J. reine angew. Math. **804** (2023), 105–154
DOI 10.1515/crelle-2023-0065

Journal für die reine und angewandte Mathematik

The integrality conjecture and the cohomology of preprojective stacks

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Abstract. We study the Borel–Moore homology of stacks of representations of preprojective algebras Π_Q, via the study of the DT theory of the undeformed 3-Calabi–Yau completion $\Pi_Q[x]$. Via a result on the supports of the BPS sheaves for $\Pi_Q[x]$-mod, we prove purity of the BPS cohomology for the stack of $\Pi_Q[x]$-modules and define BPS sheaves for stacks of Π_Q-modules. These are mixed Hodge modules on the coarse moduli space of Π_Q-modules that control the Borel–Moore homology and geometric representation theory associated to these stacks. We show that the hypercohomology of these objects is pure and thus that the Borel–Moore homology of stacks of Π_Q-modules is also pure. We transport the cohomological wall-crossing and integrality theorems from DT theory to the category of Π_Q-modules. We use our results to prove positivity of a number of "restricted" Kac polynomials, determine the critical cohomology of $\mathrm{Hilb}_n(\mathbb{A}^3)$, and the Borel–Moore homology of genus one character stacks, as well as providing various applications to the cohomological Hall algebras associated to Borel–Moore homology of stacks of modules over preprojective algebras, including the PBW theorem, and torsion-freeness.

1. Introduction

1.1. Background. This paper concerns the Borel–Moore homology of stacks of representations of preprojective algebras Π_Q, which play a prominent role in many branches of mathematics, and which we study through the prism of cohomological Donaldson–Thomas (DT) theory and BPS cohomology. The Borel–Moore homology of stacks of finite-dimensional Π_Q-modules occurs as the underlying vector space of the cohomological Hall algebra containing all raising operators for the cohomology of Nakajima quiver varieties [33, 34], which themselves can be presented as certain stacks of semistable representations of preprojective algebras. More generally, stacks of representations of preprojective algebras model the local geometry of complex 2-Calabi–Yau categories possessing good moduli spaces [9], for example

During the writing of this paper, I was a postdoctoral researcher at EPFL, supported by the Advanced Grant "Arithmetic and physics of Higgs moduli spaces" No. 320593 of the European Research Council. During redrafting of the paper, I was supported by the starter grant "Categorified Donaldson–Thomas theory" No. 759967 of the European Research Council. I was also supported by a Royal Society university research fellowship.

coherent sheaves on K3 and abelian surfaces, Higgs bundles on smooth projective curves, local systems on Riemann surfaces, and moduli of semistable objects in Kuznetsov components.

Via dimensional reduction, we study the Borel–Moore homology of the stack of Π_Q-modules by relating it to the BPS sheaves for the stack of objects in the 3-Calabi–Yau completion $\mathscr{C}_{\Pi_Q}$ (as defined by Keller [25]) of the category of Π_Q-modules. This paper is devoted to understanding the BPS sheaves (as defined in [10]) of the 3CY categories $\mathscr{C}_{\Pi_Q}$ formed this way. By studying these BPS sheaves and the associated BPS cohomology, we prove a number of theorems regarding the Borel–Moore homology of stacks of Π_Q-representations, Nakajima quiver varieties, stacks of coherent sheaves on surfaces, as well as vanishing cycle cohomology of $\mathrm{Hilb}_n(\mathbb{A}^3)$, and vanishing cycle cohomology of stacks of objects in $\mathscr{C}_{\Pi_Q}$.

1.2. Purity. In DT theory, as well as many of the other subjects this paper touches on, we are typically interested in *motivic* invariants. See e.g. [22, 27] and references therein for extensive background on motivic DT theory. This means that we are interested in invariants $\tilde{\chi}$ of objects in a triangulated category $\mathscr{D}$ that factor through the Grothendieck group of $\mathscr{D}$; if $V' \to V \to V''$ is a distinguished triangle in $\mathscr{D}$, then we require that $\tilde{\chi}(V) = \tilde{\chi}(V') + \tilde{\chi}(V'')$. Alternatively, by "motivic", people mean invariants of varieties X such that if $U \subset X$ is open, with complement Z, then the *cut and paste* relation $\tilde{\chi}(X) = \tilde{\chi}(U) + \tilde{\chi}(Z)$ holds. The link between the two meanings is provided by the distinguished triangle

$$\mathrm{H}_c(U,\mathbb{Q}) \to \mathrm{H}_c(X,\mathbb{Q}) \to \mathrm{H}_c(Z,\mathbb{Q})$$

so that a motivic invariant in the first sense induces one in the second sense.

A very basic example of a motivic invariant is the *Euler characteristic* of a complex of vector spaces $\chi(V) = \sum_{i\in\mathbb{Z}}(-1)^i \dim(V^i)$. A basic example of a *non*-motivic invariant is the Poincaré polynomial $P(V,q) = \sum_{i\in\mathbb{Z}} \dim(V^i)q^i$; since the connecting morphisms in a long exact sequence of vector spaces may be nonzero, the Poincaré polynomial may not satisfy the cut and paste relation.

Recall that a mixed Hodge structure on a rational vector space V is the data of an ascending weight filtration $W_\bullet V$, along with a descending Hodge filtration $F^\bullet V_{\mathbb{C}}$ of the complexification, such that the Hodge filtration induces a weight n Hodge structure on the nth piece $\mathrm{Gr}^W_n(V)$ of the associated graded object with respect to the weight filtration. Given $\mathscr{L}$, a cohomologically graded mixed Hodge structure, one defines its Hodge series, E series, and weight series, respectively, by

$$\begin{aligned}\mathtt{h}(\mathscr{L},x,y,z) &= \sum_{a,b,c\in\mathbb{Z}} \dim(\mathrm{Gr}^b_F(\mathrm{Gr}^W_{b+c}(\mathrm{H}^a(\mathscr{L}))))x^b y^c z^a,\\ \mathrm{E}(\mathscr{L},x,y) &= \mathtt{h}(\mathscr{L},x,y,-1),\\ \chi_{\mathrm{wt}}(\mathscr{L},q^{1/2}) &= \mathrm{E}(\mathscr{L},q^{1/2},q^{1/2}).\end{aligned}$$

Since both the E series and weight series involve an alternating sum over cohomological degrees, they are motivic invariants.

We say that a cohomologically graded mixed Hodge structure $\mathscr{L}$ is *pure* if its ath cohomologically graded piece is pure of weight a, i.e. if $\mathrm{Gr}^W_b\,\mathrm{H}^a(\mathscr{L}) = 0$ for $b \neq a$. Our interest in pure mixed Hodge structures comes from the fact that if $\mathscr{L}$ is pure, then $P(\mathscr{L},q) = \chi_{\mathrm{wt}}(\mathscr{L},q)$. Moreover, when the Borel–Moore homology of a stack is pure, we have a much better chance of being able to calculate it, as we will demonstrate in this paper.

1.3. The purity theorem. Let Q be a quiver with set of vertices Q_0 and arrows Q_1. The quiver $\overline{Q}$, which is the double of Q, is obtained by adding an arrow a^* for every arrow of a, with the reverse orientation. Then the preprojective algebra is defined as the quotient of the free path algebra of $\overline{Q}$,

$$\Pi_Q := \mathbb{C}\overline{Q}/\Big\langle \sum_{a\in Q_1} [a,a^*]\Big\rangle. \tag{1.1}$$

We define $\mathbb{N} := \mathbb{Z}_{\geq 0}$. Let $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector for $\overline{Q}$. Define

$$X(\overline{Q})_{\mathbf{d}} = \prod_{a \text{ an arrow of } \overline{Q}} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}_{\mathrm{source}(a)}}, \mathbb{C}^{\mathbf{d}_{\mathrm{target}(a)}}).$$

This space is symplectic, via the natural isomorphism $X(\overline{Q})_{\mathbf{d}} \cong \mathrm{T}^*(X(Q)_{\mathbf{d}})$. This symplectic manifold carries an action of the gauge group $\mathrm{GL}_{\mathbf{d}} := \prod_{i\in Q_0} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C})$, with moment map

$$\mu_{Q,\mathbf{d}}\colon X(\overline{Q})_{\mathbf{d}} \to \mathfrak{gl}_{\mathbf{d}} := \prod_{i\in Q_0} \mathfrak{gl}_{\mathbf{d}_i}(\mathbb{C}), \qquad \rho \mapsto \sum_{a\in Q_1} [\rho(a), \rho(a^*)].$$

We identify $\mathfrak{gl}_{\mathbf{d}_i}(\mathbb{C})$ with the dual vector space $\mathfrak{gl}_{\mathbf{d}_i}(\mathbb{C})^\vee$ via the trace pairing. The stack $\mathfrak{M}(\Pi_Q)_{\mathbf{d}}$ of Π_Q-representations with dimension vector $\mathbf{d}$ is isomorphic to the stack-theoretic quotient $\mu_{Q,\mathbf{d}}^{-1}(0)/\mathrm{GL}_{\mathbf{d}}$. Our first main result is the following.

Theorem A ([7, Conjecture 3.1]). *Fix a quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then the mixed Hodge structure on $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) := \mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})^\vee$ is pure, of Tate type.*

We prove a more general version of Theorem A, concerning Borel–Moore homology of stacks of semistable Π_Q-modules: see Section 6 and Theorem 6.4.

In Theorem A, the symbol $\vee$ denotes the dual in the category of cohomologically graded mixed Hodge structures. Purity means that Deligne's mixed Hodge structure on each cohomologically graded piece $\mathrm{H}^n_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$ is pure of weight n, and the statement that a cohomologically graded mixed Hodge structure $\mathcal{L}$ is of Tate type is the statement that we can write $\mathcal{L} = \bigoplus_{m,n\in\mathbb{Z}} (\mathbb{L}^m[n])^{\oplus a_{m,n}}$, for some set of numbers $a_{m,n} \in \mathbb{N}$, with $\mathbb{L} := \mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$ given the usual weight 2 pure Hodge structure, concentrated in cohomological degree 2. Purity is the further statement that $a_{m,n} = 0$ for $n \neq 0$.

Theorem A concerns compactly supported cohomology. Since $\mu_{Q,\mathbf{d}}^{-1}(0)$ is a cone, and hence homotopic to a point, there is an isomorphism $\mathrm{H}(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) \cong \mathrm{H}(\mathrm{BGL}_{\mathbf{d}}, \mathbb{Q})$ in usual singular cohomology, and it is known this cohomology is pure [11]. On the other hand, compactly supported cohomology is not preserved by homotopy equivalence, and the highly singular nature of $\mu_{Q,\mathbf{d}}^{-1}(0)/\mathrm{GL}_{\mathbf{d}}$ means that its compactly supported cohomology is a great deal more complicated than its cohomology. In fact, purity requires an essentially new type of argument, requiring the full force of cohomological DT theory. In particular, outside of finite type Q, there is no way known (to date) of proving this purity statement without invoking the cohomological integrality theorem for the DT theory of quivers with potential, along with dimensional reduction.

1.3.1. Okounkov's conjecture. Theorem A is a singular stack-theoretic cousin of the result that the cohomology of Nakajima quiver varieties is pure, with Hodge polynomial expressible as a polynomial in xyz^2 (this can be obtained by combining the proof of [18, Theorem 1] with [20, Theorem 6.1.2 (3)]). In fact, we recover this result (Corollary 6.8). The purity of Nakajima quiver varieties provides one of the main motivations for the purity statement in Theorem A.

In a little more detail, it is conjectured that the cohomological Hall algebra $\mathcal{A}_{\Pi_Q}$ obtained by taking the direct sum of $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$ across all dimension vectors $\mathbf{d}$ is isomorphic to the positive half of the Yangian $\mathcal{Y}_{\mathtt{MO},Q}$ constructed by Maulik and Okounkov in [29]. This in turn would imply that the graded dimensions of $\mathfrak{g}_{\mathtt{MO},Q}$ are given by Kac polynomials, as conjectured by Okounkov. Since the algebra $\mathcal{Y}_{\mathtt{MO},Q}$ is defined as a subalgebra of the endomorphism algebra of the cohomology of Nakajima quiver varieties, the purity of $\mathcal{Y}_{\mathtt{MO},Q}$ follows from purity for these quiver varieties. Our purity theorem provides evidence towards the conjecture that $\mathcal{A}_{\Pi_Q} \cong \mathcal{Y}^+_{\mathtt{MO},Q}$.

1.4. From DT theory to symplectic geometry. Consider the following general setup, of which our situation with $X(\overline{Q})_{\mathbf{d}}$ being acted on by $\mathrm{GL}_{\mathbf{d}}$ is a special case. Let X be a complex symplectic manifold, with the affine algebraic group G acting on it via a Hamiltonian action, with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}^*$. Then define the function

$$\overline{g}\colon X \times \mathfrak{g} \to \mathbb{C}, \quad (x, \zeta) \mapsto \mu(x)(\zeta). \tag{1.2}$$

This function is G-invariant and so defines a function $g\colon (X \times \mathfrak{g})/G \to \mathbb{C}$ on the stack-theoretic quotient. Via dimensional reduction [5, Theorem A.1], there is a natural isomorphism in compactly supported cohomology $\mathrm{H}_c(\mu^{-1}(0)/G, \mathbb{Q}) \otimes \mathbb{L}^{\dim(\mathfrak{g})} \cong \mathrm{H}_c((X \times \mathfrak{g})/G, \phi_g\mathbb{Q})$, where $\phi_g\mathbb{Q}$ is the mixed Hodge module complex of vanishing cycles for g. This explains the appearance of vanishing cycles in what follows.

Note that $\phi_g\mathbb{Q}$ is supported on the critical locus of g. A guiding principle for DT theory (e.g. as expressed in [45]) is that a given moduli stack $\mathfrak{N}$ of coherent sheaves on a Calabi–Yau 3-fold can be locally expressed as the critical locus of a function g on some smooth ambient stack $\mathfrak{M}$. DT invariants are then defined by taking invariants, factoring through the Grothendieck group of mixed Hodge structures, of

$$\mathrm{H}_c(\mathfrak{M}, \phi_g\mathbb{Q}) = \mathrm{H}_c(\mathrm{crit}(g), \phi_g\mathbb{Q}) = \mathrm{H}_c(\mathfrak{N}, \phi_g\mathbb{Q}).$$

The link between DT theory and symplectic geometry is completed by the observation of [14, Section 4.2] (see also [32]) that, associated to any quiver Q, there is a tripled quiver with potential $(\widetilde{Q}, \widetilde{W})$ such that $(X(\overline{Q})_{\mathbf{d}} \times \mathfrak{gl}_{\mathbf{d}})/\mathrm{GL}_{\mathbf{d}}$ is identified with the smooth stack of $\mathbf{d}$-dimensional representations of $\mathbb{C}\widetilde{Q}$, and the critical locus of the function $\mathfrak{Tr}(\widetilde{W})$ (which is the function g from (1.2)) is exactly the substack of representations belonging to the category of representations of the Jacobi algebra[1] $\mathrm{Jac}(\widetilde{Q}, \widetilde{W})$ associated to the pair $(\widetilde{Q}, \widetilde{W})$.

Putting all of this together, the cohomological DT theory of $\mathrm{Jac}(\widetilde{Q}, \widetilde{W})$ gives us a tool for understanding the compactly supported cohomology of $\mathfrak{M}(\Pi_Q)$, i.e. there is an isomorphism of cohomologically graded mixed Hodge structures

$$\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{\dim(\mathrm{GL}_{\mathbf{d}})} \cong \mathrm{H}_c\big(\mathfrak{M}(\mathrm{Jac}(\widetilde{Q}, \widetilde{W}))_{\mathbf{d}}, \phi_{\mathfrak{Tr}(\widetilde{W})}\mathbb{Q}\big).$$

We use cohomological DT theory to prove powerful theorems regarding the right-hand side, and deduce results regarding the left-hand side.

[1] The definition of $\mathrm{Jac}(\widetilde{Q}, \widetilde{W})$ is recalled in Section 2.1.

1.5. BPS sheaves and their supports. We prove Theorem A via an analysis of BPS sheaves. These were introduced in [10], in the course of the proof of the relative cohomological integrality/PBW theorem for the critical cohomological Hall algebras introduced by Kontsevich and Soibelman [28]. This theorem states that, for a symmetric quiver Q' with potential W' and stability condition ζ, the direct image of the mixed Hodge module of vanishing cycles for the function $\mathrm{Tr}(W')$ along the morphism $\mathtt{JH}$ from the moduli stack of ζ-semistable $\mathbb{C}Q'$-modules to the coarse moduli space is obtained by taking the free symmetric algebra generated by an explicitly defined mixed Hodge module $\mathcal{BPS}^{\zeta}_{Q',W'}$, called the *BPS sheaf*, tensored with a half Tate twist of $\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})$. The *BPS cohomology* $\mathrm{BPS}^{\zeta}_{Q',W'}$ is defined to be the hypercohomology of this sheaf.

Although the direct image of the mixed Hodge module of vanishing cycles along $\mathtt{JH}$ is concentrated in infinitely many cohomological degrees, this BPS sheaf is a genuine mixed Hodge module, i.e. its underlying complex of constructible sheaves is a perverse sheaf. It follows that, for every $\mathbf{d}\in\mathbb{N}^{Q_0}$, the BPS cohomology $\mathrm{BPS}^{\zeta}_{Q',W',\mathbf{d}}$ lives in bounded degrees.

Unless the pair Q', W' is quite special, it is difficult to actually determine $\mathcal{BPS}^{\zeta}_{Q',W'}$. In this paper, we show that, for the quiver $\widetilde{Q}$ with potential $\widetilde{W}$ appearing in the previous section, the situation is much better. A key role is played by a support lemma, Lemma 4.1, which imposes strong restrictions on the support of the BPS sheaves for $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$ for Q any quiver.

Lemma 1.1 (Lemma 4.1). *Let x be a point in $\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}$ corresponding to a $\mathbb{C}\widetilde{Q}$-module ρ, and let x lie in the support of $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$. Let Λ be the set of generalised eigenvalues of the operators $\rho(\omega_i)$, with i the vertices of Q. Then Λ contains only one element.*

This is a crucial lemma on the way to proving purity of BPS cohomology for $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$. In combination with this purity result, the lemma also enables us to provide some of the first nontrivial calculations of BPS sheaves; see in particular Section 5, lifting the work of Behrend, Bryan and Szendrői on motivic degree zero invariants to the level of BPS sheaves. This lemma is also one of the crucial ingredients in proving the purity of the BPS sheaves $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W}}$ themselves, and the definition of the "less perverse filtration": see [8, 9] for developments in this direction.

1.6. 2d BPS sheaves. Aside from purity of BPS cohomology, one of the main applications of the support lemma is that it enables us to define *2d BPS sheaves*.

Theorem B. *Let $m\colon \mathbb{A}^1\times\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}\to\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}$ be the morphism extending a Π_Q-module to a $\mathbb{C}\widetilde{Q}$-module by letting each of the extra loops ω_i act via scalar multiplication by $z\in\mathbb{A}^1$. Then there is a Verdier self-dual mixed Hodge module $\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}$ on $\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}$, which we call the* 2d BPS sheaf*, such that*

$$\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}\cong m_*(\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})\boxtimes\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}).$$

The pure intersection complex $\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})$ is defined in Section 3.1. The 2d BPS sheaves enjoy a number of properties.

