we have a family of Klein bottles in $L(4m,2m\pm1)$ $(m\geq 2)$ parameterized by $\mathscr{G}\colon\mathbb{RP}^1\to\mathscr{X}_{G_\pm}$,

$$\mathscr{G}([a_1,a_2]) := \operatorname{spt}\big(\tau(a_1j+a_2k,E(1j))\big).$$

We can define the map

$$\Phi_1 = \mathscr{G}\colon\mathbb{RP}^1\to\mathscr{X}_{G_\pm},$$

which is a sweepout in the sense of Almgren–Pitts since $\tau(-1j,E(1j))$ is $\tau(1j,E(1j))$ with the opposite orientation. Hence $\mathscr{G}^*\iota^*(\bar\lambda)\neq 0\in H^1(\mathbb{RP}^1;\mathbb{Z}_2)$,

$$\alpha := \iota^*(\bar\lambda)\neq 0\in H^1(\mathscr{X}_{G_\pm};\mathbb{Z}_2),$$

and we can similarly define

$$\mathscr{P}_1 := \{\Phi\colon X \to \mathscr{X}_{G\pm} \mid \Phi^*(\alpha) \neq 0 \in H^1(X;\mathbb{Z}_2)\} \tag{7.5}$$

with $\Phi_1 \in \mathscr{P}_1$.

7.2. Proof of Theorem 1.5. Using Theorem 5.3 and Proposition 5.4, we have the following corollary, which gives the existence of min-max and minimizing minimal Klein bottles.

Corollary 7.2. *Suppose $\mathscr{P}_k$ is given by* (7.4) *or* (7.5)*, and $\Pi \subset \mathscr{P}_k$ is a homotopy class of sweepouts in $\mathscr{X}_{G\pm}$. Then $\mathbf{L}(\Pi) > 0$, and the G-connected min-max minimal G-surfaces $\{\Gamma_j\}_{j=1}^J$ with integer multiplicities $\{m_j\}_{j=1}^J$ given by Theorem* 5.3 *satisfy that*

(i) *there is exactly one G-component, say Γ_1, that is a $G_\pm$-separating minimal torus with multiplicity one, i.e. $\Gamma_1 \in \mathscr{X}_{G\pm}$ and $m_1 = 1$;*

(ii) *any other G-component Γ_j, $j \geq 2$, is a G-invariant disjoint union of $\#G = 4m$ minimal 2-spheres in M.*

Moreover, suppose $\{\Sigma_k\}_{k\in\mathbb{N}} \subset \mathscr{X}_{G\pm}$ is an area minimizing sequence in $\mathscr{X}_{G\pm}$ satisfying

$$\lim_{k\to\infty} \mathcal{H}^2(\Sigma_k) = \inf\{\mathcal{H}^2(\Gamma) \mid \Gamma \in \mathscr{X}_{G\pm}\};$$

then, up to a subsequence, Σ_k converges as varifolds to an area minimizing minimal torus $\Sigma_0 \in \mathscr{X}_{G\pm}$ with multiplicity one.

Proof. Firstly, $\mathbf{L}(\Pi) > 0$ since $\Pi \subset \mathscr{P}_k$ are k-sweepouts in the sense of Almgren–Pitts. By Theorem 5.3, $\{\Gamma_j\}_{j=1}^J$ satisfies the genus bound (4.1), which implies each Γ_j is either

(a) a $G_\pm$-separating minimal torus in $\mathscr{X}_{G\pm}$ with multiplicity one, i.e. $\pi(\Gamma_j)$ is a minimal Klein bottle in $L(4m, 2m \pm 1)$; or

(b) a connected G-invariant minimal torus not in $\mathscr{X}_{G\pm}$ with multiplicity one, i.e. $\pi(\Gamma_j)$ is a minimal torus in $L(4m, 2m \pm 1)$; or

(c) a G-invariant disjoint union of minimal spheres, i.e. $\pi(\Gamma_j)$ is a minimal sphere in lens spaces $L(4m, 2m \pm 1)$. (Note that there is no embedded RP^2 in $L(4m, 2m \pm 1)$; see Geiges–Thies [16].)

Combining Proposition 5.4 with the fact that every two elements in $\mathscr{X}_{G\pm}$ have nonempty intersections, we know that there is exactly one element of $\{\Gamma_j\}_{j=1}^J$ that is in $\mathscr{X}_{G\pm}$, which gives (i). Then, by (4.1), every other G-component Γ_j, $j \geq 2$, is in case (c). Note that there is no embedded RP^2 in $L(4m, 2m \pm 1)$, and the only nontrivial finite effective free action on S^2 is $\mathbb{Z}_2$ with the quotient homeomorphic to RP^2. Hence Γ_j, $j \geq 2$, has $\#G = 4m$ components with G permuting them.