(i) They categorify the Kac polynomials; we elaborate upon this in Section 8.

(ii) They are Verdier self-dual (see Section 4.2), which we expect to have a role in producing geometric doubles of BPS Lie algebras.

(iii) Their hypercohomology carries a Lie algebra structure, the *BPS Lie algebra* $\mathfrak{g}_{\Pi_Q}$.

(iv) They are *pure* as mixed Hodge modules, enabling us to relate generators of $\mathfrak{g}_{\Pi_Q}$ to intersection cohomology.

These last two properties are explained and explored in the paper [8], which is devoted to the further study of 2d BPS sheaves.

1.7. Serre subcategories. A Serre subcategory $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod is a full subcategory such that, for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathbb{C}\overline{Q}$-modules, M is in $\mathcal{S}$ if and only if M' and M'' are. Note that a module M is in $\mathcal{S}$ if and only if all of the subquotients in its Jordan–Hölder filtration are in $\mathcal{S}$, or equivalently if its semisimplification is in $\mathcal{S}$. So restricting attention to $\mathfrak{M}(\mathbb{C}\overline{Q})^{\mathcal{S}}$, which is defined to be the substack of $\mathbb{C}\overline{Q}$-modules belonging to $\mathcal{S}$, is the same as restricting to the preimage of a particular subspace under the semisimplification map from the stack of $\mathbb{C}\overline{Q}$-modules to the coarse moduli space $\mathcal{M}(\overline{Q})$.

Because many of our results can be stated in the category of mixed Hodge modules[2)] on $\mathcal{M}(\overline{Q})$, we can prove results on the Borel–Moore homology of $\mathfrak{M}(\Pi_Q)^{\mathcal{S}}$ via restriction functors and base change. Working with the BPS sheaf $\mathcal{BPS}_{\Pi_Q}$, as opposed to its hypercohomology, enables us to calculate the compactly supported cohomology of substacks of $\mathfrak{M}(\Pi_Q)$ corresponding to Serre subcategories, leading to e.g. applications for character stacks.

1.8. Structural results. We prove two general structural results (Theorems C and D) regarding the compactly supported cohomology of stacks $\mathcal{M}(\Pi_Q)^{\mathcal{S}}$ for arbitrary finite quiver Q and Serre subcategory $\mathcal{S}$. The first is a kind of cohomological wall-crossing isomorphism.

Theorem C. *Let Q be a quiver, let $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod be a Serre subcategory, let $\zeta \in \mathbb{H}_+^{Q_0}$ be a stability condition, and let ϱ be the slope function defined with respect to ζ. Then there is an isomorphism of $\mathbb{N}^{Q_0}$-graded mixed Hodge structures*

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \cong \bigotimes_{\theta\in(-\infty,\infty)} \Big(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0} \,\big|\, \substack{\mathbf{d}=0 \text{ or} \\ \varrho(\mathbf{d})=\theta}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \Big),$$

where

$$(\mathbf{d}',\mathbf{d}'') := \sum_{i \text{ a vertex of } Q} \mathbf{d}'_i\mathbf{d}''_i - \sum_{a \text{ an arrow of } Q} \mathbf{d}'_{\mathrm{source}(a)}\mathbf{d}''_{\mathrm{target}(a)},$$

and $\mathfrak{M}(\Pi_Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}$ is the stack of $\mathbf{d}$-dimensional ζ-semistable Π_Q-modules in $\mathcal{S}$.

Taking the Hodge series of both sides of this isomorphism yields the equality

$$\begin{aligned} (1.3)\quad &\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathtt{h}\big(\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}), x, y, z\big)(xyz^2)^{(\mathbf{d},\mathbf{d})} t^{\mathbf{d}} \\ &= \prod_{\theta\in(-\infty,\infty)} \Big(1 + \sum_{\varrho(\mathbf{d})=\theta} \mathtt{h}\big(\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}, \mathbb{Q}), x, y, z\big)(xyz^2)^{(\mathbf{d},\mathbf{d})} t^{\mathbf{d}}\Big) \end{aligned}$$

regardless of whether the compactly supported cohomology of $\mathfrak{M}(\Pi_Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}$ is pure. We explain how a specialisation of a special case of equation (1.3) yields Hausel's formula for the Betti polynomials of Nakajima quiver varieties [18] in Section 7.3.

2) In cohomological DT theory, this is what is meant by the "relative" in the relative integrality conjecture.

1.8.1. PBW/integrality isomorphism. Fix a quiver Q, a stability condition $\zeta \in \mathbb{H}_+^{Q_0}$, a slope $\theta \in (-\infty, \infty)$, and a Serre subcategory $\mathcal{S}$ of the category of $\mathbb{C}\overline{Q}$-modules. We write

$$\mathcal{A}^{\mathcal{S},\zeta}_{\Pi_Q,\theta} := \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0} \,\middle|\, \substack{\mathbf{d}=0 \text{ or} \\ \varrho(\mathbf{d})=\theta}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{-(\mathbf{d},\mathbf{d})}.$$

This graded mixed Hodge module carries a Hall algebra structure; see Section 9.1 for details.

Theorem D. *Let ϱ be the slope function defined with respect to a stability condition $\zeta \in \mathbb{H}_+^{Q_0}$, let $\theta \in (-\infty, \infty)$ be a slope. Define the 2d BPS sheaf $\mathcal{BPS}^{\zeta}_{\Pi_Q,\theta}$ as in Theorem* B *and the* BPS cohomology *to be the mixed Hodge structure*

$$\mathrm{BPS}^{\mathcal{S},\zeta}_{\Pi_Q,\theta} := \bigoplus_{\substack{0\neq\mathbf{d}\in\mathbb{N}^{Q_0} \\ \varrho(\mathbf{d})=\theta}} \mathrm{H}_c(\mathcal{M}(\overline{Q})^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}}, \mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}})^{\vee}.$$

Then there is an isomorphism

$$\mathrm{JH}^{\zeta}_{\theta,!}\Big(\bigoplus_{\mathbf{d}\in\Lambda^{\zeta}_{\theta}} \mathbb{Q}_{\mathfrak{M}(\Pi_Q)_{\mathbf{d}}} \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})}\Big) \cong \mathrm{Sym}_{\boxtimes_{\oplus}}(\mathcal{BPS}^{\zeta}_{\Pi_Q,\theta} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})^{\vee}). \tag{1.4}$$

Moreover, there is a PBW isomorphism

$$\mathrm{Sym}(\mathrm{BPS}^{\mathcal{S},\zeta}_{\Pi_Q,\theta} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})) \xrightarrow{\cong} \mathcal{A}^{\mathcal{S},\zeta}_{\Pi_Q,\theta}. \tag{1.5}$$

Since $\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}$ is Verdier self-dual by Theorem B, $\mathrm{BPS}^{\mathcal{S},\zeta}_{\Pi_Q,\mathbf{d}}$ is the hypercohomology of the !-restriction of the BPS sheaf on the coarse moduli space of ζ-semistable $\mathbf{d}$-dimensional $\mathbb{C}\widetilde{Q}$-modules to the subspace of points representing modules in $\mathcal{S}$.

1.9. Positivity of restricted Kac polynomials. For an arbitrary quiver Q, it was proven by Kac in [23] that, for each dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, there is a polynomial $\mathtt{a}_{Q,\mathbf{d}}(q) \in \mathbb{Z}[q]$ which is equal to the number of absolutely indecomposable $\mathbf{d}$-dimensional representations of Q over the field of order q, whenever q is equal to a prime power.

In the case of the degenerate stability condition, for which *all* modules are semistable of the same slope, and so the superscript ζ and the subscript θ can be dropped, (1.5) gives

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{-(\mathbf{d},\mathbf{d})} \cong \mathrm{Sym}(\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})). \tag{1.6}$$

Taking weight series of both sides of (1.6) yields

$$\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \chi_{\mathrm{wt}}(\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}), q^{1/2}) q^{-(\mathbf{d},\mathbf{d})} t^{\mathbf{d}} = \mathrm{Exp}\Big(\sum_{\mathbf{d}\neq 0} \mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{-1/2})(1-q)^{-1} t^{\mathbf{d}}\Big),$$

where $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2}) := \chi_{\mathrm{wt}}(\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q,\mathbf{d}}, q^{1/2})$ is by definition the "$\mathcal{S}$-restricted Kac polynomial". We have used the plethystic exponential

$$\begin{aligned} \mathrm{Exp}\colon \mathbb{Z}((q^{1/2}))[[t_i \mid i \in Q_0]]_+ &\to \mathbb{Z}((q^{1/2}))[[t_i \mid i \in Q_0]], \\ \sum_{i\in\mathbb{Z},\, \mathbf{d}\in\mathbb{N}^{Q_0}} b_{i,\mathbf{d}} q^{i/2} t^{\mathbf{d}} &\mapsto \prod_{i\in\mathbb{Z},\, \mathbf{d}\in\mathbb{N}^{Q_0}} (1-q^{i/2}t^{\mathbf{d}})^{-b_{i,\mathbf{d}}}, \end{aligned}$$

where the $+$ subscript means that $b_{i,\mathbf{0}} = 0$ for all $i \in \mathbb{Z}$.

The mere existence of isomorphism (1.6) can tell us something highly nontrivial about $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2})$ without knowing how to calculate it. Namely, if the left-hand side of (1.6) is pure, then the BPS cohomology $\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q,\mathbf{d}}$ must also be pure, and so $\mathtt{a}^{\mathcal{S}}_{\mathbf{d}}(q^{1/2})$ has positive coefficients (expressed as a polynomial in $-q^{1/2}$). In particular, for the case $\mathcal{S} = \mathbb{C}\overline{Q}$-mod, the $\mathcal{S}$-restricted Kac polynomial is the same as Kac's original polynomial, and our purity theorem (Theorem A) implies Kac's positivity conjecture, originally proved by Hausel, Letellier and Rodriguez-Villegas [19].

In [3, 43], Bozec, Schiffmann and Vasserot define the subcategory of nilpotent, *-semi-nilpotent and *-strongly semi-nilpotent $\mathbb{C}\overline{Q}$-representations by demanding nilpotence of certain paths in $\mathbb{C}\overline{Q}$; see Section 7.1 for definitions. By the above method, in Section 8, we prove positivity of *all* of the resulting polynomials.

Theorem E (Theorem 8.2, Remark 8.4). *Let Q be an arbitrary finite quiver, and let $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. Setting $\mathcal{S}$ to be any out of the full subcategory of nilpotent, *-semi-nilpotent, or *-strongly semi-nilpotent $\mathbb{C}\overline{Q}$-representations, the $\mathcal{S}$-restricted Kac polynomial $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q)$ has positive coefficients.*

1.10. Conventions. For G a complex algebraic group, we set $\mathrm{H}_G := \mathrm{H}(\mathrm{B}G, \mathbb{Q})$. All functors are assumed to be derived unless explicitly stated otherwise. All quivers are assumed to be finite. For X a complex variety, or global quotient stack, we continue to denote

$$\mathrm{H}^{\mathrm{BM}}(X, \mathbb{Q}) := \mathrm{H}_c(X, \mathbb{Q})^{\vee}.$$

We continue to write $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Wherever an object appears with a subscript that is a bold Roman letter, that letter refers to a dimension vector, and $\bullet_{\mathbf{d}}$ is the subobject corresponding to that dimension vector. If any such object appears with a Greek letter such as θ as a subscript, then θ will refer to a slope, and $\bullet_\theta$ will refer to the subobject corresponding to dimension vectors of slope θ. Finally, if an expected subscript is missing altogether, then the entire object is intended.

For $\mathcal{D}$ a triangulated category equipped with a t structure, we define the total cohomology functor $\mathcal{H}(\bullet) := \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(\bullet)[-i]$. We generally use capital Roman letters to refer to spaces of representations before taking any kind of quotient, calligraphic letters to refer to GIT moduli spaces, and Fraktur letters to refer to moduli stacks. Where a space or object is defined with respect to a stability condition ζ, that stability condition will appear as a superscript. In the event that the superscript is missing, we assume that ζ is the degenerate King stability condition $(i, \dots, i) \in \mathbb{H}_+^{Q_0}$. With respect to this stability condition, all representations have the same slope and are semistable, semisimple representations are the polystable representations, and the stable representations are exactly the simple ones.

2. Quiver representations

2.1. Quivers and potentials. Throughout the paper, Q will be used to denote a finite quiver, i.e. a pair of finite sets Q_0 and Q_1 (the vertices and arrows, respectively), and a pair of maps $s, t\colon Q_1 \to Q_0$ (taking an arrow to its source and target, respectively). We denote by $\mathbb{C}Q$ the path algebra of Q, i.e. the algebra over $\mathbb{C}$ having as a basis the paths in Q, with structure constants for the multiplication given by concatenation of paths. For each vertex $i \in Q_0$, we denote by $e_i \in \mathbb{C}Q$ the "lazy" path of length 0 starting and ending at i.

A *potential* on a quiver Q is an element $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]_{\mathrm{vect}}$, where the vect subscript means that we take the quotient by the linear span of the set of commutators. A potential is given by a linear combination of cyclic words in Q, where two cyclic words are considered to be the same if one can be cyclically permuted to the other. If W is a single cyclic word and $a \in Q_1$, we define

$$\partial W/\partial a = \sum_{\substack{W=cac' \\ c \text{ and } c' \text{ paths in } Q}} c'c,$$

and we extend this definition linearly to general W. We define the *Jacobi algebra*

$$\mathrm{Jac}(Q, W) := \mathbb{C}Q/\langle \partial W/\partial a \mid a \in Q_1 \rangle$$

associated to the quiver with potential (Q, W). We will often abbreviate "quiver with potential" to just "QP".

Given a quiver Q, we denote by $\overline{Q}$ the quiver obtained by doubling Q. This is defined by setting $\overline{Q}_0 := Q_0$ and $\overline{Q}_1 = \{a, a^* \mid a \in Q_1\}$, and extending s and t to maps $\overline{Q}_1 \to \overline{Q}_0$ by setting $s(a^*) = t(a)$ and $t(a^*) = s(a)$. We denote by Π_Q the preprojective algebra of Q, defined as in (1.1).

We denote by $\tilde{Q}$ the quiver obtained from Q by setting

$$\tilde{Q}_0 := Q_0, \quad \tilde{Q}_1 := \overline{Q}_1 \coprod \{\omega_i \mid i \in Q_0\},$$

where each ω_i is an arrow satisfying $s(\omega_i) = t(\omega_i) = i$. If a quiver Q is fixed, we define the potential $\tilde{W}$ as in [14, Section 4.2] and [32] by setting $\tilde{W} = \sum_{a \in Q_1}[a, a^*] \sum_{i \in Q_0} \omega_i$. If A is an algebra, we denote by A-mod the category of finite-dimensional A-modules.

Proposition 2.1. *Define $\mathcal{C}_{\Pi_Q}$ to be the category whose objects are pairs (M, f), where M is a finite-dimensional Π_Q-module and $f \in \mathrm{End}_{\Pi_Q\text{-mod}}(M)$, and define*

$$\mathrm{Hom}_{\mathcal{C}_{\Pi_Q}}((M, f), (M', f'))$$

to be the subspace of morphisms $g \in \mathrm{Hom}_{\Pi_Q\text{-mod}}(M, M')$ such that $f'g = gf$. Then there is an isomorphism of categories

$$\mathcal{C}_{\Pi_Q} \cong \mathrm{Jac}(\tilde{Q}, \tilde{W})\text{-mod}.$$

Proof. From the relations $\partial\tilde{W}/\partial\omega_i$, for $i \in Q_0$, we deduce that the natural inclusion $\mathbb{C}\overline{Q} \subset \mathbb{C}\tilde{Q}$ induces an inclusion $\Pi_Q \subset \mathrm{Jac}(\tilde{Q}, \tilde{W})$. Therefore, a $\mathrm{Jac}(\tilde{Q}, \tilde{W})$-module is given by a Π_Q-module M, along with linear maps $M(\omega_i) \in \mathrm{End}_{\mathbb{C}}(e_i \cdot M)$ satisfying

$$M(\partial\tilde{W}/\partial a) = M(a^*)M(\omega_{s(a^*)}) - M(\omega_{t(a^*)})M(a^*) = 0,$$
$$M(\partial\tilde{W}/\partial a^*) = M(\omega_{t(a)})M(a) - M(a)M(\omega_{s(a)}) = 0.$$

These are precisely the conditions for the elements $\{M(\omega_i)\}_{i \in Q_0}$ to define an endomorphism of M, considered as a Π_Q-module. □

2.2. Moduli spaces. Given an algebra A, presented as a quotient $A = \mathbb{C}Q/I$ of a free path algebra by a two-sided ideal $I \subset \mathbb{C}Q_{\geq 1}$ generated by paths of length at least one, and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $\mathfrak{M}(A)_{\mathbf{d}}$ the stack of $\mathbf{d}$-dimensional complex representations of A. This is a finite type Artin stack. In the case $A = \mathbb{C}Q$, we abbreviate $\mathfrak{M}(\mathbb{C}Q)_{\mathbf{d}}$ to $\mathfrak{M}(Q)_{\mathbf{d}}$. This stack is naturally isomorphic to the quotient stack $X(Q)_{\mathbf{d}}/\mathrm{GL}_{\mathbf{d}}$, where

$$X(Q)_{\mathbf{d}} := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}), \quad \mathrm{GL}_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C})$$

and the action is by simultaneous conjugation. We define $\mathfrak{gl}_{\mathbf{d}} = \prod_{i \in Q_0} \mathfrak{gl}_{\mathbf{d}_i}(\mathbb{C})$ and define

$$\mu_{Q,\mathbf{d}}\colon X(\overline{Q})_{\mathbf{d}} \to \mathfrak{gl}_{\mathbf{d}}, \quad \rho \mapsto \sum_{a \in Q_1} [\rho(a), \rho(a^*)].$$

As substacks of $\mathfrak{M}(\overline{Q})_{\mathbf{d}}$, there is an equality $\mu_{Q,\mathbf{d}}^{-1}(0)/\mathrm{GL}_{\mathbf{d}} = \mathfrak{M}(\Pi_Q)_{\mathbf{d}}$. As in the introduction, we define the function

$$\mathrm{Tr}(\widetilde{W})_{\mathbf{d}}\colon X(\widetilde{Q})_{\mathbf{d}} \to \mathbb{C}, \quad \rho \mapsto \mathrm{Tr}\Big(\sum_{a \in Q_1} [\rho(a), \rho(a^*)] \sum_{i \in Q_0} \rho(\omega_i)\Big)$$

and denote by $\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}\colon \mathfrak{M}(\widetilde{Q})_{\mathbf{d}} \to \mathbb{C}$ the induced function. As substacks of $\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}$, there are equalities

$$\mathrm{crit}(\mathrm{Tr}(\widetilde{W})_{\mathbf{d}})/\mathrm{GL}_{\mathbf{d}} = \mathfrak{M}(\mathrm{Jac}(\widetilde{Q}, \widetilde{W}))_{\mathbf{d}} = \mathrm{crit}(\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}). \tag{2.1}$$

We define $\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\omega\text{-nilp}} \subset \mathfrak{M}(\widetilde{Q})_{\mathbf{d}}$ to be the reduced stack defined by the vanishing of the functions $\mathfrak{Tr}(\rho(\omega_i)^m)$ for $i \in Q_0$ and $1 \leq m \leq \mathbf{d}_i$. The geometric points of $\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\omega\text{-nilp}}$ over a field extension $K \supset \mathbb{C}$ correspond to $\mathbf{d}$-dimensional $K\widetilde{Q}$ representations ρ such that, for each $i \in Q_0$, the endomorphism $\rho(\omega_i)$ is a nilpotent K-linear endomorphism.

A *stability condition* for Q is defined to be an element of $\mathbb{H}_+^{Q_0}$, where

$$\mathbb{H}_+ := \{r \exp(i\pi\phi) \in \mathbb{C} \mid r > 0,\ 0 < \phi \leq 1\}.$$

For a fixed stability condition $\zeta \in \mathbb{H}_+^{Q_0}$, we define the *central charge*

$$Z\colon \mathbb{N}^{Q_0} \setminus \{0\} \to \mathbb{H}_+, \quad \mathbf{d} \mapsto \mathbf{d} \cdot \zeta.$$

We define the slope of a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ by setting

$$\varrho(\mathbf{d}) := \begin{cases} -\Re e(Z(\mathbf{d}))/\Im m(Z(\mathbf{d})) & \text{if } \Im m(Z(\mathbf{d})) \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

If ρ is a representation of Q, we define $\varrho(\rho) := \varrho(\dim(\rho))$. A representation ρ is called ζ-*semistable* if, for all proper subrepresentations $\rho' \subset \rho$, we have $\varrho(\rho') \leq \varrho(\rho)$, and ρ is called ζ-*stable* if the inequality is strict. We will always assume that our stability conditions are *King stability conditions*, meaning that, for each $1_i \in \mathbb{N}^{Q_0}$ in the natural generating set,

$$\Im m(Z(1_i)) = 1 \quad \text{and} \quad \Re e(Z(1_i)) \in \mathbb{Q}.$$

If ζ is a King stability condition, then for each $\mathbf{d} \in \mathbb{N}^{Q_0}$, there is a geometric invariant theory (GIT) coarse moduli space of ζ-semistable Q-representations of dimension $\mathbf{d}$, constructed in [26], which we denote $\mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} := X(Q)^{\zeta\text{-ss}}_{\mathbf{d}} /\!/_{\chi(\zeta)} \mathrm{GL}_{\mathbf{d}}$. Here $X(Q)^{\zeta\text{-ss}}_{\mathbf{d}} \subset X(Q)_{\mathbf{d}}$ is the open subscheme whose geometric points correspond to ζ-semistable Q-representations.

We denote by

$$\mathtt{JH}^{\zeta}_{Q,\mathbf{d}} : \mathfrak{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} \to \mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} \tag{2.2}$$

the morphism from the stack to the coarse moduli space. At the level of points, this map takes a semistable representation ρ to the direct sum of the subquotients appearing in the Jordan–Hölder filtration of ρ, considered as an object in the category of ζ-semistable representations of slope $\varrho(\mathbf{d})$. If there is no ambiguity, we omit the subscript Q from the definition of $\mathtt{JH}$.

We denote by $q^{\zeta}_{Q,\mathbf{d}} : \mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} \to \mathcal{M}(Q)_{\mathbf{d}}$ the morphism from the GIT quotient to the affinisation. This morphism is proper, as can be seen from the construction of the domain via GIT. At the level of points, $q^{\zeta}_{Q,\mathbf{d}}$ takes a ζ-semistable module to its semisimplification.

We define two pairings on $\mathbb{Z}^{Q_0}$,

$$(\mathbf{d},\mathbf{e})_Q := \sum_{i\in Q_0} \mathbf{d}_i \mathbf{e}_i - \sum_{a\in Q_1} \mathbf{d}_{s(a)}\mathbf{e}_{t(a)} \quad \langle \mathbf{d},\mathbf{e}\rangle_Q := (\mathbf{d},\mathbf{e})_Q - (\mathbf{e},\mathbf{d})_Q.$$

Again, we will drop the subscript Q when the choice of quiver is obvious from the context. For $\theta \in (-\infty,\infty)$ a slope, we denote by $\Lambda^{\zeta}_{\theta} \subset \mathbb{N}^{Q_0}$ the submonoid of dimension vectors $\mathbf{d}$ such that $\mathbf{d} = 0$ or $\varrho(\mathbf{d}) = \theta$. A stability condition $\zeta \in \mathbb{H}^{Q_0}_{+}$ is θ-*generic* if, for all $\mathbf{d},\mathbf{e} \in \Lambda^{\zeta}_{\theta}$, $\langle \mathbf{d},\mathbf{e}\rangle = 0$, and we say that ζ is *generic* if it is θ-generic for all θ. A quiver Q is a *symmetric* if, for any two vertices $i,j \in Q_0$, the number of arrows a with $s(a) = i$ and $t(a) = j$ is equal to the number of arrows with $s(a) = j$ and $t(a) = i$. For Q a quiver, we define the *degenerate* stability condition $\zeta = (i,\dots,i) \in \mathbb{H}^{Q_0}_{+}$. If Q is symmetric, then all stability conditions $\zeta \in \mathbb{H}^{Q_0}_{+}$ are generic. The degenerate stability condition is generic if and only if Q is symmetric. In particular, for all quivers Q, the degenerate stability condition is generic for $\overline{Q}$ and $\widetilde{Q}$.

We denote by $\dim^{\zeta} : \mathcal{M}(Q)^{\zeta\text{-ss}} \to \mathbb{N}^{Q_0}$ the map taking a polystable quiver representation to its dimension vector, and define $\mathrm{Dim}^{\zeta} := \dim^{\zeta} \circ \mathtt{JH}^{\zeta}_{Q}$, where $\mathtt{JH}^{\zeta}_{Q}$ is as in (2.2).

If $\mathcal{S}$ is a Serre subcategory of the category of $\mathbb{C}Q$-mod, we denote by

$$\iota' : \mathcal{M}(Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}} \hookrightarrow \mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}}$$

the inclusion of the polystable $\mathbb{C}Q$ modules that are objects of $\mathcal{S}$. We only consider choices of $\mathcal{S}$ for which this is a morphism of varieties. We set

$$\mathfrak{M}(Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}} = \mathcal{M}(Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}} \times_{\mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}}} \mathfrak{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}}$$

and denote the inclusion $\iota : \mathfrak{M}(Q)^{\mathcal{S},\zeta\text{-ss}}_{\mathbf{d}} \hookrightarrow \mathfrak{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}}$.

3. Cohomological DT theory

3.1. Vanishing cycles and mixed Hodge modules. Let X be a smooth complex variety, and let f be a regular function on it. Set $X_0 = f^{-1}(0)$ and $X_{<0} = f^{-1}(\mathbb{R}_{<0})$. We define the nearby cycle functor as the following composition of (derived) functors:

$$\psi_f := (X_0 \to X)_*(X_0 \to X)^*(X_{<0} \to X)_*(X_{<0} \to X)^*,$$

and we define the functor

$$\phi_f^{\mathfrak{p}} = \operatorname{cone}((X_0 \to X)_*(X_0 \to X)^* \to \psi_f)[-1].$$

Alternatively, define $X_{\leq 0} = f^{-1}(\mathbb{R}_{\leq 0})$, and define the (underived) functor $\Gamma_{X_{\leq 0}}$ by setting

$$\Gamma_{X_{\leq 0}}\mathcal{F}(U) = \ker(\mathcal{F}(U) \to \mathcal{F}(U \setminus X_{\leq 0})).$$

Then we can define $\phi_f^{\mathfrak{p}}\mathcal{F} = (R\Gamma_{X_{\leq 0}}\mathcal{F})_{X_0}$. We define $\psi_f^{\mathfrak{p}} := \psi_f[-1]$.

If X is a quasiprojective complex variety, and so there is a closed embedding $X \subset Y$ inside a smooth complex variety, and f extends to a function $\overline{f}$ on Y, we define $\phi_f^{\mathfrak{p}} = i^*\phi_{\overline{f}}^{\mathfrak{p}} i_*$, where $i\colon X \to Y$ is the embedding. For a complex variety X, we define as in [38, 39] the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X. See [37] for an overview of the theory. There is an exact functor rat: $\mathcal{D}(\mathrm{MHM}(X)) \to \mathcal{D}(\mathrm{Perv}(X))$ which takes a complex of mixed Hodge modules $\mathcal{F}$ to its underlying complex of perverse sheaves, and commutes with f_*, $f_!$, f^*, $f^!$, $\mathbb{D}_X$ and tensor product. In addition, the functors $\phi_f^{\mathfrak{p}}$ and $\psi_f^{\mathfrak{p}}$ lift to exact functors for the category of mixed Hodge modules. We denote by ϕ_f the lift of $\phi_f^{\mathfrak{p}}$.

Remark 3.1. If f is a regular function on the smooth variety X, then

$$\operatorname{supp}(\phi_f^{\mathfrak{p}}\mathbb{Q}_X) = \operatorname{supp}(\phi_f\mathbb{Q}_X) = \operatorname{crit}(f).$$

We define $\mathcal{D}^b(\mathrm{MHM}(X))$ to be the bounded derived category of mixed Hodge modules on X. If X is connected, we define $\mathcal{D}^{\geq}(\mathrm{MHM}(X))$ to be the inverse limit of the diagram of categories

$$\cdots \longrightarrow \mathcal{D}^b(\mathrm{MHM}(X)) \xrightarrow{\tau^{\leq n}} \mathcal{D}^b(\mathrm{MHM}(X)) \xrightarrow{\tau^{\leq n-1}} \mathcal{D}^b(\mathrm{MHM}(X)) \longrightarrow \cdots.$$

Explicitly, an object of $\mathcal{D}^{\geq}(\mathrm{MHM}(X))$ is given by a $\mathbb{Z}$-tuple of objects $\mathcal{F}_n$ in $\mathcal{D}^{\geq}(\mathrm{MHM}(X))$, along with isomorphisms $\tau^{\leq n-1}\mathcal{F}_n \cong \mathcal{F}_{n-1}$. For $\mathcal{F}$ an object in $\mathcal{D}^{\geq}(\mathrm{MHM}(X))$, we write $\tau^{\leq n}\mathcal{F} = \mathcal{F}_n$ and $\mathcal{H}^n(\mathcal{F}) = \mathcal{H}^n(\mathcal{F}_n)$. For an object $\mathcal{F}$ of $\mathcal{D}^{\geq}(\mathrm{MHM}(X))$, the cohomological amplitude of the objects $\mathcal{F}_n$ are universally bounded below.

Similarly, we define $\mathcal{D}^{\leq}(\mathrm{MHM}(X))$ to be the inverse limit of the diagram

$$\cdots \longrightarrow \mathcal{D}^b(\mathrm{MHM}(X)) \xrightarrow{\tau^{\geq n}} \mathcal{D}^b(\mathrm{MHM}(X)) \xrightarrow{\tau^{\geq n+1}} \mathcal{D}^b(\mathrm{MHM}(X)) \longrightarrow \cdots.$$

For general X, we define $\mathcal{D}^{\geq}(\mathrm{MHM}(X)) := \prod_{X' \in \pi_0(X)} \mathcal{D}^{\geq}(\mathrm{MHM}(X'))$ and $\mathcal{D}^{\leq}(\mathrm{MHM}(X))$ similarly. A mixed Hodge module $\mathcal{F}$ comes with a filtration $\cdots \subset W_i\mathcal{F} \subset W_{i+1}\mathcal{F} \subset \cdots$, the weight filtration, which is equal to the usual weight filtration if $\mathcal{F}$ is a genuine mixed Hodge module. We say that $\mathcal{F} \in \mathrm{MHM}(X))$, with $\heartsuit = b, \leq, \geq$, we say that $\mathcal{F}$ is pure of weight n if $\mathcal{H}^i(\mathcal{F})$ is pure of weight $i+n$ for all i, or we just call $\mathcal{F}$ "*pure*" if each $\mathcal{H}^i(\mathcal{F})$ is pure of weight i.

We define $\mathbb{L} := \mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$, considered as a cohomologically graded mixed Hodge structure, i.e. as a pure cohomologically graded mixed Hodge structure concentrated in cohomological degree two. We formally add a tensor square root $\mathbb{L}^{1/2}$ of $\mathbb{L}$ to this category.

Remark 3.2. This may be achieved either purely formally, or by embedding $\mathrm{MHM}(X)$ inside the category of monodromic mixed Hodge structures; both approaches are explained in [28]. Since we will only consider cohomology of vanishing cycle complexes that may be dimensionally reduced as in Theorem 3.4, the natural monodromy operators on the resulting mixed Hodge structures will be trivial so that (apart from this square root) we stay essentially within the category of mixed Hodge modules and will refrain from elaborating upon the theory of monodromic mixed Hodge here (see [10, §2.1] for a thorough introduction in the context of DT theory).

Convention 3.3. Let X be a complex variety such that each connected component contains a connected dense smooth locus. In this paper, we will shift the definition of the intersection complex mixed Hodge module for X so that it is pure of weight zero, while its underlying element in $\mathcal{D}^b(\mathrm{Perv}(X))$ is a perverse sheaf. This we achieve by setting

$$\mathcal{IC}_X(\mathbb{Q}) := \sum_{Z\in\pi_0(X)} \mathrm{IC}_{Z_{\mathrm{reg}}}(\mathbb{Q}) \otimes \mathbb{L}^{-\dim(Z)/2}.$$

If X is a smooth connected variety, we set $\mathrm{H}_{(c)}(X,\mathbb{Q})_{\mathrm{vir}} := \mathrm{H}_{(c)}(X,\mathcal{IC}_X(\mathbb{Q}))$. Since the smooth stack $\mathrm{B}\mathbb{C}^*$ has complex dimension -1, we extend this notation in the natural way by setting $\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}} := \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})\otimes\mathbb{L}^{1/2}$ and $\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}} := \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})^\vee\otimes\mathbb{L}^{-1/2}$.

3.2. Pushforwards from stacks. Assume that X is a complex variety, acted on by an algebraic group G, following [10, Section 2] how to define $p_*\phi_f\mathbb{Q}_{X/G}\in\mathrm{Ob}(\mathcal{D}^{\geq}(\mathrm{MHM}(Y)))$. We recall the definition for the case in which X is connected – the general definition is obtained by taking the direct sum over connected components. The definition is a minor modification of Totaro's well-known construction [46].

Let $V_0\subset V_1\subset\cdots$ be an ascending chain of G-representations, and let $U_0\subset U_1\subset\cdots$ be an ascending sequence of closed inclusions of G-equivariant varieties, with each $U_i\subset X\times V_i$ an open dense subvariety. We assume that $\lim_{i\mapsto\infty}(\mathrm{codim}_{X\times V_i}((X\times V_i)\setminus U_i))=\infty$, that G acts freely on U_i for all i, and that the principal bundle $U_i\to U_i/G$ exists in the category of complex varieties. We define $X_i := U_i/G$ and denote by $p_i\colon X_i\to Y$ and $f_i\colon X_i\to\mathbb{C}$ the induced maps. We define

$$\tau^{\leq n}(p_*\phi_f\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\leq n}(p_{i,*}\phi_{f_i}\mathbb{Q}_{X_i})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2},$$
$$\tau^{\geq n}(p_!\phi_f\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\geq n}(p_{i,!}\phi_{f_i}\mathbb{Q}_{X_i}\otimes\mathbb{L}^{-\dim(U_i)})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2},$$

where the limit is constructed, and exists, as in [10, §2.2]. Similarly, we define

$$\tau^{\leq n}(p_*\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\leq n}(p_{i,*}\mathbb{Q}_{X_i})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2},$$
$$\tau^{\geq n}(p_!\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\geq n}(p_{i,!}\mathbb{Q}_{X_i}\otimes\mathbb{L}^{-\dim(U_i)})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2}.$$

This can be seen as a special case of the previous construction, with $f=0$.

Let $Z\subset X$ be a subvariety, preserved by the G-action, and denote by $\iota\colon Z/G\hookrightarrow X/G$ the inclusion of stacks. We obtain inclusions $\iota_i\colon Z_i := (U_i\cap(Z\times V_i))/G\to X_i$ and we define the restricted pushforward of vanishing cycle cohomology

$$\tau^{\leq n}(p_*\iota_*\iota^!\phi_f\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\leq n}(p_{i,*}\iota_{i,*}\iota_i^!\phi_{f_i}\mathbb{Q}_{X_i})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2},$$
$$\tau^{\geq n}(p_!\iota_!\iota^*\phi_f\mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty}\tau^{\geq n}(p_{i,!}\iota_{i,!}\iota_i^*\phi_{f_i}\mathbb{Q}_{X_i}\otimes\mathbb{L}^{-\dim(U_i)})\otimes\mathbb{L}^{(\dim(G)-\dim(X))/2}.$$

As a particular case, setting Y to be a point, we obtain

$$\mathrm{H}_c^n(Z/G, \phi_f \mathcal{IC}_{X/G}(\mathbb{Q})) := \lim_{i\mapsto\infty} \mathrm{H}_c^n(Z_i, \iota_i^* \phi_{f_i} \mathbb{Q}_{X_i} \otimes \mathbb{L}^{-\dim(U_i)}) \otimes \mathbb{L}^{(\dim(G)-\dim(X))/2}.$$

3.3. Dimensional reduction. Given a decomposition $X = X' \times \mathbb{A}^n$ of varieties, let $\mathbb{C}^*$ act on X via the product of the trivial action on X', and the scaling action on $\mathbb{A}^n$. Assume that $f \in \Gamma(X)$ has weight one. Denote by $\pi\colon X \to X'$ the natural projection. Then we can write $f = \sum_{1\leq i\leq n} \pi^* f_i \cdot x_i$, where f_i are functions on X', and x_i are coordinates for $\mathbb{A}^n$. Define $Z' = Z(f_1, \dots, f_n)$ to be the shared vanishing locus of all the functions $f_1, \dots, f_n$, and denote $Z = \pi^{-1}(Z')$. Note that $Z \subset X_0 := f^{-1}(0)$, so we can postcompose the canonical natural transformation $\nu_f\colon \phi_f \to (X_0 \to X)_*(X_0 \to X)^*$ with the restriction map

$$(X_0 \to X)_*(X_0 \to X)^* \to (Z \to X)_*(Z \to X)^*$$

to obtain a natural transformation $\upsilon\colon \phi_f \to (Z \to X)_*(Z \to X)^*$.