Suppose $\{\Sigma_k\}_{k\in\mathbb{N}} \subset \mathscr{X}_{G\pm}$ is an area minimizing sequence in $\mathscr{X}_{G\pm}$. It then follows from the G-invariance of Σ_k and [37, Theorem 1] that Σ_k converges (up to a subsequence) as varifolds to a disjoint union of embedded G-connected minimal G-surfaces $\{\Gamma_j\}_{j=1}^J$ with integer multiplicities $\{m_j\}_{j=1}^J$ so that the genus bound (4.1) is also valid. Next, combining the proof of Proposition 5.4 and the above arguments, we know that there is exactly one G-component, say Γ_1, satisfying $\Gamma_1 \in \mathscr{X}_{G\pm}$ and $m_j = 1$. Finally, we notice $J = 1$ since

$$\inf\{\mathcal{H}^2(\Gamma) \mid \Gamma \in \mathscr{X}_{G\pm}\} \leq \mathcal{H}^2(\Gamma_1) \leq \lim_{k\to\infty} \mathcal{H}^2(\Sigma_k). \qquad \square$$

Next, we can use the Lusternik–Schnirelmann theory to show the following result.

Theorem 7.3. *Given any Riemannian metric g_L on $L(4m, 2m \pm 1)$ with $m \geq 1$, let $M, G, G_\pm$ be given by* (7.2)*, and let g_M be the G-invariant pull-back metric on M. Suppose $\mathscr{X}_{G_\pm}$ contains only finitely many elements that are minimal in (M, g_M). Then*

$$0 < 4M_0 < \mathbf{L}(\mathscr{P}_1) < \mathbf{L}(\mathscr{P}_2) < \mathbf{L}(\mathscr{P}_3) \text{ if } m = 1, \quad \text{and} \quad 0 < 4mM_0 < \mathbf{L}(\mathscr{P}_1) \text{ if } m \geq 2,$$

where $4mM_0 = \inf\{\mathcal{H}^2(\Gamma) \mid \Gamma \in \mathscr{X}_{G_\pm}\}$, and $\mathscr{P}_k$ is defined in (7.4)*,* (7.5).

Proof. Using Corollary 7.2 in place of Corollary 6.1 and [6, Proposition 2.3], the proof in Theorem 6.2 can be taken almost verbatim to show the desired results. □

At the end of this subsection, we prove Theorem 1.5.

Proof of Theorem 1.5. Firstly, given any Riemannian metric g_L on $L(4m, 2m \pm 1)$, we have an embedded area minimizing Klein bottle denoted as Σ_0 by the second part of Corollary 7.2. Let $M = S^3$, $G = \mathbb{Z}_{4m}$ and $G_\pm \subset G$ be defined as in (7.2). Endow M with the pull-back metric g_M so that (M, g_M) is locally isometric to $(L(4m, 2m \pm 1), g_L)$.

Case I: *g_L is a metric with positive Ricci curvature.* In this case, g_M is a G-invariant Riemannian metric on $M = S^3$ with positive Ricci curvature. Hence, similar to (6.1), the embedded Frankel property indicates there is no embedded G-invariant union of minimal 2-spheres in (M, g_M), i.e. there is no minimal 2-sphere in $(L(4m, 2m \pm 1), g_L)$.

Without loss of generality, we assume that $(L(4m, 2m \pm 1), g_L)$ contains only finitely many distinct minimal embedded Klein bottles, i.e.

$$(M, g_M) \text{ contains finitely many minimal tori in } \mathscr{X}_{G_\pm}. \tag{7.6}$$

Sub-case I (A): $m = 1$. Combining (7.6) with Theorem 7.3, $\{\mathscr{P}_k\}_{k=1}^3$ in (7.4) satisfies

$$0 < 4\mathcal{H}^2(\Sigma_0) < \mathbf{L}(\mathscr{P}_1) < \mathbf{L}(\mathscr{P}_2) < \mathbf{L}(\mathscr{P}_3). \tag{7.7}$$

For each $k \in \{1, 2, 3\}$, although $\mathscr{P}_k$ may contain many different homotopy classes, we see from (7.6) and Corollary 7.2 (i)–(ii) that the min-max values of the homotopy classes in $\mathscr{P}_k$ must be stabilized to $\mathbf{L}(\mathscr{P}_k)$, and moreover, $\mathbf{L}(\mathscr{P}_k)$ is realized by the area of an embedded G-connected minimal torus $\Gamma_k \in \mathscr{X}_{G_\pm}$ with multiplicity one, i.e. $\mathbf{L}(\mathscr{P}_k) = \mathcal{H}^2(\Gamma_k)$.

Therefore, we have found four distinct embedded minimal Klein bottles in $(L(4, 1), g_L)$.

Sub-case I (B): $m \geq 2$. The proof in Sub-case I (A) would carry over for Σ_0 and $\mathscr{P}_1$ in (7.5) to find two distinct embedded minimal Klein bottles in $(L(4m, 2m \pm 1), g_L)$.