Theorem 3.4 ([5, Theorem A.1]). *$\pi_! \upsilon \pi^*$ is a natural isomorphism.*

This is a cohomological analogue of the dimensional reduction theorem of [1]. It implies (see [5, Corollary A.7]) that if X is the total space of a G-equivariant affine fibration $\pi\colon X \to X'$ for G an algebraic group, and $S \subset X'$ is a G-invariant subspace of the base, there is a natural isomorphism in compactly supported cohomology

$$\mathrm{H}_c(\pi^{-1}(S)/G, \phi_f \mathbb{Q}_{X/G}) \cong \mathrm{H}_c((S \cap Z')/G, \mathbb{Q}) \otimes \mathbb{L}^{\dim(\pi)}.$$

3.4. Integrality and PBW isomorphisms. Let Q be a finite quiver. We consider $\mathbb{N}^{Q_0}$-graded mixed Hodge structures as mixed Hodge modules on the space $\mathbb{N}^{Q_0}$ in the obvious way: a mixed Hodge module on a point is just a polarisable mixed Hodge structure, and $\mathbb{N}^{Q_0}$ is a union of points $\mathbf{d} \in \mathbb{N}^{Q_0}$, and so a mixed Hodge module on $\mathbb{N}^{Q_0}$ is given by a formal direct sum $\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathcal{L}_{\mathbf{d}}$ of mixed Hodge structures.

The GIT quotient $\mathcal{M}(Q)^{\zeta\text{-ss}}$ is a monoid with monoid morphism $\oplus$ taking a pair of points representing polystable representations ρ, ρ' to the point representing their direct sum $\rho \oplus \rho'$. This morphism is finite [31, Lemma 2.1]. A unit for the monoid morphism is provided by the inclusion $\mathcal{M}(Q)_0^{\zeta\text{-ss}} \hookrightarrow \mathcal{M}(Q)^{\zeta\text{-ss}}$, which at the level of complex points, corresponds to the inclusion of the zero module. The morphism $\dim^\zeta\colon \mathcal{M}(Q)^{\zeta\text{-ss}} \to \mathbb{N}^{Q_0}$, taking a representation to its dimension vector, is a morphism of monoids, where the morphism $+\colon \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0} \to \mathbb{N}^{Q_0}$ is the usual addition map. If W is a potential for Q, there is an induced function $\mathcal{T}r(W)^\zeta\colon \mathcal{M}(Q)^{\zeta\text{-ss}} \to \mathbb{C}$ satisfying $\mathcal{T}r(W)^\zeta \circ \mathtt{JH}^\zeta = \mathfrak{T}\mathrm{r}(W)^\zeta$.

If X is a commutative monoid in the category of locally finite type complex schemes, with finite type monoid morphism $\tau\colon X \times X \to X$, then by [30, Theorem 1.9], the categories carry symmetric monoidal structures defined by $\mathcal{F} \boxtimes_\tau \mathcal{G} := \tau_*(\mathcal{F} \boxtimes \mathcal{G})$. In particular, the categories $\mathcal{D}^{\geq}(\mathrm{MHM}(\mathcal{M}(Q)^{\zeta\text{-ss}}))$ and $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathcal{M}(Q)^{\zeta\text{-ss}}))$ carry symmetric monoidal structures defined by $\mathcal{F} \boxtimes_\oplus \mathcal{G} := \oplus_*(\mathcal{F} \boxtimes \mathcal{G})$.

The following theorem allows for the definition of BPS sheaves and BPS cohomology. It is a cohomological lift of the property known in DT theory as *integrality*.

Theorem 3.5 ([10, Theorem A][3)]). *Fix a QP (Q,W) such that*

$$\mathrm{crit}(\mathrm{Tr}(W)) \subset \mathrm{Tr}(W)^{-1}(0),$$

a slope $\theta \in (-\infty, \infty)$, and a θ-generic stability condition ζ. For nonzero $\mathbf{d} \in \Lambda_\theta^\zeta$, where $\Lambda_\theta^\zeta \subset \mathbb{N}^{Q_0}$ is as in Section 2.2, define the mixed Hodge module on $\mathcal{M}(Q)_{\mathbf{d}}^{\zeta\text{-ss}}$,

$$\mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta} = \begin{cases} \phi_{\mathcal{T}r(W)_{\mathbf{d}}^{\zeta}} \mathcal{IC}_{\mathcal{M}(Q)_{\mathbf{d}}^{\zeta\text{-ss}}}(\mathbb{Q}) & \text{if } \mathcal{M}(Q)_{\mathbf{d}}^{\zeta\text{-st}} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and define

$$\mathcal{BPS}_{Q,W,\theta}^{\zeta} := \bigoplus_{\mathbf{d} \in \Lambda_\theta^\zeta} \mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta}.$$

Then there are isomorphisms of objects in $\mathcal{D}^{\geq}(\mathrm{MHM}(\mathcal{M}(Q)_\theta^{\zeta\text{-ss}}))$, $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathcal{M}(Q)_\theta^{\zeta\text{-ss}}))$, respectively,

$$\mathtt{JH}_{\theta,*}^{\zeta} \phi_{\mathfrak{Tr}(W)_\theta^\zeta} \mathcal{IC}_{\mathfrak{M}(Q)_\theta^{\zeta\text{-ss}}}(\mathbb{Q}) \cong \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{Q,W,\theta}^{\zeta} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}}), \tag{3.1}$$

$$\mathtt{JH}_{\theta,!}^{\zeta} \phi_{\mathfrak{Tr}(W)_\theta^\zeta} \mathcal{IC}_{\mathfrak{M}(Q)_\theta^{\zeta\text{-ss}}}(\mathbb{Q}) \cong \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{Q,W,\theta}^{\zeta} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}}). \tag{3.2}$$

Since Verdier duality naturally commutes with the vanishing cycles functor, and since $\mathcal{IC}_{\mathcal{M}(Q)_{\mathbf{d}}^{\zeta\text{-ss}}}(\mathbb{Q})$ is Verdier self-dual, the BPS sheaf is Verdier self-dual: there is an isomorphism

$$\mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta} \cong \mathbb{D}\mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta}.$$

3.4.1. (3d) BPS cohomology. Let $\mathcal{S}$ be a Serre subcategory of the category of $\mathbb{C}Q$-modules. Recall that we denote by $\iota' \colon \mathcal{M}(Q)^{\mathcal{S},\zeta\text{-ss}} \hookrightarrow \mathcal{M}(Q)^{\zeta\text{-ss}}$ the inclusion of objects in $\mathcal{S}$. We define the *BPS cohomology*

$$\begin{aligned} \mathrm{BPS}_{Q,W,\mathbf{d}}^{\mathcal{S},\zeta} &:= \mathrm{H}(\mathcal{M}(Q)_{\mathbf{d}}^{\mathcal{S},\zeta\text{-ss}}, \iota'^{!} \mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta}) \\ &\cong \mathrm{H}_c(\mathcal{M}(Q)_{\mathbf{d}}^{\mathcal{S},\zeta\text{-ss}}, \iota'^{*} \mathcal{BPS}_{Q,W,\mathbf{d}}^{\zeta})^{\vee}, \end{aligned}$$

where the isomorphism follows from Verdier self-duality of the BPS sheaf.

The cohomologically graded mixed Hodge structure

$$\mathcal{A}_{Q,\theta}^{\mathcal{S},\zeta} := \mathrm{Dim}_*^{\zeta} \iota_* \iota^! \phi_{\mathfrak{Tr}(W)_\theta^\zeta} \mathcal{IC}_{\mathfrak{M}(Q)_\theta^{\zeta\text{-ss}}}(\mathbb{Q})$$

carries a Hall algebra multiplication, defined in [5,28], via pullback and pushforward of vanishing cycle sheaves; see Section 9.1 for a generalisation of the construction. Applying the natural transformation $\tau^{\leq 1} \to \mathrm{id}$ to (3.1), we obtain the morphism

$$\mathcal{BPS}_{Q,W,\theta}^{\zeta} \otimes \mathbb{L}^{1/2} \to \mathtt{JH}_{\theta,*}^{\zeta} \phi_{\mathfrak{Tr}(W)_\theta^\zeta} \mathcal{IC}_{\mathfrak{M}(Q)_\theta^{\zeta\text{-ss}}}(\mathbb{Q}). \tag{3.3}$$

---

3) Technically, the result quoted from [10] is in fact stated for $\mathcal{H}(\bullet)$ of the LHS of (3.1). That there is an isomorphism $\mathcal{H}(\mathrm{LHS}) \cong \mathrm{LHS}$ is a consequence of approximation by projective morphisms and the decomposition theorem; see [10]. In addition, for general Q, W, the result makes use of the symmetric monoidal structure on the category of monodromic mixed Hodge modules, which we ignore, following Remark 3.2.

Applying $\mathrm{H}\,\iota'_*\iota'^!$ to (3.3), we obtain the embedding $\mathrm{BPS}^{\mathcal{S},\zeta}_{Q,W,\theta} \otimes \mathbb{L}^{1/2} \hookrightarrow \mathcal{A}^{\mathcal{S},\zeta}_{Q,W,\theta}$. Since $\mathrm{H}_{\mathbb{C}^*}$ acts on the target, this extends to a morphism

$$g\colon \mathrm{BPS}^{\mathcal{S},\zeta}_{Q,W,\theta} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}} \to \mathcal{A}^{\mathcal{S},\zeta}_{Q,W,\theta}.$$

Theorem 3.6 (PBW theorem [10, Theorem C]). *The morphism*

$$\mathrm{Sym}(\mathrm{BPS}^{\mathcal{S},\zeta}_{Q,W,\theta} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}) \to \mathcal{A}^{\mathcal{S},\zeta}_{Q,W,\theta}$$

extending g via the Hall algebra multiplication on the target is an isomorphism.

3.5. Framed moduli spaces. Let Q be a quiver. For the moment, we do not assume that Q is symmetric. Let $\mathbf{d},\mathbf{f} \in \mathbb{N}^{Q_0}$. Following [10, Section 3.3], we extend Q to $Q_{\mathbf{f}}$ via

$$(Q_{\mathbf{f}})_0 := Q_0 \cup \{\infty\}, \quad (Q_{\mathbf{f}})_1 := Q_1 \cup \{\beta_{i,m} \mid i \in Q_0,\ 1 \le m \le \mathbf{f}_i\}$$

and

$$s(\beta_{i,m}) = \infty, \quad t(\beta_{i,m}) = i.$$

Given a King stability condition ζ for Q, and a slope $\theta \in (-\infty,\infty)$, we extend ζ to a stability condition $\zeta^{(\theta)}$ for $Q_{\mathbf{f}}$ by fixing the slope

$$-\Re e(\zeta^{(\theta)}_\infty)/\Im m(\zeta^{(\theta)}_\infty) = \theta + \epsilon$$

for sufficiently small positive ϵ. Let $\mathbf{d} \in \Lambda^\zeta_\theta$. Then a $(1,\mathbf{d})$-dimensional representation ρ of $Q_{\mathbf{f}}$ is $\zeta^{(\theta)}$-semistable if and only if it is $\zeta^{(\theta)}$-stable, and this holds if and only if the underlying Q-representation of ρ is ζ-semistable, and for all proper $Q_{\mathbf{f}}$-subrepresentations $\rho' \subset \rho$, if $\dim(\rho')_\infty = 1$, then the underlying Q-representation of ρ' is nonzero and has slope strictly less than θ.

We denote by $\mathcal{M}(Q)^\zeta_{\mathbf{f},\mathbf{d}} = X(Q_{\mathbf{f}})^{\zeta^{(\theta)}\text{-ss}}_{(1,\mathbf{d})}/\mathrm{GL}_{\mathbf{d}}$ the fine moduli space of $\mathbf{f}$-framed ζ-semistable representations of Q of dimension $\mathbf{d}$, or in other words, the fine moduli space of $\zeta^{(\theta)}$-stable $(1,\mathbf{d})$-dimensional representations of $Q_{\mathbf{f}}$. We denote by $\pi^\zeta_{\mathbf{f},\mathbf{d}}\colon \mathcal{M}(Q)^\zeta_{\mathbf{f},\mathbf{d}} \to \mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}}$ the induced map from the quotient. The following is the version of the integrality theorem (Theorem 3.5) for moduli spaces of stable framed modules; the proof follows the proof of [10, Theorem 4.10], to which we refer for more details.

Proposition 3.7. *Let ζ be a θ-generic stability condition, and assume that*

$$\mathrm{crit}(\mathrm{Tr}(W)) \subset \mathrm{Tr}(W)^{-1}(0).$$

There is an isomorphism in the category $\mathcal{D}^{\le}(\mathrm{MHM}(\mathcal{M}(Q)^{\zeta\text{-ss}}_\theta))$,

$$\pi^\zeta_{Q,\mathbf{f},\theta,!}\Big(\bigoplus_{\mathbf{d}\in\Lambda^\zeta_\theta} \phi_{\mathcal{T}r(W)^\zeta_{\mathbf{f},\mathbf{d}}}\mathbb{Q}_{\mathcal{M}(Q)^\zeta_{\mathbf{f},\mathbf{d}}} \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_Q/2}\Big)$$
$$\cong \mathrm{Sym}_{\boxtimes_\oplus}\Big(\bigoplus_{\mathbf{d}\in\Lambda^\zeta_\theta} \mathcal{BPS}^\zeta_{Q,W,\mathbf{d}} \otimes \mathrm{H}(\mathbb{CP}^{\mathbf{f}\cdot\mathbf{d}-1},\mathbb{Q})^\vee \otimes \mathbb{L}^{-1/2}\Big).$$

4. Purity and supports

In this section, we prove Theorem A. A crucial role in the proof is played by the support lemma (Lemma 4.1), which also enables us to prove Theorem B.

4.1. Proof of Theorem A. Fix a quiver Q. We define $(\widetilde{Q},\widetilde{W})$ as in Section 2.1. Define

$$\mathrm{BPS}^{\vee}_{\widetilde{Q},\widetilde{W}} := \dim_! \mathcal{BPS}_{\widetilde{Q},\widetilde{W}} = \mathrm{H}_c(\mathcal{M}(\widetilde{Q}), \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}),$$
$$\mathrm{BPS}^{\omega\text{-nilp},\vee}_{\widetilde{Q},\widetilde{W}} := \dim_!(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}(\widetilde{Q})^{\omega\text{-nilp}}})$$

the compactly supported cohomology and the restricted compactly supported cohomology, respectively, of the BPS sheaf from Theorem 3.5. Recall that $\dim\colon \mathcal{M}(\widetilde{Q}) \to \mathbb{N}^{Q_0}$ is the map taking a semisimple representation to its dimension vector. As explained at the beginning of Section 3.4, we may consider both of the above objects equivalently as mixed Hodge module complexes on the discrete space $\mathbb{N}^{Q_0}$, or $\mathbb{N}^{Q_0}$-graded mixed Hodge structures. We will prove Theorem A using the following three lemmas.

Lemma 4.1 (Support lemma). *Let $x \in \mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}$ lie in the support of $\mathcal{BPS}^{\zeta\text{-ss}}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$, corresponding to a $\mathbf{d}$-dimensional semisimple $\mathbb{C}\widetilde{Q}$ representation ρ. Then the union of the multisets $\bigcup_{i\in Q_0}\{\lambda_{i,1},\ldots,\lambda_{i,\mathbf{d}_i}\}$ of generalised eigenvalues of $\rho(\omega_i)$ contains only one distinct element $\lambda \in \mathbb{C}$. The action of $\sum_{i\in Q_0}\omega_i$ on ρ is by multiplication by the constant λ.*

Lemma 4.2. *There are isomorphisms of $\mathbb{N}^{Q_0}$-graded mixed Hodge structures*

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mu^{-1}_{Q,\mathbf{d}}(0)/\mathrm{GL}_{\mathbf{d}},\mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \tag{4.1}$$
$$\cong \mathrm{Dim}_! \,\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}(\mathbb{Q})$$
$$\cong \mathrm{Sym}_{\boxtimes_+}(\mathrm{BPS}^{\omega\text{-nilp},\vee}_{\widetilde{Q},\widetilde{W}} \otimes \mathbb{L} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}), \tag{4.2}$$
$$\mathrm{Dim}_!\big((\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}(\mathbb{Q}))|_{\mathfrak{M}(\widetilde{Q})^{\omega\text{-nilp}}}\big) \tag{4.3}$$
$$\cong \mathrm{Sym}_{\boxtimes_+}(\mathrm{BPS}^{\omega\text{-nilp},\vee}_{\widetilde{Q},\widetilde{W}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}).$$

Lemma 4.3 ([7, Theorem 3.4]). *The $\mathbb{N}^{Q_0}$-graded mixed Hodge structure*

$$\mathrm{Dim}_!\big((\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}(\mathbb{Q}))|_{\mathfrak{M}(\widetilde{Q})^{\omega\text{-nilp}}}\big)$$

is pure, of Tate type.

Assuming Lemmas 4.1, 4.2 and 4.3, we argue as follows.

Proof of Theorem A. First, note that a graded mixed Hodge structure $\mathcal{F}$ is pure, of Tate type, if and only if $\mathrm{Sym}(\mathcal{F})$ is. Lemma 4.3 and (4.3) thus imply that $\mathrm{BPS}^{\omega\text{-nilp},\vee}_{\widetilde{Q},\widetilde{W}}$ is pure, of Tate type. A tensor product of pure mixed Hodge modules is pure, and so $\mathrm{BPS}^{\omega\text{-nilp},\vee}_{\widetilde{Q},\widetilde{W}} \otimes \mathbb{L}$ is also pure, of Tate type. It follows from (4.1) and (4.2) that

$$\mathrm{Dim}_!\,\phi_{\mathfrak{Tr}(W)}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}(\mathbb{Q}) \quad \text{and} \quad \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mu^{-1}_{Q,\mathbf{d}}(0)/\mathrm{GL}_{\mathbf{d}},\mathbb{Q})$$

are pure, of Tate type, and the theorem follows. □

Before coming to the proof of Lemmas 4.1, 4.2 and 4.3, we note the following.

Corollary 4.4. *For all $\mathbf{d} \in \mathbb{N}^{Q_0}$, the BPS cohomology $\mathrm{BPS}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ is pure, of Tate type.*

Proof. From Theorem (3.6), we deduce that there is an inclusion of mixed Hodge structures $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathbb{L}^{1/2} \hookrightarrow \mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})^\vee \otimes \mathbb{L}^{-(\mathbf{d},\mathbf{d})}$. By Theorem A, the target is pure, of Tate type, and so $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathbb{L}^{1/2}$ is also pure, of Tate type. It follows that the Tate twist $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}$ is pure, of Tate type. □

The proof of both Lemmas 4.2 and 4.3 will use the dimensional reduction theorem, recalled as Theorem 3.4. Let Q^+ be obtained from $\tilde{Q}$ by deleting all of the arrows a^*, and let Q^{op} be obtained from $\tilde{Q}$ by deleting all the arrows a and all the loops ω_i. We decompose

$$X(\tilde{Q})_{\mathbf{d}} = X(Q^+)_{\mathbf{d}} \times X(Q^{\mathrm{op}})_{\mathbf{d}}.$$

If we let $\mathbb{C}^*$ act on $X(\tilde{Q})_{\mathbf{d}}$ via the trivial action on $X(Q^+)_{\mathbf{d}}$ and the weight one action on $X(Q^{\mathrm{op}})_{\mathbf{d}}$, then $\mathrm{Tr}(\tilde{W})_{\mathbf{d}}$ is $\mathbb{C}^*$-equivariant in the manner required to apply Theorem 3.4. In the notation of Theorem 3.4, we have that $Z' \subset X(Q^+)_{\mathbf{d}}$ is determined by the vanishing of the matrix-valued functions, $\partial W/\partial a^* = a\omega_{s(a)} - \omega_{t(a)}a$ for $a \in Q_1$. Concretely, the stack $Z'/\mathrm{GL}_{\mathbf{d}}$ is isomorphic to the stack of pairs (ρ, f), where ρ is a $\mathbf{d}$-dimensional Q-representation, and $f\colon \rho \to \rho$ is an endomorphism in the category of Q-representations.

We fix $X(Q^+)^{\omega\text{-nilp}}_{\mathbf{d}} \subset X(Q^+)_{\mathbf{d}}$ to be the subspace of representations such that each $\rho(\omega_i)$ is nilpotent. We deduce from Theorem 3.4 that there is a natural isomorphism in compactly supported cohomology

$$\begin{aligned}(4.4)\qquad \mathrm{Dim}_!\big((\phi_{\mathfrak{Tr}(W)}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})_{\mathbf{d}}}(\mathbb{Q}))|_{\mathfrak{M}(\tilde{Q})^{\omega\text{-nilp}}_{\mathbf{d}}}\big)\\ \cong \mathrm{H}_c\big((Z' \cap X(Q^+)^{\omega\text{-nilp}}_{\mathbf{d}})/\mathrm{GL}_{\mathbf{d}}, \mathbb{Q}\big).\end{aligned}$$

Lemma 4.3 is proved in [7] by analysing the right-hand side of (4.4). Note that there is no overall Tate twist in (4.4) – the Tate twist in the definition of the left-hand side is cancelled by the Tate twist appearing in Theorem 3.4.

The first isomorphism in Lemma 4.2 is obtained in similar fashion. Let $L \subset \tilde{Q}$ be the quiver obtained by deleting all of the arrows a and a^*, for $a \in Q_1$. Then we can decompose

$$X(\tilde{Q})_{\mathbf{d}} \cong X(\overline{Q})_{\mathbf{d}} \times X(L)_{\mathbf{d}},$$

and let $\mathbb{C}^*$ act on $X(\tilde{Q})_{\mathbf{d}}$ via the trivial action on $X(\overline{Q})_{\mathbf{d}}$ and the scaling action on $X(L)_{\mathbf{d}}$. This time the role of Z' in Theorem 3.4 is played by $\mu^{-1}_{\overline{Q},\mathbf{d}}(0) \subset X(\overline{Q})_{\mathbf{d}}$, and we deduce that

$$(4.5)\qquad \mathrm{Dim}_!\, \phi_{\mathfrak{Tr}(\tilde{W})_{\mathbf{d}}}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})_{\mathbf{d}}}(\mathbb{Q}) \cong \mathrm{H}_c(\mu^{-1}_{\overline{Q},\mathbf{d}}(0)/\mathrm{GL}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})}.$$

Proof of Lemma 4.2. Since the map $\dim\colon \mathcal{M}(\tilde{Q}) \to \mathbb{N}^{Q_0}$ is a morphism of commutative monoids, with proper monoid maps $\oplus$ and $+$, respectively, by [30, Section 1.12], there is a natural equivalence of functors $\dim_! \mathrm{Sym}_{\boxtimes_\oplus} \cong \mathrm{Sym}_{\boxtimes_+} \dim_!$. We denote by

$$\iota'_{\mathbf{d}}\colon \mathcal{M}(\tilde{Q})^{\omega\text{-nilp}}_{\mathbf{d}} \hookrightarrow \mathcal{M}(\tilde{Q})_{\mathbf{d}}$$

the inclusion. Taking the direct sum over all $\mathbf{d} \in \mathbb{N}^{Q_0}$, applying base change, and using the relative cohomological integrality theorem (Theorem 3.5),

$$\begin{aligned}&\mathrm{Dim}_!\big((\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})}(\mathbb{Q}))|_{\mathfrak{M}(\tilde{Q})^{\omega\text{-nilp}}}\big)\\ &\cong \dim_! \iota'^* \mathrm{JH}_! \phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})}(\mathbb{Q})\\ &\cong \dim_! \iota'^* \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{\tilde{Q},\tilde{W}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}})\end{aligned}$$

$$\cong \dim_! \operatorname{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{\tilde{Q},\tilde{W}}|_{\mathcal{M}(\tilde{Q})^{\omega\text{-nilp}}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})$$
$$\cong \operatorname{Sym}_{\boxtimes_+}(\dim_! \mathcal{BPS}_{\tilde{Q},\tilde{W}}|_{\mathcal{M}(\tilde{Q})^{\omega\text{-nilp}}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})$$
$$\cong \operatorname{Sym}_{\boxtimes_+}(\dim_! \mathcal{BPS}_{\tilde{Q},\tilde{W}}|_{\mathcal{M}(\tilde{Q})^{\omega\text{-nilp}}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}),$$

giving isomorphism (4.3). Taking the direct sum of isomorphisms (4.5) over $\mathbf{d} \in \mathbb{N}^{Q_0}$ gives isomorphism (4.1). Applying $\dim_!$ to (3.2), we have the isomorphisms

$$\begin{aligned}\dim_! \mathrm{JH}_! \phi_{\mathfrak{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})}(\mathbb{Q}) &\cong \dim_! \operatorname{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{\tilde{Q},\tilde{W}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})\\ &\cong \operatorname{Sym}_{\boxtimes_+} \dim_!(\mathcal{BPS}_{\tilde{Q},\tilde{W}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})\\ &\cong \operatorname{Sym}_{\boxtimes_+}(\mathrm{BPS}_{\tilde{Q},\tilde{W}} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}).\end{aligned}$$

To prove the existence of isomorphism (4.2), then it is sufficient to prove that

$$\mathrm{BPS}_{\tilde{Q},\tilde{W}} \cong \mathrm{BPS}^{\omega\text{-nilp}}_{\tilde{Q},\tilde{W}} \otimes \mathbb{L}.$$

Fix a dimension vector $\mathbf{d}$. We let $\mathbb{A}^1$ act on $\mathcal{M}(\tilde{Q})_{\mathbf{d}}$ as follows:

$$z \cdot \rho(a) = \begin{cases} \rho(a) + z\mathrm{id}_{\mathbf{d}_i \times \mathbf{d}_i} & \text{if } a = \omega_i \text{ for some } i,\\ \rho(a) & \text{otherwise.}\end{cases}$$

Then $\mathcal{T}r(\tilde{W})_{\mathbf{d}}$ is invariant with respect to the $\mathbb{A}^1$-action and it follows that the underlying perverse sheaf of $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}$ can be obtained from an $\mathbb{A}^1$-equivariant MHM via the forgetful map. Let $\mathcal{BPS}'_{\tilde{Q},\tilde{W},\mathbf{d}}$ be the restriction of $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}$ to the locus $\mathcal{M} \subset \mathcal{M}(\tilde{Q})_{\mathbf{d}}$, where the union of the sets of generalised eigenvalues of all of the ω_i has only one element, and let

$$m\colon \mathbb{A}^1 \times \mathcal{M}(\tilde{Q})^{\omega\text{-nilp}}_{\mathbf{d}} \xrightarrow{\cong} \mathcal{M}$$

be the restriction of the action map. This is an isomorphism since, for a module represented by a point in $\mathcal{M}$, there exists a $z \in \mathbb{C}$ such that, adding $z \cdot \mathrm{Id}_{\mathbf{d}_i \times \mathbf{d}_i}$ to the action of each of the ω_i, they all become nilpotent. We have

$$\mathcal{BPS}'_{\tilde{Q},\tilde{W},\mathbf{d}} \cong m_*(\mathbb{Q}_{\mathbb{A}^1} \boxtimes \mathcal{BPS}^{\omega\text{-nilp}}_{\tilde{Q},\tilde{W}}).$$

By the support lemma (Lemma 4.1), we have $\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} = \mathcal{BPS}'_{\tilde{Q},\tilde{W},\mathbf{d}}$, and so we deduce that

$$\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} \cong \mathrm{BPS}^{\omega\text{-nilp}}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes (\mathbb{A}^1 \to \mathrm{pt})_! \mathbb{Q}_{\mathbb{A}^1} \cong \mathrm{BPS}^{\omega\text{-nilp}}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathbb{L},$$

as required. □

We complete the proof of Theorem A by proving the support lemma.

Proof of Lemma 4.1. To ease the notation, we prove the lemma under the assumption that ζ is the degenerate stability condition: the proof for the general case is unchanged. Since the support of $\mathcal{BPS}_{\tilde{Q},\tilde{W}}$ is the same as the support of the underlying perverse sheaf, and all complexes that we encounter in the following proof are quasi-isomorphic to their total cohomology, throughout the proof, we work in the category of cohomologically graded perverse sheaves.

Let $x \in \mathcal{M}(\tilde{Q})_{\mathbf{d}}$ be a point corresponding to a semisimple $\mathbb{C}\tilde{Q}$-module ρ, and assume that there are at least two distinct eigenvalues ϵ_1, ϵ_2 for the set of operators $\{\rho(\omega_i) \mid i \in Q_0\}$.

Assume, for a contradiction, that $x \in \mathrm{supp}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}})$ so that, in particular,

$$x \in \mathrm{supp}(\mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})} \mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}(\mathbb{Q})),$$

and so, by (2.1) and Remark 3.1, there exists a $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$ module with semisimplification given by ρ, and so ρ is a semisimple $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-module.

Under our assumptions, there are disjoint (analytic) open sets $U_1, U_2 \subset \mathbb{C}$ with $\epsilon_1 \in U_1$ and $\epsilon_2 \in U_2$, and with all of the generalised eigenvalues of ρ contained in $U_1 \cup U_2$. Given an (analytic) open set $U \subset \mathbb{C}$, we denote by $\mathcal{M}^U(\widetilde{Q})_{\mathbf{d}} \subset \mathcal{M}(\widetilde{Q})_{\mathbf{d}}$ the subspace consisting of those ρ such that all of the generalised eigenvalues of $\{\rho(\omega_i) \mid i \in Q_0\}$ belong to U, and we define $\mathfrak{M}^U(\widetilde{Q})$ similarly. Given a point $x \in \mathfrak{M}^{U_1 \cup U_2}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))$, the associated $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-module M admits a canonical direct sum decomposition $M = M_1 \oplus M_2$ for which all of the eigenvalues of all of the ω_i, restricted to M_i, belong to U_i.[4] In particular, there is an isomorphism of complex analytic stacks

$$\mathfrak{M}^U(\mathrm{Jac}(\widetilde{Q},\widetilde{W})) \cong \mathfrak{M}^{U_1}(\mathrm{Jac}(\widetilde{Q},\widetilde{W})) \times \mathfrak{M}^{U_2}(\mathrm{Jac}(\widetilde{Q},\widetilde{W})). \tag{4.6}$$

By Lemma 4.5, proved below, there is an isomorphism

$$\begin{aligned} \mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}|_{\mathfrak{M}^{U_1 \cup U_2}(\widetilde{Q})} \cong \mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})} \mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}|_{\mathfrak{M}^{U_1}(\widetilde{Q})} \\ \boxtimes_{\oplus} \mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})} \mathcal{IC}_{\mathfrak{M}(\widetilde{Q})}|_{\mathfrak{M}^{U_2}(\widetilde{Q})}. \end{aligned} \tag{4.7}$$

Applying (3.1) to the right-hand side of (4.7), we have isomorphisms

$$\begin{aligned} &\mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}|_{\mathfrak{M}^{U_1 \cup U_2}(\widetilde{Q})} \\ &\quad \cong \mathrm{Sym}_{\boxtimes_{\oplus}}\big((\mathcal{BPS}_{\widetilde{Q},\widetilde{W}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})|_{\mathcal{M}^{U_1}(\widetilde{Q})}\big) \\ &\qquad \boxtimes_{\oplus} \mathrm{Sym}_{\boxtimes_{\oplus}}\big((\mathcal{BPS}_{\widetilde{Q},\widetilde{W}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})|_{\mathcal{M}^{U_2}(\widetilde{Q})}\big) \\ &\quad \cong \mathrm{Sym}_{\boxtimes_{\oplus}}\big((\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1}(\widetilde{Q})} \oplus \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_2}(\widetilde{Q})}) \\ &\qquad\qquad \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}\big). \end{aligned} \tag{4.8}$$

On the other hand, restricting the isomorphism of (3.1) to the left-hand side of (4.7) yields

$$\mathtt{JH}_* \phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}|_{\mathfrak{M}^{U_1 \cup U_2}(\widetilde{Q})} \cong \mathrm{Sym}_{\boxtimes_{\oplus}}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1 \cup U_2}(\widetilde{Q})} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}). \tag{4.9}$$

Comparing (4.8) and (4.9), we deduce that

$$\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1 \cup U_2}(\widetilde{Q})} \cong \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1}(\widetilde{Q})} \oplus \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_2}(\widetilde{Q})}.$$

We deduce that

$$\begin{aligned} \mathrm{supp}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1 \cup U_2}(\widetilde{Q})}) &= \mathrm{supp}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1}(\widetilde{Q})} \oplus \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_2}(\widetilde{Q})}) \\ &= \mathrm{supp}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_1}(\widetilde{Q})}) \cup \mathrm{supp}(\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}|_{\mathcal{M}^{U_2}(\widetilde{Q})}) \\ &\subset \mathcal{M}^{U_1}(\widetilde{Q}) \cup \mathcal{M}^{U_2}(\widetilde{Q}), \end{aligned}$$

and so, since $x \in \mathcal{M}^{U_1 \cup U_2}(\widetilde{Q}) \setminus (\mathcal{M}^{U_1}(\widetilde{Q}) \cup \mathcal{M}^{U_2}(\widetilde{Q}))$, the restriction of $\mathcal{BPS}_{\widetilde{Q},\widetilde{W}}$ to x is zero, which is the required contradiction.

[4] Note that this is not true of a general point in $\mathfrak{M}^{U_1 \cup U_2}(\widetilde{Q})$; the crucial fact is that the operation $\sum_{i \in Q_0} \rho(\omega_i) \cdot$ defines a module homomorphism for a $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-module ρ as $\sum_{i \in Q_0} \omega_i$ is central in $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$.

For the final statement of the lemma, it suffices to prove that if ρ is a simple $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-module, then $\sum_{i\in Q_0}\rho(\omega_i)$ acts via scalar multiplication. In the decomposition of ρ into generalised eigenspaces for the action of the operator $\sum_{i\in Q_0}\rho(\omega_i)\cdot$, we have already shown that there is only one generalised eigenvalue, which we denote λ. Then ρ is filtered by the nilpotence degree of the nilpotent operator $\Psi:=\sum_{i\in Q_0}\rho(\omega_i)\cdot-\lambda\,\mathrm{Id}_\rho$, and so since ρ is simple, $\Psi=0$ and we are done. □

4.2. Proof of Theorem B. Firstly, by Lemma 4.1, the support of $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ lies in the image of the morphism $\mathbb{A}^1\times\mathcal{M}(\overline{Q})^{\zeta\text{-ss}}\to\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}$ defined as in the statement of Theorem B. The support of $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ also lies within the locus of polystable $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-modules, so by Proposition 2.1, the support of $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ lies within the image of

$$m\colon\mathbb{A}^1\times\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}\hookrightarrow\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}.$$

We have seen in the proof of Lemma 4.2 that $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ is $\mathbb{A}^1$-equivariant, where the $\mathbb{A}^1$-action on the subspace $\mathbb{A}^1\times\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}$ is via translation in the first factor. It follows that we can write

$$\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}\cong\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})\boxtimes\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}. \tag{4.10}$$

Finally, $\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}=\phi_{\mathfrak{Tr}(W)}\mathcal{IC}_{\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}$ is Verdier self-dual [10], as is $\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})$. So from (4.10), we deduce

$$\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})\boxtimes\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}\cong\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})\boxtimes\mathbb{D}\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}$$

and Verdier self-duality of $\mathcal{BPS}^{\zeta}_{\Pi_Q,\mathbf{d}}$ follows.

We finish this section with the technical lemma appearing in the proof of Lemma 4.1. Fix a decomposition $\mathbf{d}=\mathbf{d}'+\mathbf{d}''$. Then, via (4.6), there is an open and closed inclusion

$$i\colon\mathfrak{M}^{U_1}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))_{\mathbf{d}'}\times\mathfrak{M}^{U_2}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))_{\mathbf{d}''}\to\mathfrak{M}^{U_1\cup U_2}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))_{\mathbf{d}}.$$

Lemma 4.5. *Let U_1,U_2 be disjoint analytic open subspaces of $\mathbb{A}^1$. There is a natural isomorphism of perverse sheaves*

$$i^*\phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}^{U_1\cup U_2}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}(\mathbb{Q})\cong\phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}^{U_1}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}'}}(\mathbb{Q})\boxtimes\phi^{\mathfrak{p}}_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\mathfrak{M}^{U_2}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}''}}(\mathbb{Q}).$$

This isomorphism does *not* follow directly from the Thom–Sebastiani isomorphism since we need to compare the vanishing cycle sheaf of the function $\mathfrak{Tr}(\widetilde{W})\boxplus\mathfrak{Tr}(\widetilde{W})$ on

$$\mathfrak{M}^{U_1}(\widetilde{Q})_{\mathbf{d}'}\times\mathfrak{M}^{U_2}(\widetilde{Q})_{\mathbf{d}''}$$

with the vanishing cycle sheaf for the function $\mathfrak{Tr}(\widetilde{W})$ on $\mathfrak{M}^{U_1\cup U_2}(\widetilde{Q})_{\mathbf{d}}$, and these ambient smooth stacks are different.

Proof of Lemma 4.5. Again, it is sufficient to prove the lemma for the degenerate stability condition (the general case then follows by restriction to the ζ-semistable locus). Writing

$$Y=X^{U_1\cup U_2}(L)_{\mathbf{d}}\times X(\overline{Q})_{\mathbf{d}},\quad B=X^{U_1}(\widetilde{Q})_{\mathbf{d}'}\times X^{U_2}(\widetilde{Q})_{\mathbf{d}''},$$

where L is the quiver with vertices Q_0 and arrows ω_i for $i \in Q_0$, we have

$$\mathfrak{M}^{U_1\cup U_2}(\widetilde{Q})_{\mathbf{d}} \cong Y/\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}, \quad \mathfrak{M}^{U_1}(\widetilde{Q})_{\mathbf{d}'} \times \mathfrak{M}^{U_1}(\widetilde{Q})_{\mathbf{d}''} \cong B/\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}.$$

The space Y is the total space of the $\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}$-equivariant vector bundle $V^+ \oplus V^-$ on B, where

$$V^+ = \prod_{a\in\overline{Q}_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}'_{s(a)}}, \mathbb{C}^{\mathbf{d}''_{t(a)}}), \quad V^- = \prod_{a\in\overline{Q}_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}'_{t(a)}}, \mathbb{C}^{\mathbf{d}''_{s(a)}}).$$

Note that $\mathrm{rank}(V^+) = \mathrm{rank}(V^-)$. Denote by $z\colon B \to Y$ the inclusion of the zero section. Writing f, g for the functions on Y, B induced by $\mathrm{Tr}(\widetilde{W})$, it is sufficient to show that there is an isomorphism of $\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}$-equivariant perverse sheaves $\phi^{\mathfrak{p}}_f \mathcal{IC}_Y(\mathbb{Q}) \cong z_*\phi^{\mathfrak{p}}_g \mathcal{IC}_B(\mathbb{Q})$.