Case II: *g_L is a bumpy metric.* In this case, every embedded G-invariant minimal surface in (M, g_M) is non-degenerate. Without loss of generality, we also assume that (M, g_M) contains finitely many G-invariant minimal tori in $\mathscr{X}_{G_\pm}$ and finitely many G-invariant union of minimal 2-spheres.

Sub-case II (A): $m = 1$. Combining the constructions of $\{\mathscr{P}_k\}_{k=1}^3$ in (7.4) and the above finiteness assumptions with Theorem 7.3, we know that (7.7) is still valid here. In addition, for each $k \in \{1, 2, 3\}$, using the arguments in Sub-case I (A) and Corollary 7.2 (i)–(ii), there are disjoint embedded G-connected minimal G-surfaces $\{\Gamma_{k,j}\}_{j=1}^{J_k}$ and integer multiplicities $\{m_{k,j}\}_{j=1}^{J_k}$ so that

$$\mathbf{L}(\mathscr{P}_k) = \sum_{j=1}^{J_k} m_{k,j}\mathscr{H}^2(\Gamma_{k,j}),$$

where $\Gamma_{k,1} \in \mathscr{X}_{G\pm}$ has multiplicity $m_{k,1} = 1$, and $\Gamma_{k,j}$ with $j \in \{2, \dots, J_k\}$ is a G-invariant disjoint union of $4m$ minimal 2-spheres.

Suppose Theorem 1.5 (i) fails, i.e. the number of distinct embedded minimal Klein bottles in $(L(4,1), g_L)$ is strictly less than 4. Clearly, this assumption and (7.7) indicate the existence of $k_0 \in \{1,2,3\}$ with $J_{k_0} \geq 2$. Hence we have a minimal Klein bottle $K = \pi(\Gamma_{k_0,1})$ and a minimal 2-sphere $S = \pi(\Gamma_{k_0,2})$ in $(L(4,1), g_L)$ with $K \cap S = \emptyset$.

By cutting $L(4,1)$ along K, we obtain a compact manifold $\widetilde{L}$ with a connected boundary $\partial\widetilde{L}$ diffeomorphic to the oriented double cover $\widetilde{K}$ of K. Then, by isotopic area minimizing S within the region in $\widetilde{L}$ enclosed between $\partial\widetilde{L} = \widetilde{K}$ and S, we have a stable (and thus strictly stable) minimal 2-sphere $S_0 \subset \widetilde{L} \setminus \partial\widetilde{L}$ so that $\widetilde{L} \setminus S_0$ has a component N with $N \cap \partial\widetilde{L} = \emptyset$. After applying [48, Proposition 8.8] (using Song's strategy [43]) to N, we can find either

- at least one more minimal sphere in N that is different from S if $S \neq S_0$; or
- at least two more minimal spheres in N if $S = S_0$.

Thus we obtain three minimal 2-spheres and an area minimizing Klein bottle in $(L(4,1), g_L)$.

Sub-case II (B): $m \geq 2$. The proof in Sub-case II (A) would carry over for Σ_0 and $\mathscr{P}_1$ in (7.5) to find either two distinct embedded minimal Klein bottles or one area minimizing Klein bottle together with three embedded minimal spheres in $(L(4m, 2m \pm 1), g_L)$. □

8. Minimal torus in lens spaces

Note that Theorem 1.6 (ii) has been proven by Ketover in [25, Theorem 4.8]. Hence, in this section, we only consider minimal tori in $L(2,1) = \mathrm{RP}^3$ and $L(p,q)$ with $q \notin \{1, p-1\}$. In particular, we always set

$$M := S^3, \quad G := \mathbb{Z}_p = \langle \xi_{p,q} \rangle, \quad G_+ = \mathbb{Z}_{p/2} := \langle \xi_{p,q}^2 \rangle, \quad G_- := G \setminus G_+,$$

where $\xi_{p,q}$ is given by (7.1), and either $(p,q) = (2,1)$ or $q \notin \{1, p-1\}$. In addition, we consider the following spaces:

- $\mathscr{X} := \{\phi(T^2) \mid \phi\colon T^2 \to M/G$ a separating embedding$\}$,
- $\mathscr{Y} := \{\pi(S^1) \mid S^1$ is a G-invariant equator (great circle) in $S^3\}$,
- $\overline{\mathscr{X}} := \mathscr{X} \cup \mathscr{Y}$,

which are endowed with the smooth unoriented topology. Note that $\phi(T^2)$ is separating if $(M/G) \setminus \phi(T^2)$ has two connected components U_1, U_2. Moreover, denote by

$$\iota\colon \overline{\mathscr{X}} \to \mathcal{Z}_2(M/G; \mathbb{Z}_2)$$

the natural inclusion, and by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_2(M/G;\mathbb{Z}_2);\mathbb{Z}_2)$.