Let $\mathbb{C}^*$ act on V^+ and V^- with weights 1 and -1, respectively; then f is $\mathbb{C}^*$-invariant. It follows that $g|_{\mathrm{Tot}(V^+)} = f \circ \pi^+$, where $\pi^+\colon \mathrm{Tot}(V^+) \to B$ is the projection. So there is a natural isomorphism

$$\phi^{\mathfrak{p}}_g \mathbb{Q}_B \cong \pi^+_! \phi^{\mathfrak{p}}_f \mathbb{Q}_{\mathrm{Tot}(V^+)}[2\,\mathrm{rank}(V^+)]. \tag{4.11}$$

We claim that the natural morphism

$$\pi_!\phi^{\mathfrak{p}}_f(\mathbb{Q}_Y \to i^+_*\mathbb{Q}_{\mathrm{Tot}(V^+)}) \tag{4.12}$$

is an isomorphism, where we denote by $i^+\colon \mathrm{Tot}(V^+) \to Y$ the inclusion. This can be checked locally on the base B. Pick $b \in B$, and let $x_1, \dots, x_\alpha, y_1, \dots, y_\beta, z_1, \dots, z_\beta$ be a set of elements of the local ring $\mathcal{O}_{X(\widetilde{Q}),b}$, providing a basis for $\mathfrak{m}_b/\mathfrak{m}_b^2$, where x_i all have weight zero, y_i have weight 1, and z_i have weight -1 for the $\mathbb{C}^*$-action. The weight -1 partial derivatives of f are provided by $\partial g/\partial y_i$, and so, since the critical locus of g (restricted to a neighbourhood of b) lies on the zero section B, it follows that we can change coordinates and pick $z_i = \partial g/\partial y_i$. Then we have $g = h + k$ in $\mathcal{O}_{X(\widetilde{Q}),b}$, with $h \in \mathbb{C}[x_1, \dots, x_\alpha]$ and $k = \sum_{1\le i\le\beta} y_i z_i$. By the Thom–Sebastiani theorem, after restricting to a neighbourhood $U \ni b$, we find $\phi^{\mathfrak{p}}_g \mathbb{Q}_{U\times V} \cong \phi^{\mathfrak{p}}_h \mathbb{Q}_U \boxtimes \phi^{\mathfrak{p}}_k \mathbb{Q}_V$ and the claim reduces to the claim that $\pi_!\phi^{\mathfrak{p}}_k(\mathbb{Q}_V \to (V^+ \hookrightarrow V)_*\mathbb{Q}_{V^+})$ is an isomorphism, which is a simple calculation, or a trivial application of the dimensional reduction theorem.

Combining (4.11) and (4.12) yields the isomorphism $\phi^{\mathfrak{p}}_g \mathbb{Q}_B \cong \pi_!\phi^{\mathfrak{p}}_f \mathbb{Q}_Y[\mathrm{codim}_Y(B)]$. Since $\phi^{\mathfrak{p}}_f \mathbb{Q}_Y$ is supported on B, we obtain the required isomorphism by applying z_* to this isomorphism and shifting cohomological degree by $\dim B$. □

4.3. Calculating $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$. We use Theorem A and existing results on the E series of $\mathfrak{M}(\Pi_Q)_{\mathbf{d}}$ to determine the compactly supported cohomology of $\mathfrak{M}(\Pi_Q)_{\mathbf{d}}$, along with its mixed Hodge structure. The E series (see Section 1.2) of $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$ was calculated in [32].

Recall the plethystic exponential defined in Section 1.9. We define the ring

$$\mathbb{Z}((X_1, \dots, X_m))[[Y_1, \dots, Y_n]]$$

of formal Laurent power series

$$g(X_1, \dots, X_m, Y_1, \dots, Y_n)$$

such that, for each $(a_1, \dots, a_n) \in \mathbb{N}^n$, the $Y_1^{a_1} \cdots Y_n^{a_n}$ coefficient of

$$g(X_1, \dots, X_m, Y_1, \dots, Y_n) X_1^{c_1} \cdots X_m^{c_m}$$

is in $\mathbb{Z}[[X_1, \dots, X_m]]$ for sufficiently large $c_1, \dots, c_m$. This is isomorphic to the Grothendieck ring of the category $\mathcal{D}^{\diamond}(\mathrm{Vect}_{\mathbb{Z}^m \oplus \mathbb{Z}^n})$, which we define to be the subcategory of the unbounded derived category of $\mathbb{Z}^m \oplus \mathbb{Z}^n$-graded vector spaces V such that

(i) for each $(\mathbf{e}, \mathbf{d}) \in \mathbb{Z}^m \oplus \mathbb{Z}^n$, the total cohomology $\mathrm{H}(V)_{\mathbf{e},\mathbf{d}}$ is finite-dimensional,

(ii) $\mathrm{H}(V_{\mathbf{e},\mathbf{d}}) \neq 0$ only if $\mathbf{d} \in \mathbb{N}^n$,

(iii) for each $\mathbf{d} \in \mathbb{N}^n$, there exists $\mathbf{e} \in \mathbb{Z}^m$ such that we have $\mathrm{H}(V)_{\mathbf{e}',\mathbf{d}} = 0$ if $\mathbf{e}'_i \leq \mathbf{e}_i$ for some $i = 1, \dots, m$.

This isomorphism is induced by the character function

$$\chi\colon [V] \mapsto \sum_{i \in \mathbb{Z}} \sum_{(\mathbf{e},\mathbf{d}) \in \mathbb{Z}^m \oplus \mathbb{Z}^n} (-1)^i \dim(\mathrm{H}^i(V)_{\mathbf{e},\mathbf{d}}) X^{\mathbf{e}} Y^{\mathbf{d}}.$$

We define $\mathcal{D}^{\diamond}(\mathrm{Vect}^{+}_{\mathbb{Z}^m \oplus \mathbb{Z}^n}) \subset \mathcal{D}^{\diamond}(\mathrm{Vect}_{\mathbb{Z}^m \oplus \mathbb{Z}^n})$ to be the full subcategory satisfying the extra condition that the total cohomology $\mathrm{H}(V)_{(\mathbf{e},0)}$ is zero for all $\mathbf{e} \in \mathbb{Z}^m$. Then χ induces an isomorphism

$$\chi\colon \mathrm{K}_0(\mathcal{D}^{\diamond}(\mathrm{Vect}^{+}_{\mathbb{Z}^m \oplus \mathbb{Z}^n})) \to \mathfrak{m}\mathbb{Z}((X_1, \dots, X_m))[[Y_1, \dots, Y_n]],$$

where $\mathfrak{m}$ is the maximal ideal generated by $Y_1, \dots, Y_n$. We may define plethystic exponentiation via the formula $\mathrm{Exp}(\chi([V])) = \chi[\mathrm{Sym}(V)]$ for $V \in \mathcal{D}^{\diamond}(\mathrm{Vect}^{+}_{\mathbb{Z}^m \oplus \mathbb{Z}^n})$. Then the E series for $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$ is given by (see [32])

$$\begin{aligned} \sum_{\mathbf{d} \in \mathbb{N}^{Q_0}} & \mathrm{E}\big(\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}), x, y\big)(xy)^{(\mathbf{d},\mathbf{d})} t^{\mathbf{d}} \\ &= \mathrm{Exp}\Big(\sum_{0 \neq \mathbf{d} \in \mathbb{N}^{Q_0}} \mathtt{a}_{Q,\mathbf{d}}(xy)(1 - x^{-1}y^{-1})^{-1} t^{\mathbf{d}}\Big). \end{aligned} \tag{4.13}$$

Here x^{-1} and y^{-1} are the invertible commuting variables, and $\{t_i\}_{i \in Q_0}$ are the other commuting variables. Each of the (xy) terms arises from the E polynomial $\mathrm{E}(\mathbb{L}, x, y) = xy$. Given a polynomial $b(q) = \sum_{i \geq 0} b_i q^i \in \mathbb{N}[q]$ and an object $\mathcal{F}$ in a tensor category $\mathscr{C}$, we define $b(\mathcal{F}) = \bigoplus_{i \in \mathbb{N}} (\mathcal{F}^{\otimes i})^{\oplus b_i}$. By Theorem A, the mixed Hodge structure on $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q})$ is entirely determined by its E series, and we deduce from (4.13) the following result.

Theorem 4.6. *There is an isomorphism of cohomologically graded,* $\mathbb{N}^{Q_0}$*-graded mixed Hodge structures*

$$\bigoplus_{\mathbf{d} \in \mathbb{N}_{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_Q} \cong \mathrm{Sym}\Big(\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}} \mathtt{a}_{Q,\mathbf{d}}(\mathbb{L}) \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})^{\vee}\Big).$$

5. Degree zero DT theory

5.1. Degree zero BPS sheaves. For $n \in \mathbb{N}$, we define $Q^{(n)}$ to be a quiver with one vertex, denoted 0, and n loops. We will be particularly interested in the quiver $Q_{\mathrm{Jor}} := Q^{(1)}$: the *Jordan quiver*. We identify $Q^{(3)} = \widetilde{Q_{\mathrm{Jor}}}$. We denote by x, y, z the three arrows of $Q^{(3)}$. Then $\widetilde{W} = x[y, z]$. The ideas in the proof of Theorem A allow us to prove rather more for the

QP $(\widetilde{Q_{\mathrm{Jor}}}, \tilde{W})$, essentially because this QP is invariant (up to sign) under permutation of the loops so that we can apply the support lemma (Lemma 4.1) three times.

Let $d \in \mathbb{N}$ with $d \geq 1$. The support of $\mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}},!}\phi_{\mathfrak{Tr}(\tilde{W})_d}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})_d}(\mathbb{Q})$ is given by the coarse moduli space of d-dimensional representations of the Jacobi algebra $\mathbb{C}[x, y, z]$, i.e. the space of semisimple representations of $\mathbb{C}[x, y, z]$. This space is in turn isomorphic to $\mathrm{Sym}^d(\mathbb{A}^3)$ since any simple representation ρ of $\mathbb{C}[x, y, z]$ is one-dimensional and determined up to isomorphism by the three complex numbers $\rho(x), \rho(y), \rho(z)$.

Theorem 5.1. *For all $d \geq 1$, there is an isomorphism of MHMs*

$$\mathcal{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d} \cong \Delta_{\mathbb{A}^3,d,*}\mathcal{IC}_{\mathbb{A}^3}(\mathbb{Q}).$$

Proof. By the same argument as for Lemma 4.1, the support of $\mathcal{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d}$ is contained in the image of the morphism

$$\Delta_{\mathbb{A}^3,d}\colon \mathbb{A}^3 \to \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})_d, \quad (z_1, z_2, z_3) \mapsto (z_1 \cdot \mathrm{Id}_{d\times d}, z_2 \cdot \mathrm{Id}_{d\times d}, z_3 \cdot \mathrm{Id}_{d\times d}).$$

By the argument in the proof of Lemma 4.2, $\mathcal{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d}$ is constant on its support, so $\mathcal{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d} \cong \Delta_{\mathbb{A}^3,d,*}\mathcal{IC}_{\mathbb{A}^3}(\mathbb{Q}) \otimes \mathcal{L}_d$ for some mixed Hodge structure $\mathcal{L}_d$. It follows that

$$\mathrm{BPS}^{\vee}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d} \cong \mathcal{L}_d^{\vee} \otimes \mathbb{L}^{3/2}. \tag{5.1}$$

On the other hand, by [1, Proposition 1.1], there is an equality

$$[\mathrm{BPS}^{\vee}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d}] = [\mathbb{L}^{3/2}]$$

in the Grothendieck ring of mixed Hodge structures. The mixed Hodge structure $\mathrm{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d}$ is pure by Corollary 4.4. We deduce that

$$\mathrm{BPS}^{\vee}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},d} \cong \mathbb{L}^{3/2},$$

and so, from (5.1), there is an isomorphism $\mathcal{L}_d \otimes \mathbb{L}^{3/2} \cong \mathbb{L}^{3/2}$, and we finally deduce that $\mathcal{L}_d \cong \mathbb{Q}$, with the standard pure weight zero mixed Hodge structure, as required. □

For any constructible inclusion $U \hookrightarrow \mathbb{C}^3$, there is an inclusion of triples of diagonal matrices with entries in U which we denote $\iota_{U,d}\colon \mathrm{Sym}^d(U) \hookrightarrow \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})_d$ as well as an inclusion $\Delta_{U,d}\colon U \hookrightarrow \mathrm{Sym}^d(U) \hookrightarrow \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})_d$ of the small diagonal. Taking disjoint unions of all these inclusions, we define the inclusions

$$\iota_U\colon \mathrm{Sym}(U) \hookrightarrow \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}}), \quad \Delta_U\colon \coprod_{d\geq 1} U \hookrightarrow \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}}).$$

We denote by $\mathfrak{M}(\mathbb{C}[x, y, z])_d^U$ the preimage of $\iota_{U,d}(\mathrm{Sym}^d(U))$ under the map

$$\mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}},d}\colon \mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})_d \to \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})_d.$$

We set $\mathfrak{Coh}^U(\mathbb{A}^3) = \left(\coprod_{d\geq 1} \mathfrak{M}(\mathbb{C}[x, y, z])_d^U\right) \amalg \mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})_0$. Then define the $\mathbb{N}$-graded, cohomologically graded mixed Hodge structure

$$\mathcal{A}_{\mathfrak{Coh}^U(\mathbb{A}^3)} := \mathrm{H}(\mathfrak{Coh}^U(\mathbb{A}^3), \phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}).$$

Combining Theorems 3.5, 3.6 and 5.1 gives the following.

Corollary 5.2. *There is an isomorphism in* $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})))$,

$$\begin{aligned}(\mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}},!}\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q}))|_{\mathrm{Sym}(U)}\\ \cong \mathrm{Sym}_{\boxtimes_\oplus}(\Delta_{U,*}\mathbb{Q}_{\coprod_{d\geq 1} U}\otimes\mathbb{L}^{-1}\otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})),\end{aligned}$$

and a PBW isomorphism of $\mathbb{N}$*-graded mixed Hodge structures*

$$\mathrm{Sym}\Big(\bigoplus_{d\in\mathbb{Z}_{\geq 1}}\mathrm{H}^{\mathrm{BM}}(U,\mathbb{Q})\otimes\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})\otimes\mathbb{L}^2\Big)\xrightarrow{\cong}\mathcal{A}_{\mathfrak{Coh}^U(\mathbb{A}^3)}. \tag{5.2}$$

Proof. We construct the first isomorphism as a special case of (3.2); via the same argument, we then realise (5.2) as a special case of Theorem 3.6. In fact, it is sufficient to construct the isomorphism in the case $U=\mathbb{C}^3$ since then the general case is given by restriction to $\iota_U(\mathrm{Sym}(U))$. In this case, since $\mathrm{supp}(\mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}},!}\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q}))=\mathrm{Sym}(\mathbb{A}^3)$, the proposed isomorphism becomes

$$\mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}},!}\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q})\cong\mathrm{Sym}_{\boxtimes_\oplus}(\Delta_{\mathbb{A}^3,*}\mathcal{IC}_{\coprod_{d\geq 1}\mathbb{A}^3}(\mathbb{Q})\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}),$$

which follows from (3.2) and Theorem 5.1. □

5.2. Applications to surfaces and character stacks. Let $j\colon V\hookrightarrow\mathbb{A}^2$ be the inclusion of a constructible subset, and write $U=V\times\mathbb{A}^1\subset\mathbb{A}^3$. We consider the commutative diagram

$$\begin{array}{ccc}\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}}) & \xrightarrow{\pi} & \mathfrak{M}(\overline{Q_{\mathrm{Jor}}})\\ \downarrow{\scriptstyle \mathtt{JH}_{\widetilde{Q_{\mathrm{Jor}}}}} & & \downarrow{\scriptstyle \mathtt{JH}_{\overline{Q_{\mathrm{Jor}}}}}\\ \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}}) & \xrightarrow{\varpi} & \mathcal{M}(\overline{Q_{\mathrm{Jor}}}),\end{array}$$

where the horizontal morphisms are the forgetful morphisms. We denote by $\mathfrak{C}_d=\mathcal{C}_d/\mathrm{GL}_d(\mathbb{C})$ the stack of commuting pairs of matrices, and set $\mathfrak{C}=\coprod_{d\in\mathbb{N}}\mathfrak{C}_d$. We define the inclusions

$$\iota_V\colon\mathrm{Sym}(V)\hookrightarrow\mathcal{M}(\overline{Q_{\mathrm{Jor}}}),\quad \Delta_V\colon\coprod_{d\geq 1}V\hookrightarrow\mathrm{Sym}(V)$$

as in the previous section. We denote by $\cup\colon\mathrm{Sym}(V)\times\mathrm{Sym}(V)\to\mathrm{Sym}(V)$ the morphism taking a pair of multisets of points to their union (so that ι_V is a morphism of monoids in the category of schemes). We define $\mathcal{F}\boxtimes_\cup\mathcal{G}:=\cup_*(\mathcal{F}\boxtimes\mathcal{G})$. We denote by $i\colon\mathfrak{C}\hookrightarrow\mathfrak{M}(\overline{Q_{\mathrm{Jor}}})$ the inclusion. By Theorem 3.4, there is an isomorphism of complexes of mixed Hodge modules

$$\pi_!\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q})\cong i_*\underline{\mathbb{Q}}_{\mathfrak{C}}. \tag{5.3}$$

We denote by $\mathfrak{Coh}^V(\mathbb{A}^2)$ the reduced substack of coherent sheaves on $\mathbb{A}^2$ set-theoretically supported on V with zero-dimensional support, and by $p\colon\mathfrak{Coh}^V(\mathbb{A}^2)\to\mathrm{Sym}(V)$ the morphism taking such a sheaf to its support, counted with multiplicity, so that p restricts to a morphism

$$p_d\colon\mathfrak{Coh}^V_d(\mathbb{A}^2)\to\mathrm{Sym}^d(V)$$

from the stack of coherent sheaves of length d. We define

$$\mathcal{A}_{\mathfrak{Coh}^V(\mathbb{A}^2)}:=\bigoplus_{d\geq 0}\mathrm{H}^{\mathrm{BM}}(\mathfrak{Coh}^V_d(\mathbb{A}^2),\mathbb{Q}).$$

Corollary 5.3. *Let $j\colon V \hookrightarrow \mathbb{A}^2$ be the inclusion of a constructible subset. Then there is an isomorphism of complexes of mixed Hodge modules*

$$p_!\mathbb{Q}_{\mathfrak{Coh}^V(\mathbb{A}^2)} \cong \mathrm{Sym}_{\cup}(\Delta_{V,*}\mathbb{Q}_{\coprod_{d\geq 1} V} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q}))$$

and a PBW isomorphism of $\mathbb{N}$-graded mixed Hodge structures

$$\mathrm{Sym}\Big(\bigoplus_{d\geq 1}\mathrm{H}^{\mathrm{BM}}(V,\mathbb{Q})\otimes\mathbb{L}\otimes\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})\Big) \xrightarrow{\cong} \mathcal{A}_{\mathfrak{Coh}^V(\mathbb{A}^2)}.$$