8.1. Sweepouts formed by tori. To begin with, we have a family of T^2 in $\mathscr{X}$ by Ketover's construction [25, Proposition 4.2].

Proposition 8.1 (Ketover [25]). *Let $L(p,q)$ denote the lens space endowed with the round metric, $p \geq 2$. Then we have*

(1) RP^3 *admits a family of Clifford tori parameterized by* $\mathbb{RP}^2 \times \mathbb{RP}^2$*;*

(2) $L(p,q)$ *with* $q \notin \{1, p-1\}$ *admits exactly one Clifford torus.*

For the sake of completeness, we adopt the notation as in Section 7 and explain the constructions of sweepouts formed by tori in RP^3 or $L(p,q)$ with $q \notin \{1, p-1\}$.

Firstly, for any $a, b \in \mathbb{S}^2$, recall that $E(b) = \partial\{x \in \mathbb{S}^2 \mid x \cdot b < 0\}$ is an oriented equator and $\tau(a, E(b))$ is an oriented Clifford torus in $\mathbb{S}^3$. Then, for any $t \in [-1,1]$, we define

$$E_t(b) := \partial\{x \in \mathbb{S}^2 \mid x \cdot b := x_1 b_1 + x_2 b_2 + x_3 b_3 < t\}$$

to be an oriented circle parallel to $E(b)$ with $E_1(b) = \{b\}$ and $E_{-1}(b) = \{-b\}$. It is known that $\tau(a, E_t(b))$ is an oriented flat torus in $\mathbb{S}^3$ parallel to the Clifford torus $\tau(a, E_0(b))$ (cf. [25, page 22]), which degenerates to an oriented great circle in $\mathbb{S}^3$ at $t = \pm 1$.

Case A: $(p,q) = (2,1)$ *and* $L(p,q) = \mathrm{RP}^3$. Using the notation in Section 7 and (7.3), $\xi_{2,1}$ is orientation preserving and the oriented (possibly degenerated) torus $\xi_{2,1}(\tau(a, E_t(b)))$ in $\mathbb{S}^3$ is the same as $\tau(a, E_t(b))$ for all $a, b \in \mathbb{S}^2$, $t \in [-1,1]$, which implies the quotient of $\mathrm{spt}(\tau(a, E_t(b)))$ is an element in $\overline{\mathscr{X}}$. Therefore, we have a continuous family of oriented tori in RP^3 parameterized by $\widetilde{\mathscr{G}}\colon [-1,1] \times \mathbb{S}^2 \times \mathbb{S}^2 \to \overline{\mathscr{X}}$ (with oriented smooth topology),

$$\widetilde{\mathscr{G}}(t,a,b) = (\tau(a, E_t(b)))/G \quad \text{for all } (t,a,b) \in [-1,1] \times \mathbb{S}^2 \times \mathbb{S}^2.$$

Note that the oriented torus $\widetilde{\mathscr{G}}(t,a,b) = \widetilde{\mathscr{G}}(t,-a,-b)$ is the oriented torus

$$\widetilde{\mathscr{G}}(-t,-a,b) = \widetilde{\mathscr{G}}(-t,a,-b)$$

with the opposite orientation for $t \in (-1,1)$. Hence $\widetilde{\mathscr{G}}$ induces a continuous map into $(\overline{\mathscr{X}}, \mathscr{Y})$ with unoriented smooth topology,

$$\mathscr{G}\colon (\overline{\mathcal{X}}, \mathcal{Y}) \to (\overline{\mathscr{X}}, \mathscr{Y}), \quad \mathscr{G}([t,a,b]) = \mathrm{spt}(\widetilde{\mathscr{G}}(t,a,b)),$$

where $[t,a,b] \in \overline{\mathcal{X}}$,

$$\overline{\mathcal{X}} := ([-1,1] \times \mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \mathcal{Y} := \partial\overline{\mathcal{X}}, \quad \mathcal{X} := \overline{\mathcal{X}} \setminus \mathcal{Y},$$

and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts on $[-1,1] \times \mathbb{S}^2 \times \mathbb{S}^2$ by $(0,0) = \mathrm{id}$,

$$\begin{aligned} (1,0) \cdot (t,a,b) &= (-t,-a,b), \\ (0,1) \cdot (t,a,b) &= (-t,a,-b), \\ (1,1) \cdot (t,a,b) &= (t,-a,-b). \end{aligned} \tag{8.1}$$

In addition, one notices that the parameter space $\overline{\mathcal{X}}$ retracts onto

$$\mathcal{X}_4 := \{[0,a,b] \in \mathcal{X} \mid a, b \in \mathbb{S}^2\} \cong \mathbb{RP}^2 \times \mathbb{RP}^2,$$

the parameter space of Clifford tori. Then, by Poincaré–Lefschetz duality,

$$H^1(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2) \cong H_4(\overline{\mathcal{X}}; \mathbb{Z}_2) \cong H_4(\mathcal{X}_4; \mathbb{Z}_2) \cong \mathbb{Z}_2 \tag{8.2}$$

has a generator $\lambda \in H^1(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$, and for any fixed $a, b \in \mathbb{S}^2$, the curve $\gamma(t) := [t, a, b] \in \overline{\mathcal{X}}$ satisfies that

$$\iota \circ \mathcal{G} \circ \gamma \colon ([-1,1], \{\pm 1\}) \to (\mathcal{Z}_2(M; \mathbb{Z}_2), \{0\})$$

is a sweepout in the sense of Almgren–Pitts, which implies $\mathcal{G}^*\iota^*(\bar{\lambda}) = \lambda \in H^1(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$.

Lemma 8.2. $\lambda^4 = \mathcal{G}^*\iota^*(\bar{\lambda}^4) \neq 0 \in H^4(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$ *and* $\lambda^5 = 0$.

The proof of Lemma 8.2 is left to Appendix A. Now, we know that

$$\iota \circ \mathcal{G} \colon (\overline{\mathcal{X}}, \mathcal{Y}) \to (\mathcal{Z}_2(M/G; \mathbb{Z}_2), \{0\})$$

is a 4-sweepout in the sense of Almgren–Pitts as $\lambda^4 = (\iota \circ \mathcal{G})^*(\bar{\lambda}^4) \neq 0$. In particular,

$$\alpha = \iota^*(\bar{\lambda}) \in H^1(\overline{\mathscr{X}}, \mathscr{Y}; \mathbb{Z}_2)$$

satisfies $\alpha^4 \neq 0 \in H^4(\overline{\mathscr{X}}, \mathscr{Y}; \mathbb{Z}_2)$. Hence we can define the collection of continuous maps

$$\mathscr{P}_k := \{\Phi \colon (X, Z) \to (\overline{\mathscr{X}}, \mathscr{Y}) \mid \Phi^*(\alpha^k) \neq 0 \in H^k(X, Z; \mathbb{Z}_2)\}, \tag{8.3}$$

where $k \in \{1, 2, 3, 4\}$ and $Z \subset X$ are any two cubical complexes.

Case B: $q \notin \{1, p-1\}$. In this case, we fix $a_0 = b_0 = (1, 0, 0) \in \mathbb{S}^3$. Similarly, by (7.3), the oriented (and possibly degenerated) torus $\xi_{p,q}(\tau(a_0, E_t(b_0)))$ in $\mathbb{S}^3$ is the same as $\tau(a_0, E_t(b_0))$ for all $t \in [-1, 1]$, which implies the quotient of $\mathrm{spt}(\tau(a_0, E_t(b_0)))$ is an element in $\overline{\mathscr{X}}$. Therefore, we have a continuous family of unoriented tori in RP^3 parameterized by

$$\mathcal{G} \colon ([-1,1], \{\pm 1\}) \to (\overline{\mathscr{X}}, \mathscr{Y}), \quad \mathcal{G}(t) := \mathrm{spt}\big(\tau(a_0, E_t(b_0))/G\big) \quad \text{for all } t \in [-1, 1].$$

Note that $\iota \circ \mathcal{G}$ is a sweepout in the sense of Almgren–Pitts. Hence

$$\alpha := \iota^*(\bar{\lambda}) \neq 0 \in H^1(\overline{\mathscr{X}}, \mathscr{Y}; \mathbb{Z}_2),$$

and we can define the collection of continuous maps

$$\mathscr{P}_1 := \{\Phi \colon (X, Z) \to (\overline{\mathscr{X}}, \mathscr{Y}) \mid \Phi^*(\alpha) \neq 0 \in H^1(X, Z; \mathbb{Z}_2)\}, \tag{8.4}$$

where $Z \subset X$ are any two cubical complexes.

8.2. Proof of Theorem 1.6. Since T^2 is orientable, we can apply [48, Theorem B] to obtain the following corollary.

Corollary 8.3. *Let g_L be a Riemannian metric on $L(p,q)$ with positive Ricci curvature, where $(p,q) = (2,1)$ or $q \notin \{1, p-1\}$. Suppose $\mathscr{P}_k$ is given by* (8.3) *or* (8.4)*, and $\Pi \subset \mathscr{P}_k$ is a homotopy class of sweepouts in $\overline{\mathscr{X}}$. Then*

$$\mathbf{L}(\Pi) = \mathcal{H}^2(\Gamma) > 0$$

for a smooth connected embedded minimal torus Γ in $(L(p,q), g_L)$.