Proof. We denote by $\iota_V\colon \mathrm{Sym}(V) \to \mathcal{M}(\overline{Q_{\mathrm{Jor}}})$ the inclusion. Then compose the isomorphisms

$$\begin{aligned}
p_!\mathbb{Q}_{\mathfrak{Coh}^V(\mathbb{A}^2)} &\cong \iota_V^*\mathrm{JH}_{\overline{Q_{\mathrm{Jor}}},!}\mathbb{Q}_{\mathfrak{C}}\\
&\cong \iota_V^*\mathrm{JH}_{\overline{Q_{\mathrm{Jor}}},!}\pi_!\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q}) &(5.4)\\
&\cong \iota_V^*\varpi_!\mathrm{JH}_{\widetilde{Q_{\mathrm{Jor}}},!}\phi_{\mathfrak{Tr}(\tilde{W})}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q_{\mathrm{Jor}}})}(\mathbb{Q})\\
&\cong \iota_V^*\varpi_!\,\mathrm{Sym}_{\boxtimes_\oplus}(\Delta_{\mathbb{A}^3,*}\mathcal{IC}_{\coprod_{d\geq 1}\mathbb{A}^3}(\mathbb{Q})\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}) &(5.5)\\
&\cong \mathrm{Sym}_{\boxtimes_\oplus}(\iota_V^*\varpi_!\Delta_{\mathbb{A}^3,*}\mathcal{IC}_{\coprod_{d\geq 1}\mathbb{A}^3}(\mathbb{Q})\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}})\\
&\cong \mathrm{Sym}_{\boxtimes_\oplus}(\varpi_!\Delta_{U,*}\mathbb{Q}_{\coprod_{d\geq 1}U}\otimes\mathbb{L}^{-1}\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q}))\\
&\cong \mathrm{Sym}_{\boxtimes_\oplus}(\Delta_{V,*}\mathbb{Q}_{\coprod_{d\geq 1}V}\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})),
\end{aligned}$$

where (5.4) comes from (5.3) and isomorphism (5.5) comes from Corollary 5.2. This gives the first isomorphism; the PBW isomorphism follows by the same argument, and (5.2). □

For an application to nonabelian Hodge theory, we set $V = (\mathbb{C}^*)^2$ in Corollary 5.2. Set $A = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}\rangle$. There are identifications

$$\mathfrak{M}(A) = \mathfrak{Coh}_{\mathrm{cpct}}((\mathbb{C}^*)^2) = \mathfrak{M}(\pi_1(\Sigma_1))$$

of substacks of $\mathfrak{M}(\overline{Q_{\mathrm{Jor}}})$, where the final stack is the stack of finite-dimensional representations of the fundamental group of a genus 1 closed Riemann surface. From Corollary 5.3, we deduce the following result.

Corollary 5.4. *There is a PBW isomorphism of $\mathbb{N}$-graded mixed Hodge structures*

$$\bigoplus_{d\in\mathbb{N}}\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\pi_1(\Sigma_1))_d,\mathbb{Q}) \cong \mathrm{Sym}\Big(\bigoplus_{d\in\mathbb{Z}_{\geq 1}}\mathrm{H}^{\mathrm{BM}}((\mathbb{C}^*)^2,\mathbb{Q})\otimes\mathbb{L}\otimes\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})\Big).$$

The CoHA structure on the left-hand side of this isomorphism is introduced and studied in [4]. Given $g, d \in \mathbb{Z}_{\geq 1}$, consider the stack-theoretic quotient

$$\mathrm{Rep}_d^{\mathrm{tw}}(\Sigma_g) := \Big\{(A_1,\dots,A_g,B_1,\dots,B_g)\in\mathrm{GL}_d(\mathbb{C})^{\times 2g} \;\Big|\; \prod_{n=1}^{g}(A_n,B_n) = \exp(2\pi i/d)\cdot\mathrm{Id}_{d\times d}\Big\}/\mathrm{GL}_d(\mathbb{C}),$$

where the action is the simultaneous conjugation action. The action of $\mathrm{GL}_d(\mathbb{C})$ on the variety in brackets is not free, but it factors through the conjugation action by $\mathrm{PGL}_d(\mathbb{C})$, which is scheme-theoretically free by [20, Corollary 2.2.7], and the quotient

$$\overline{\mathrm{Rep}}_d^{\mathrm{tw}}(\Sigma_g) := \left\{ (A_1, \dots, A_g, B_1, \dots, B_g) \in \mathrm{GL}_d(\mathbb{C})^{\times 2g} \;\middle|\; \prod_{n=1}^{g} (A_n, B_n) = \exp(2\pi i/d) \cdot \mathrm{Id}_{d\times d} \right\} / \mathrm{PGL}_d(\mathbb{C})$$

is a smooth quasiprojective variety. It follows that there is an isomorphism

$$\mathrm{H}(\mathrm{Rep}_d^{\mathrm{tw}}(\Sigma_g), \mathbb{Q}) \cong \mathrm{H}(\overline{\mathrm{Rep}}_d^{\mathrm{tw}}(\Sigma_g, \mathbb{Q})) \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q}).$$

For $g = 1$, we have by [20, Theorem 2.2.17] that $\overline{\mathrm{Rep}}_d^{\mathrm{tw}}(\Sigma_1) \cong (\mathbb{C}^*)^2$. In general, we have the following conjecture.

Conjecture 5.5 ([4, Conjecture 1.1]). *There is an isomorphism of* $\mathbb{N}$*-graded cohomologically graded mixed Hodge structures*

$$\bigoplus_{d\in\mathbb{N}} \mathrm{H}_c\big(\mathfrak{M}(\pi_1(\Sigma_g))_d, \mathbb{Q}\big) \otimes \mathbb{L}^{(1-g)n^2} \cong \mathrm{Sym}\Big(\bigoplus_{d\geq 1} \mathrm{H}_c(\overline{\mathrm{Rep}}_d^{\mathrm{tw}}(\Sigma_g), \mathbb{Q}) \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q}) \otimes \mathbb{L}^{(1-g)n^2}\Big).$$

From Corollary 5.4, we deduce the $g = 1$ part of the following; the $g = 0$ case follows from [28, Section 1].

Theorem 5.6. *Conjecture* 5.5 *is true for* $g \leq 1$.

6. Generalisations of the purity theorem

6.1. The wall-crossing isomorphism in DT theory. The wall-crossing isomorphism in cohomological DT theory (e.g. [10, Theorem B]) provides a powerful way to deduce purity of Borel–Moore homology of moduli spaces of semistable quiver representations, for some stability condition ζ, from purity of Borel–Moore homology for some other stability condition ζ' (see e.g. [6] for an application of this principle for quantum cluster algebras). We will use this idea to prove a generalisation of Theorem A incorporating stability conditions.

Fix a quiver Q and a stability condition $\zeta \in \mathbb{H}_+^{Q_0}$. Let ρ be a finite-dimensional $\mathbb{C}Q$-module; then ρ admits a unique *Harder–Narasimhan* filtration $0 = \rho^0 \subset \cdots \subset \rho^s = \rho$ such that each ρ^t/ρ^{t-1} is ζ-semistable and the slopes $\varrho(\rho^1/\rho^0), \dots, \varrho(\rho^s/\rho^{s-1})$ are strictly descending. Given a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by

$$\mathrm{HN}_{\mathbf{d}} := \left\{ (\mathbf{d}^1, \dots, \mathbf{d}^s) \in (\mathbb{N}^{Q_0})^s \setminus \{0\} \;\middle|\; \varrho(\mathbf{d}^1) > \varrho(\mathbf{d}^2) > \cdots > \varrho(\mathbf{d}^s),\ \sum_{j\leq s} \mathbf{d}^j = \mathbf{d} \right\}$$

the set of Harder–Narasimhan types for $\mathbb{C}Q$-modules of dimension $\mathbf{d}$. For

$$\alpha = (\mathbf{d}^1, \dots, \mathbf{d}^s) \in \mathrm{HN}_{\mathbf{d}},$$

we denote $\mathbf{d}^j$ by α^j, and write $s(\alpha) = s$. For each $\alpha \in \mathrm{HN}_{\mathbf{d}}$, there is a locally closed quasiprojective subvariety $X(Q)_{[\alpha]} \subset X(Q)$ for which the closed points correspond exactly to those

$\mathbb{C}Q$-modules ρ of Harder–Narasimhan type α. For $\alpha \in \mathrm{HN}_{\mathbf{d}}$, define by $X(Q)_\alpha \subset X(Q)_{\mathbf{d}}$ the subspace of linear maps preserving the Q_0-graded flag $0 \subset \mathbb{C}^{\alpha^1} \subset \mathbb{C}^{\alpha^1+\alpha^2} \subset \cdots \subset \mathbb{C}^{\mathbf{d}}$ and such that each subquotient is ζ-semistable, and denote by $\mathrm{P}_\alpha \subset \mathrm{GL}_{\mathbf{d}}$ the subgroup preserving this same flag. Then the natural map $X(Q)_\alpha/\mathrm{P}_\alpha \to X(Q)_{[\alpha]}/\mathrm{GL}_{\mathbf{d}}$ is an isomorphism. We set $\mathfrak{M}(Q)_\alpha := X(Q)_\alpha/\mathrm{P}_\alpha$ and denote by $i_\alpha\colon \mathfrak{M}(Q)_\alpha \to \mathfrak{M}(Q)_{\mathbf{d}}$ the locally closed inclusion of substacks. By [35, Proposition 3.4], there is a decomposition into locally closed substacks $\mathfrak{M}(Q)_{\mathbf{d}} \cong \coprod_{\alpha\in\mathrm{HN}_{\mathbf{d}}} \mathfrak{M}(Q)_\alpha$.

Theorem 6.1. *For Q a quiver, $W \in \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$ a potential, and stability condition ζ, there is an isomorphism in $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathcal{M}(Q)))$,*

$$\text{(6.1)}\quad \mathrm{JH}_!\phi_{\mathfrak{Tr}(W)}\mathcal{IC}_{\mathfrak{M}(Q)}(\mathbb{Q}) \cong \bigoplus_{\substack{\mathbf{d}\in\mathbb{N}^{Q_0}\\ \alpha\in\mathrm{HN}_{\mathbf{d}}}} \Big(\boxtimes_{\oplus,1\leq j\leq s(\alpha)} q^{\zeta}_{\alpha^j,!}\mathrm{JH}^{\zeta}_{\alpha^j,!}\phi_{\mathfrak{Tr}(W)^{\zeta}_{\alpha^j}}\mathcal{IC}_{\mathfrak{M}(Q)^{\zeta\text{-ss}}_{\alpha^j}}(\mathbb{Q})\Big)\otimes\mathbb{L}^{f(\alpha)/2},$$

where $f((\mathbf{d}^1,\ldots,\mathbf{d}^s)) := \sum_{1\leq j'<j''\leq s}\langle \mathbf{d}^{j'},\mathbf{d}^{j''}\rangle$ and $q^{\zeta}_{\mathbf{d}}\colon \mathcal{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} \to \mathcal{M}(Q)_{\mathbf{d}}$ is the affinisation morphism. Taking the direct image to $\mathbb{N}^{Q_0}$, there is an isomorphism

$$\text{(6.2)}\quad \mathrm{Dim}_!\,\phi_{\mathfrak{Tr}(W)}\mathcal{IC}_{\mathfrak{M}(Q)}(\mathbb{Q}) \cong \bigoplus_{\substack{\mathbf{d}\in\mathbb{N}^{Q_0}\\ \alpha\in\mathrm{HN}_{\mathbf{d}}}} \Big(\boxtimes_{1\leq j\leq s(\alpha)} \mathrm{Dim}^{\zeta}_{\alpha^j,!}\,\phi_{\mathfrak{Tr}(W)^{\zeta}_{\alpha^j}}\mathcal{IC}_{\mathfrak{M}(Q)^{\zeta\text{-ss}}_{\alpha^j}}(\mathbb{Q})\Big)\otimes\mathbb{L}^{f(\alpha)/2}$$

in $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathbb{N}^{Q_0}))$.

If Q is symmetric, the function f in the above proposition is identically zero.

Corollary 6.2. *For any stability condition $\zeta \in \mathbb{H}_+^{Q_0}$, the cohomologically graded mixed Hodge structure*

$$\mathrm{H}_c(\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}, \phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\mathbf{d}}}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}(\mathbb{Q})) \in \mathcal{D}^{\leq}(\mathrm{MHS})$$

is pure, of Tate type.

Proof. For each $\mathbf{d} \in \mathbb{N}^{\widetilde{Q}_0}$, the Harder–Narasimhan type $(\mathbf{d})$ contributes the summand

$$\text{(6.3)}\quad \mathrm{H}_c(\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}, \phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\mathbf{d}}}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}(\mathbb{Q}))$$

to the right-hand side of (6.2), and so we deduce that, as a sub-mixed Hodge module of a mixed Hodge module that is both an ordinary mixed Hodge module and pure, of Tate type by Lemma 4.2 and Theorem A, the mixed Hodge module (6.3) is a pure element of $\mathcal{D}^{\leq}(\mathrm{MHS})$, of Tate type. □

6.2. Purity for stacks of semistable Π_Q-modules. Fix a quiver Q and a dimension vector $\mathbf{d}$. There is a natural projection

$$\tau_{Q,\mathbf{d}}\colon \mathfrak{M}(\widetilde{Q})_{\mathbf{d}} \to \mathfrak{M}(\overline{Q})_{\mathbf{d}}$$

induced by forgetting $\rho(\omega_i)$ for all $i \in Q_0$. Let $\zeta \in \mathbb{H}_+^{Q_0}$ be a stability condition. The inclusion $\tau^{-1}_{Q,\mathbf{d}}(\mathfrak{M}(\overline{Q})^{\zeta\text{-ss}}_{\mathbf{d}}) \subset \mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}$ is strict in general. We nonetheless have the following useful lemma, which enables us to prove purity for stacks of semistable Π_Q-modules.

Lemma 6.3. *For Q an arbitrary finite quiver, $\zeta \in \mathbb{H}_+^{Q_0}$ a stability condition, $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector, and $\tau_{Q,\mathbf{d}}\colon \mathfrak{M}(\widetilde{Q})_{\mathbf{d}} \to \mathfrak{M}(\overline{Q})_{\mathbf{d}}$ the natural projection, the inclusion*

$$(6.4)\qquad \big(\tau_{Q,\mathbf{d}}^{-1}(\mathfrak{M}(\overline{Q})_{\mathbf{d}}^{\zeta\text{-ss}}) \cap \operatorname{crit}(\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta})\big) \hookrightarrow \big(\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}} \cap \operatorname{crit}(\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta})\big)$$

is the identity.

Proof. Let ρ be a $\operatorname{Jac}(\widetilde{Q},\widetilde{W})$-representation represented by a closed point of the complement of inclusion (6.4). Then, via Proposition 2.1, ρ corresponds to a pair (M,f), where M is a Π_Q-module and $f \in \operatorname{End}_{\Pi_Q}(M)$. By assumption, the Harder–Narasimhan filtration of M, considered as a Π_Q-module, is nontrivial, i.e. it takes the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s = M,$$

where $s \geq 2$. Since each $\mu(M_j/M_{j-1})$ for $j \geq 2$ has slope strictly less than $\mu(M_1)$, each $\operatorname{Hom}_{\Pi_Q\text{-mod}}(M_1, M_j/M_{j-1}) = 0$, and so the restriction $f|_{M_1}\colon M_1 \to M$ factors through the inclusion $M_1 \subset M$. So the pair $(M_1, f|_{M_1})$ is a proper subobject of the pair (M,f) in the category $\mathcal{C}_{\Pi_Q}$ of Proposition 2.1. But then, by Proposition 2.1, ρ is not a ζ-semistable $\widetilde{Q}$-representation, a contradiction. □

Theorem 6.4. *Let Q be a finite quiver, let $\zeta \in \mathbb{H}_+^{Q_0}$ be a stability condition, and let $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. There is a natural isomorphism in $\mathcal{D}^{\leq}(\mathrm{MHS})$,*

$$\mathrm{H}_c(\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}, \phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta}} \mathbb{Q}_{\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}}) \cong \mathrm{H}_c\big((\mu_{Q,\mathbf{d}}^{-1}(0) \cap X(\overline{Q})_{\mathbf{d}}^{\zeta\text{-ss}})/\mathrm{GL}_{\mathbf{d}}, \mathbb{Q}\big) \otimes \mathbb{L}^{\mathbf{d}\cdot\mathbf{d}},$$

and so, by Theorem 6.5*, taking duals, the mixed Hodge structure $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}^{\zeta\text{-ss}}, \mathbb{Q})$ is pure, of Tate type.*

Proof. Write $V = \tau_{Q,\mathbf{d}}^{-1}(\mathfrak{M}(\overline{Q})_{\mathbf{d}}^{\zeta\text{-ss}})$. By Theorem 3.4, there is an isomorphism

$$(6.5)\qquad \mathrm{H}_c\big((\mu_{Q,\mathbf{d}}^{-1}(0) \cap X(\overline{Q})_{\mathbf{d}}^{\zeta\text{-ss}})/\mathrm{GL}_{\mathbf{d}}, \mathbb{Q}\big) \otimes \mathbb{L}^{\mathbf{d}\cdot\mathbf{d}} \cong \mathrm{H}_c(V, \phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta}} \mathbb{Q}_V).$$

There are equalities

$$\begin{aligned}\operatorname{supp}(\phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta}} \mathbb{Q}_V) &= \big(V \cap \operatorname{crit}(\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta})\big) \\ &= \big(\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}} \cap \operatorname{crit}(\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta})\big) \quad \text{(Lemma 6.3)} \\ &= \operatorname{supp}(\phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}} \mathbb{Q}_{\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}}).\end{aligned}$$

Thus the natural map $\mathrm{H}_c(V, \phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta}} \mathbb{Q}_V) \to \mathrm{H}_c(\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}, \phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}^{\zeta}} \mathbb{Q}_{\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}})$ is an isomorphism. Combining (6.5) and this isomorphism with Corollary 6.2, we deduce the result. □

6.3. Framed quivers. For Q' a quiver, $\mathbf{f}, \mathbf{d} \in \mathbb{N}^{Q'_0}$, and $\zeta \in \mathbb{H}_+^{Q'_0}$ a stability condition, recall from Section 3.5 the construction of the moduli space $\mathcal{M}(Q')_{\mathbf{f},\mathbf{d}}^{\zeta}$ of $\mathbf{f}$-framed ζ-semistable $\mathbf{d}$-dimensional Q'-representations. We consider this construction for $Q' = \widetilde{Q}$, the tripled quiver associated to a quiver Q. We define

$$\pi_{\widetilde{Q},\mathbf{f},\mathbf{d}}^{\zeta}\colon \mathcal{M}(\widetilde{Q})_{\mathbf{f},\mathbf{d}}^{\zeta} \to \mathcal{M}(\widetilde{Q})_{\mathbf{d}}^{\zeta\text{-ss}}$$

to be the map forgetting the framing and remembering the associated graded object of the Jordan–Hölder filtration (in the category of ζ-semistable $\widetilde{Q}$-representations) of the underlying $\widetilde{Q}$-representation.