Proof. Firstly, we know that $\mathbf{L}(\Pi) > 0$ since Π is a family of sweepouts in the sense of Almgren–Pitts. Then, combining [48, Theorem B] with the embedded Frankel property, there exists a connected smooth embedded minimal surface Γ in $(L(p,q), g_L)$ so that

$$\mathbf{L}(\Pi) = m\mathcal{H}^2(\Gamma) \quad \text{for some integer } m \in \mathbb{Z}_+,$$

and (by the genus bound) Γ is either a minimal

(a) torus with multiplicity one, or

(b) 2-sphere, or

(c) real projective plane with a stable connected double cover (only if $L(p,q) \cong \mathrm{RP}^3$), or

(d) Klein bottle with a stable connected double cover (only if $(p,q) = (4m, 2m \pm 1)$).

Note that the pull-back metric $g_M = \pi^* g_L$ on $M = S^3$ also has positive Ricci curvature. Hence (c), (d) cannot happen. Additionally, if Γ is in case (b), then $\pi^{-1}(\Gamma)$ is a $\#G = p$ union of minimal spheres in S^3, since 2-spheres only have trivial finite covers. This contradicts the embedded Frankel property in (M, g_M). Therefore, Γ has to be in case (a). □

Recall that $\mathbf{L}(\mathscr{P}_k) := \inf_{\Phi \in \mathscr{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathcal{H}^2(\Phi(x))$.

Theorem 8.4. *Let g_L be a Riemannian metric on $L(p,q)$ with positive Ricci curvature, where $(p,q) = (2,1)$ or $q \notin \{1, p-1\}$. Suppose $\mathscr{X}$ contains only finitely many elements that are minimal in $(L(p,q), g_L)$. Then*

$$0 < \mathbf{L}(\mathscr{P}_1) < \mathbf{L}(\mathscr{P}_2) < \mathbf{L}(\mathscr{P}_3) < \mathbf{L}(\mathscr{P}_4)$$

when $(p,q) = (2,1)$, and $0 < \mathbf{L}(\mathscr{P}_1)$ when $q \notin \{1, p-1\}$.

Proof. This result follows from the Lusternik–Schnirelmann theory. One can easily use Corollary 8.3 and [20, Lemma 5.3] to extend the proof of [34, Theorem 6.1] and [20, Theorem 5.2]. □

We mention that, for a general Riemannian metric g_L on $L(p,q)$, the above result is still valid if $(L(p,q), g_L)$ contains only finitely many minimal 2-spheres, minimal real projective planes, minimal Klein bottles and minimal tori.

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. By combining Corollary 8.3 and Theorem 8.4, we find four embedded minimal tori in $L(2,1) = \mathrm{RP}^3$ and one embedded minimal torus in $L(p,q)$ with $q \neq \{1, p-1\}$. It remains to show the second part of (i), namely to find one more embedded minimal torus in RP^3 under the additional assumption of bumpiness.

To begin with, we recall the setup of White's manifold structure theorem [51]. Let M, N be compact smooth Riemannian manifolds such that $\dim N < \dim M$. Given an open subset Γ of C^q Riemannian metrics on M, consider the space $\mathcal{M}$ of ordered pairs $(\gamma, [N])$, where $\gamma \in \Gamma$ and $[N]$ denotes the diffeomorphism classes of minimal embeddings of a Riemannian manifold N into M with respect to γ. It was shown that $\mathcal{M}$ is a separable C^2 Banach manifold and the projection map

$$\Pi \colon \mathcal{M} \to \Gamma, \quad (\gamma, [N]) \mapsto \gamma$$

is smooth Fredholm with Fredholm index 0.

Now, let N be an unoriented torus, $M = \mathrm{RP}^3$, and let Γ be the set of all C^4 Riemannian metrics on RP^3 with positive Ricci curvature. Since Γ is connected [19] and $\Pi\colon \mathcal{M} \to \Gamma$ is proper [11], we can calculate the mapping degree d of Π using [51, Theorem 5.1]. Denote by γ_0 the round metric on RP^3. By Proposition 8.1 (1), the set of all Clifford tori in $(\mathrm{RP}^3, \gamma_0)$ is diffeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$. As a direct corollary of Lawson's conjecture [9, 10], every embedded minimal torus in $(\mathrm{RP}^3, \gamma_0)$ is congruent to the Clifford torus. Hence $\Sigma = \Pi^{-1}(\gamma_0)$ is diffeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$ with $\chi(\Sigma) = 1$. Together with the fact that every Clifford torus in $(\mathrm{RP}^3, \gamma_0)$ has nullity 4 and Morse index 1, we see $d = (-1)^{\mathrm{index}(\Sigma)}\chi(\Sigma) = -1$. For generic metrics γ of positive Ricci curvature on RP^3, we further have

$$\sum_{S \in \Pi^{-1}(\gamma)} (-1)^{\mathrm{index}(S)} = d = -1,$$

which implies that the number of elements in $\Pi^{-1}(\gamma)$ is odd. Given that we already found four embedded minimal tori in RP^3, there must be at least a fifth one. This completes the proof. □

A. Proof of Lemma 8.2

Proof of Lemma 8.2. Using the notation in Section 8.1, Case A, for any k-dimensional closed submanifold N in $\overline{\mathcal{X}}$, we denote by $[N] \in H_k(\overline{\mathcal{X}}; \mathbb{Z}_2)$ the images of the fundamental classes of N under the inclusion into $\overline{\mathcal{X}}$, and denote by $[N]^* \in H^{5-k}(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$ its Poincaré–Lefschetz dual. It then follows from the Poincaré–Lefschetz duality (8.2) that the generator $\lambda \in H^1(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$ coincides with $[\mathcal{X}_4]^*$, where $\mathcal{X}_4 := \{[0, a, b] \in \overline{\mathcal{X}}\} \cong \mathbb{RP}^2 \times \mathbb{RP}^2$ is the parameter space of Clifford tori. We now use the transversal intersection product to visualize the cup product λ^k, $k = 2, 3, 4, 5$.

Firstly, after regarding $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{S}^2$ as vectors in $\mathbb{R}^3$, we can define

$$\mathcal{X}_4' := \{[a \cdot b, a, b] \in \overline{\mathcal{X}} \mid a, b \in \mathbb{S}^2\},$$
$$\mathcal{X}_3 := \mathcal{X}_4 \cap \mathcal{X}_4' = \{[0, a, b] \in \overline{\mathcal{X}} \mid a, b \in \mathbb{S}^2 \text{ with } a \cdot b = 0\},$$

where $a \cdot b = \sum_{i=1}^3 a_i b_i$ is the inner product in $\mathbb{R}^3$. Since $\{(a \cdot b, a, b) \mid a, b \in \mathbb{S}^2\}$ is a 4-submanifold of $[-1, 1] \times \mathbb{S}^2 \times \mathbb{S}^2$ invariant under the $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$-action in (8.1), we know $\mathcal{X}_4'$ is a closed 4-submanifold of $\overline{\mathcal{X}}$ and is homotopic to $\mathcal{X}_4$ through $(s, [t, a, b]) \mapsto [st, a, b]$, $s \in [0, 1]$. In addition, for any $[0, a, b] \in \mathcal{X}_3 = \mathcal{X}_4 \cap \mathcal{X}_4'$, we have $a \perp b$, and thus there exists a curve $\sigma \subset \mathbb{S}^2$ with $\sigma(0) = a$ and $\sigma'(0) = b$. Hence $\widetilde{\sigma}(s) = [\sigma(s) \cdot b, \sigma(s), b]$ is a curve in $\mathcal{X}_4'$ so that $\widetilde{\sigma}'(0)$ has a component $(\sigma \cdot b)'(0) \cdot \frac{\partial}{\partial t} = \frac{\partial}{\partial t}$, which implies $\mathcal{X}_4'$ is transversal to $\mathcal{X}_4$ along their intersection $\mathcal{X}_3$. Therefore, by [17, Theorem 3.6] (see also [7, VI, Theorem 11.9]), we have $\lambda^2 = [\mathcal{X}_4]^* \smile [\mathcal{X}_4']^* = [\mathcal{X}_4 \cap \mathcal{X}_4']^* = [\mathcal{X}_3]^* \in H^2(\overline{\mathcal{X}}, \mathcal{Y}; \mathbb{Z}_2)$.

Next, set $e_1 := (1, 0, 0)$, $e_2 := (0, 1, 0)$, $e_3 := (0, 0, 1)$. For any $a, b \in \mathbb{S}^2$ with $a \cdot b = 0$, we can take the cross product vector $v = (v_1, v_2, v_3) := a \times b$ in $\mathbb{R}^3$, i.e.

$$v_1(a, b) := a_1 b_3 - a_3 b_2, \quad v_2(a, b) := a_3 b_1 - a_1 b_3, \quad v_3(a, b) = a_1 b_2 - a_2 b_1.$$

Then $\{a, b, v\}$ forms an orthonormal basis of $\mathbb{R}^3$. Define then

$$\mathcal{X}_3' := \{[v_1(a, b), a, b] \in \overline{\mathcal{X}} \mid a, b \in \mathbb{S}^2 \text{ with } a \cdot b = 0\},$$
$$\mathcal{X}_2 := \mathcal{X}_4 \cap \mathcal{X}_3' = \{[0, a, b] \in \overline{\mathcal{X}} \mid a, b \in \mathbb{S}^2 \text{ with } a \cdot b = 0,\ v_1(a, b) = 0\}.$$

Since $\{(v_1(a,b),a,b) \mid a,b \in \mathbb{S}^2,\ a\cdot b = 0\}$ is a closed 3-submanifold of $[-1,1]\times\mathbb{S}^2\times\mathbb{S}^2$ invariant under the $(\mathbb{Z}_2\oplus\mathbb{Z}_2)$-action in (8.1), it is clear that $\mathcal{X}_3'$ is a closed 3-submanifold of $\overline{\mathcal{X}}$ and is homotopic to $\mathcal{X}_3$. Additionally, for any $a,b\in\mathbb{S}^2$ with $a\cdot b=0$ and $v_1(a,b)=0$, we can rotate a,b and $v=a\times b$ together around the $(v\times e_1)$-axis to obtain curves $a(\theta)$, $b(\theta)$, $v(\theta)=a(\theta)\times b(\theta)\subset\mathbb{S}^2$ so that

$$a(0)=a,\quad b(0)=b,\quad v(0)=v,\quad\text{and}\quad v_1(\theta)=v_1(a(\theta),b(\theta))=\sin(\theta).$$

Hence it follows that $\widetilde{\sigma}(\theta)=[v_1(\theta),a(\theta),b(\theta)]$ is a curve in $\mathcal{X}_3'$ so that $\widetilde{\sigma}'(0)$ has a component $v_1'(0)\cdot\frac{\partial}{\partial t}=\frac{\partial}{\partial t}$, which implies $\mathcal{X}_3'$ is transversal to $\mathcal{X}_4$ along their intersection $\mathcal{X}_2$. Therefore, we have

$$\lambda^3=[\mathcal{X}_4]^*\smile[\mathcal{X}_3']^*=[\mathcal{X}_4\cap\mathcal{X}_3']^*=[\mathcal{X}_2]^*\in H^3(\overline{\mathcal{X}},\mathcal{Y};\mathbb{Z}_2).$$

Moreover, define

$$\mathcal{X}_2':=\{[v_2(a,b),a,b]\in\overline{\mathcal{X}}\mid a,b\in\mathbb{S}^2\text{ with }a\cdot b=0,\ v_1(a,b)=0\},$$
$$\mathcal{X}_1:=\mathcal{X}_4\cap\mathcal{X}_2'=\{[0,a,b]\in\overline{\mathcal{X}}\mid a,b\in\mathbb{S}^2\text{ with }a\cdot b=0,\ v_1(a,b)=v_2(a,b)=0\}.$$

Similarly, one notices that $\mathcal{X}_2'$ is a closed 2-submanifold of $\overline{\mathcal{X}}$ and is homotopic to $\mathcal{X}_2$. Given any $a,b\in\mathbb{S}^2$ with $a\cdot b=0$ and $v_1(a,b)=v_2(a,b)=0$, one can use the rotation around the e_1-axis ($\pm e_1=v\times e_2$) to show that $\mathcal{X}_2'$ is transversal to $\mathcal{X}_4$ along their intersection $\mathcal{X}_1$. Hence $\lambda^4=[\mathcal{X}_4]^*\smile[\mathcal{X}_2']^*=[\mathcal{X}_4\cap\mathcal{X}_2']^*=[\mathcal{X}_1]^*\in H^4(\overline{\mathcal{X}},\mathcal{Y};\mathbb{Z}_2)$.

Finally, we notice that $\mathcal{X}_1$ is a nontrivial closed curve in $\mathcal{X}_4\subset\mathcal{X}$ given by

$$\theta\mapsto[0,(\cos(\theta/2),\sin(\theta/2),0),(-\sin(\theta/2),\cos(\theta/2),0)]\quad\text{for all }\theta\in[0,2\pi].$$

Hence $[\mathcal{X}_1]\neq 0\in H_1(\mathcal{X}_4;\mathbb{Z}_2)\cong H_1(\overline{\mathcal{X}};\mathbb{Z}_2)$, and thus $\lambda^4=[\mathcal{X}_1]^*\neq 0$ in $H^4(\overline{\mathcal{X}},\mathcal{Y};\mathbb{Z}_2)$. Moreover, one verifies that $\mathcal{X}_1':=\{[a_1b_1,a,b]\mid[0,a,b]\in\mathcal{X}_1\}$ is homotopic to $\mathcal{X}_1$ and transversal to $\mathcal{X}_4$, and $\mathcal{X}_1'\cap\mathcal{X}_4$ has two points, which implies

$$\lambda^5=[\mathcal{X}_1'\cap\mathcal{X}_4]^*=0\in H^5(\overline{\mathcal{X}},\mathcal{Y};\mathbb{Z}_2).\qquad\square$$

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Xingzhe Li, Department of Mathematics, Cornell University, Ithaca, NY 14850, USA
e-mail: xl833@cornell.edu

Tongrui Wang, School of Mathematical Sciences, Shanghai Jiao Tong University,
800 Dongchuan RD, Minhang District, Shanghai, 200240, P. R. China
https://orcid.org/0000-0001-7608-2186
e-mail: wangtongrui@sjtu.edu.cn

Xuan Yao, Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
e-mail: xy1216@princeton.edu

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