Theorem 6.5. *Fix a finite quiver Q, dimension vectors $\mathbf{d}, \mathbf{f} \in \mathbb{N}^{Q_0}$, and a King stability condition $\zeta \in \mathbb{H}_+^{Q_0}$. Then the $\mathbb{N}^{Q_0}$-graded mixed Hodge structure on the vanishing cycle cohomology $\mathrm{H}_c(\mathcal{M}(\widetilde{Q})^{\zeta}_{\mathbf{f},\mathbf{d}}, \phi_{\mathcal{T}r(\widetilde{W})^{\zeta}_{\mathbf{f},\mathbf{d}}} \mathcal{IC}_{\mathcal{M}(\widetilde{Q})^{\zeta}_{\mathbf{f},\mathbf{d}}}(\mathbb{Q}))$ on the fine moduli space of ζ-semistable $\mathbf{f}$-framed $\mathbb{C}\widetilde{Q}$-modules is pure, of Tate type.*

Proof. Applying $\dim^{\zeta}_{\theta,!}$ to the isomorphism of Proposition 3.7, we obtain

$$(\dim^{\zeta}_{\theta} \circ \pi^{\zeta}_{\widetilde{Q},\mathbf{f},\theta})_! \bigoplus_{\mathbf{d} \in \Lambda^{\zeta}_{\theta}} \phi_{\mathcal{T}r(\widetilde{W})^{\zeta}_{\mathbf{f},\mathbf{d}}} \mathbb{Q}_{\mathcal{M}(\widetilde{Q})^{\zeta}_{\mathbf{f},\mathbf{d}}} \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_{\widetilde{Q}}/2} \cong \mathrm{Sym}_{\boxtimes_+}\Big(\bigoplus_{\mathbf{d}\in\Lambda^{\zeta}_{\theta}} \mathrm{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}} \otimes \mathrm{H}(\mathbb{P}^{\mathbf{f}\cdot\mathbf{d}-1}, \mathbb{Q})^{\vee} \otimes \mathbb{L}^{-1/2}\Big). \tag{6.6}$$

On the other hand, from Corollary 4.4, each of the complexes $\mathrm{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ is pure. The purity of the right-hand side of (6.6) follows, and so does the theorem. □

6.4. Critical cohomology of Hilb($\mathbb{A}^3$). We consider again the special case in which $Q = Q_{\mathrm{Jor}}$, and so $\widetilde{Q}$ is a quiver with one vertex and three loops, which we label x, y, z, and $\widetilde{W} = x[y, z]$. Setting $\mathbf{f} = 1$, there is a natural isomorphism of schemes (see [1])

$$\mathcal{M}(\widetilde{Q})_{1,n} \cap \mathrm{crit}(\mathcal{T}r(\widetilde{W})_n) \cong \mathrm{Hilb}_n(\mathbb{A}^3), \tag{6.7}$$

where the right-hand side of (6.7) is the usual Hilbert scheme parameterising codimension n ideals $I \subset \mathbb{C}[x, y, z]$. The following is then a corollary of Theorem 6.5.

Corollary 6.6. *The mixed Hodge structure $\mathrm{H}_c(\mathrm{Hilb}_n(\mathbb{A}^3), \phi_{\mathcal{T}r(\widetilde{W})_n} \mathbb{Q}_{\mathcal{M}(\widetilde{Q})_{1,n}})$ is pure, of Tate type for all n.*

It follows from our purity result that the Hodge polynomial

$$\mathtt{h}\big(\mathrm{H}_c(\mathrm{Hilb}_n(\mathbb{A}^3), \phi_{\mathcal{T}r(\widetilde{W})_d} \mathbb{Q}_{\mathcal{M}(\widetilde{Q})_{1,n}}), x, y, z\big)$$

is equal to the weight polynomial

$$\chi_{\mathrm{wt}}\big(\mathrm{H}_c(\mathrm{Hilb}_n(\mathbb{A}^3), \phi_{\mathcal{T}r(\widetilde{W})_n}), q\big)$$

after the substitution $q^2 = xyz^2$. We deduce from [1, Theorem 2.7] the following generating function equation:

$$\sum_{n\geq 0} \mathtt{h}\big(\mathrm{H}_c(\mathrm{Hilb}_n(\mathbb{A}^3), \phi_{\mathcal{T}r(\widetilde{W})_n} \mathbb{Q}_{\mathcal{M}(\widetilde{Q})_{1,n}}), x, y, z\big)(xyz^2)^{-n-n^2} t^n = \prod_{n=1}^{\infty} \prod_{k=0}^{n-1} (1 - (xyz^2)^{1-k} t^n)^{-1}.$$

Indeed, we can determine the critical cohomology of $\mathrm{Hilb}_n(\mathbb{A}^3)$ itself.

Corollary 6.7. *There is an isomorphism of $\mathbb{N}$-graded, cohomologically graded mixed Hodge structures,*

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}_c(\mathrm{Hilb}_n(\mathbb{A}^3), \phi_{\mathcal{T}r(\widetilde{W})_n} \mathbb{Q}_{\mathcal{M}(\widetilde{Q})_{1,n}}) \otimes \mathbb{L}^{-n-n^2} \cong \mathrm{Sym}\Big(\bigoplus_{n\geq 1} \bigoplus_{0\leq k\leq n-1} \mathbb{L}^{1-k}\Big).$$

Proof. By Corollary 6.6, the left-hand side of the expression in the corollary is pure, of Tate type, as is the right-hand side (by definition). A cohomologically graded mixed Hodge structure that is pure, of Tate type, is determined by its weight polynomial. The required equality of weight polynomials follows from the main result of [1], following on from the earlier paper [12], where an in-depth analysis of the case $n = 4$ was undertaken. □

6.5. Nakajima quiver varieties. Let Q be an arbitrary quiver, and let $\zeta \in \mathbb{H}_+^{Q_0}$ be a stability condition. Let $\mathbf{f} \in \mathbb{N}^{Q_0}$ be a framing vector. Throughout this section, we assume that $\mathbf{f} \neq 0$. Consider the quiver $\widetilde{Q_\mathbf{f}}$, where the tilde covers the $\mathbf{f}$ as well as the Q; this is the quiver obtained by framing the quiver Q to form $Q_\mathbf{f}$, *then* doubling, and then adding a loop ω_i at every vertex (including the vertex ∞).

Fix a slope $\theta \in (-\infty, \infty)$. We define the stability condition $\zeta^{(\theta)}$ as in Section 3.5. Assume that $\mathbf{d} \in \Lambda_\theta^\zeta \subset \mathbb{N}^{Q_0}$. Then a $(1, \mathbf{d})$-dimensional $\widetilde{Q_\mathbf{f}}$-representation ρ is $\zeta^{(\theta)}$-stable if and only if the underlying $\widetilde{Q}$-representation is ζ-semistable, and for every proper subrepresentation $\rho' \subset \rho$ such that $\dim(\rho')_\infty = 1$, the underlying $\widetilde{Q}$-representation of ρ' has slope strictly less than θ. In addition, $\zeta^{(\theta)}$-stability for $\widetilde{Q_\mathbf{f}}$-representations of dimension $(1, \mathbf{d})$ is equivalent to $\zeta^{(\theta)}$-semistability.

For each of the vertices $i \in Q_0$, the condition $\mu_{(1,\mathbf{d})}(\rho) = 0$ imposes the conditions

$$(6.8)\qquad T_i := \sum_{t(a)=i} \rho(a)\rho(a^*) - \sum_{s(a)=i} \rho(a^*)\rho(a) + \sum_{i \in Q_0} \sum_{1 \leq n \leq \mathbf{f}_i} \rho(\beta_{i,n})\rho(\beta_{i,n}^*) = 0$$

which are the usual Nakajima quiver variety relations [33, 34], while at the vertex ∞, the relation imposed is

$$(6.9)\qquad T_\infty := -\sum_{i \in Q_0} \sum_{1 \leq n \leq \mathbf{f}_i} \rho(\beta_{i,n}^*)\rho(\beta_{i,n}) = 0.$$

By cyclic invariance of the trace, $\sum_{i \in (Q_\mathbf{f})_0} \mathrm{Tr}(T_i) = 0$, and so $T_\infty = \mathrm{Tr}(T_\infty) = 0$ follows already from relations (6.8), and (6.9) is redundant. It follows that

$$(\mu_{Q_\mathbf{f},(1,\mathbf{d})}^{-1}(0) \cap X(\overline{Q}_\mathbf{f})_{(1,\mathbf{d})}^{\zeta^{(\theta)}\text{-ss}})/\mathrm{GL}_\mathbf{d}$$

is the usual Nakajima quiver variety, which we will denote $\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f})$. There is an isomorphism

$$(6.10)\qquad \mathrm{H}_c\big((\mu_{Q_\mathbf{f},(1,\mathbf{d})}^{-1}(0) \cap X(\overline{Q}_\mathbf{f})_{(1,\mathbf{d})}^{\zeta^{(\theta)}\text{-ss}})/\mathrm{GL}_{(1,\mathbf{d})}, \mathbb{Q}\big) \cong \mathrm{H}_c(\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f}), \mathbb{Q}) \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q}).$$

Each $\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f})$ is smooth, and so we have

$$\mathrm{H}_c(\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f}), \mathbb{Q}) \cong \mathrm{H}(\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f}), \mathbb{Q})^\vee \otimes \mathbb{L}^{\dim(\mathbf{M}^\zeta(\mathbf{d},\mathbf{f}))},$$

and we recover the following corollary.

Corollary 6.8. *For an arbitrary quiver Q, nonzero dimension vectors $\mathbf{f}, \mathbf{d} \in \mathbb{N}^{Q_0}$, and a King stability condition $\zeta \in \mathbb{H}_+^{Q_0}$, $\mathrm{H}(\mathbf{M}^\zeta(\mathbf{d}, \mathbf{f}), \mathbb{Q})$ is pure, of Tate type.*

7. The PBW and wall-crossing isomorphisms

7.1. Serre subcategories. Let $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod be a Serre subcategory of the category of finite-dimensional $\mathbb{C}\overline{Q}$-modules, i.e. we choose a property P of $\mathbb{C}\overline{Q}$-modules such that, for every short exact sequence $0 \to M' \to M \to M'' \to 0$ inside $\mathbb{C}\overline{Q}$-mod, M' and M'' have property P if and only if M does. Then $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod is the full subcategory of modules having property P. We assume that there is an inclusion of algebraic stacks $\iota\colon \mathfrak{M}(\overline{Q})^{\mathcal{S}} \hookrightarrow \mathfrak{M}(\overline{Q})$ which induces the inclusion of the objects of $\mathcal{S}$ into the objects of $\mathbb{C}\overline{Q}$-mod after passing to $\mathbb{C}$-points.

The standard construction for P is as follows. For a quiver Q, let $C(Q)$ denote the set of equivalence classes of cycles in Q, i.e. the set of cyclic paths, where if ll' and $l'l$ are both cyclic paths, they are considered to be equivalent. For every cycle $c \in C(Q)$, we pick a constructible subset $U_c \subset \mathbb{C}$, and we say that a $\mathbb{C}\overline{Q}$-module ρ has property P if and only if the generalised eigenvalues of $\rho(\overline{c})$ belong to U_c, for each $\overline{c}$ a representative of $c \in C(U)$.

Example 7.1. Setting all $U_c = \{0\}$, $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod is the subcategory of nilpotent modules, i.e. those modules M for which there exists some $n \in \mathbb{N}$ such that $\mathbb{C}\overline{Q}_{\geq n} \cdot M = 0$.

Example 7.2. Setting

$$U_c = \begin{cases} \mathbb{C} & \text{if } c \in C(Q), \\ \{0\} & \text{otherwise,} \end{cases}$$

we obtain the condition for the Lusztig nilpotent variety if Q has no loops. In general, the Serre subcategory $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod determined by this choice of U_c is the subcategory of modules M for which there exists a filtration by Q_0-graded vector spaces $0 \subset L^1 \subset \cdots \subset L^n$ of the underlying Q_0-graded vector space of M such that $a \cdot L^s \subset L^s$ for all s, and $a^* \cdot L^s \subset L^{s-1}$. This second property is obviously of Serre type. It is introduced under the name of *-semi-nilpotency in [3].

Example 7.3 ([2]). Set $U_c = \mathbb{C}$ if c is composed entirely of loops in Q, and 0 otherwise. A $\mathbb{C}\overline{Q}$-module is called *-strongly semi-nilpotent[5)] if it possesses a filtration as in Example 7.2, for which each subquotient L^s/L^{s-1} is supported at a single vertex. These are exactly the modules in the Serre subcategory corresponding to the above choices of U_c.

7.2. Proof of Theorems C and D. Applying the functor $\dim_! \overline{\iota}^*$ to the isomorphism constructed in the next theorem yields Theorem C.

Theorem 7.4. *Pick a stability condition $\zeta \in \mathbb{H}_+^{Q_0}$. There is an isomorphism*

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{JH}_{\overline{Q},\mathbf{d},!} \mathbb{Q}_{\mathfrak{M}(\Pi_Q)_{\mathbf{d}}} \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \cong \boxtimes_{\oplus, \theta \in (-\infty,\infty)} \bigoplus_{\mathbf{d} \in \Lambda_\theta^\zeta} \big(q^\zeta_{\overline{Q},\mathbf{d},!} \mathrm{JH}^\zeta_{\overline{Q},\mathbf{d},!} \mathbb{Q}_{\mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}} \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \big)$$

in $\mathcal{D}^{\leq}(\mathrm{MHM}(\mathcal{M}(\overline{Q})))$.

5) In fact, this is the modified terminology of [3].

Proof of Theorem 7.4. We consider the commutative diagram

$$\begin{array}{ccccc}
\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}} & \xhookleftarrow{j} & V & \xrightarrow{\tau_{Q,\mathbf{d}}} & \mathfrak{M}(\overline{Q})^{\zeta\text{-ss}}_{\mathbf{d}} \\
\downarrow{\scriptstyle \mathtt{JH}^{\zeta}_{\widetilde{Q},\mathbf{d}}} & & & & \downarrow{\scriptstyle \mathtt{JH}^{\zeta}_{\overline{Q},\mathbf{d}}} \\
\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}} & & & & \mathcal{M}(\overline{Q})^{\zeta\text{-ss}}_{\mathbf{d}} \\
\downarrow{\scriptstyle q^{\zeta}_{\widetilde{Q},\mathbf{d}}} & & & & \downarrow{\scriptstyle q^{\zeta}_{\overline{Q},\mathbf{d}}} \\
\mathcal{M}(\widetilde{Q})_{\mathbf{d}} & & \xrightarrow{\tau'_{Q,\mathbf{d}}} & & \mathcal{M}(\overline{Q})_{\mathbf{d}}
\end{array}$$

with V defined as in the proof of Theorem 6.4. By Theorem 3.4, there are isomorphisms

$$\begin{aligned}
&\mathtt{JH}^{\zeta}_{\overline{Q},\mathbf{d},!}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}}\otimes\mathbb{L}^{\mathbf{d}\cdot\mathbf{d}} \cong \mathtt{JH}^{\zeta}_{\overline{Q},\mathbf{d},!}\tau_{Q,\mathbf{d},!}\phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\mathbf{d}}}\mathbb{Q}_V, \\
&\mathtt{JH}_{\overline{Q},\mathbf{d},!}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}}\otimes\mathbb{L}^{\mathbf{d}\cdot\mathbf{d}} \cong \tau'_{Q,\mathbf{d},!}\mathtt{JH}_{\widetilde{Q},\mathbf{d},!}\phi_{\mathfrak{Tr}(\widetilde{W})_{\mathbf{d}}}\mathbb{Q}_{\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}}.
\end{aligned}\tag{7.1}$$

By Lemma 6.3, the support of $\phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\mathbf{d}}}\mathbb{Q}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}$ is contained in the image of the natural inclusion j, and so from (7.1) and the above commutative diagram, we obtain the isomorphism

$$q^{\zeta}_{\overline{Q},\mathbf{d},!}\mathtt{JH}^{\zeta}_{\overline{Q},\mathbf{d},!}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}} \cong \tau'_{Q,\mathbf{d},!}q^{\zeta}_{\widetilde{Q},\mathbf{d},!}\mathtt{JH}^{\zeta}_{\widetilde{Q},\mathbf{d},!}\phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\mathbf{d}}}\mathbb{Q}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}}.$$

Thus, applying $\tau'_{Q,!}$ to isomorphism (6.1) applied to the QP $(\widetilde{Q},\widetilde{W})$ yields the required isomorphism. □

Proof of Theorem D. Let $m\colon \mathbb{A}^1\times\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}\to\mathcal{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\mathbf{d}}$ be the morphism extending a Π_Q-module to a $\mathrm{Jac}(\widetilde{Q},\widetilde{W})$-module by letting all of the loops ω_i act by multiplication by a fixed scalar in $\mathbb{A}^1$. Then, by Theorem B, there is an isomorphism

$$\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\mathbf{d}} \cong m_*(\mathcal{IC}_{\mathbb{A}^1}(\mathbb{Q})\boxtimes\mathcal{BPS}^{\zeta}_{\Pi_Q}),\tag{7.2}$$

where $\mathcal{BPS}^{\zeta}_{\Pi_Q}$ is a mixed Hodge module on $\mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathfrak{M}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))^{\zeta\text{-ss}}_{\mathbf{d}} & \xrightarrow{\tau_{\mathbf{d}}} & \mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}} \\
\downarrow{\scriptstyle \widetilde{\mathtt{JH}}^{\zeta}_{\mathbf{d}}} & & \downarrow{\scriptstyle \overline{\mathtt{JH}}^{\zeta}_{\mathbf{d}}} \\
\mathcal{M}(\mathrm{Jac}(\widetilde{Q},\widetilde{W}))^{\zeta\text{-ss}}_{\mathbf{d}} & \xrightarrow{\tau'_{\mathbf{d}}} & \mathcal{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}.
\end{array}$$

Arguing as in the proof of Theorem 7.4, there are isomorphisms

$$\begin{aligned}
\bigoplus_{\mathbf{d}\in\Lambda^{\zeta}_{\theta}}\overline{\mathtt{JH}}^{\zeta}_{\mathbf{d},!}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)^{\zeta\text{-ss}}_{\mathbf{d}}}\otimes\mathbb{L}^{(\mathbf{d},\mathbf{d})} &\cong \overline{\mathtt{JH}}^{\zeta}_{\theta,!}\tau_{\theta,!}\phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\theta}}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\theta}}(\mathbb{Q}) \\
&\cong \tau'_{\theta,!}\widetilde{\mathtt{JH}}^{\zeta}_{\theta,!}\phi_{\mathfrak{Tr}(\widetilde{W})^{\zeta}_{\theta}}\mathcal{IC}_{\mathfrak{M}(\widetilde{Q})^{\zeta\text{-ss}}_{\theta}}(\mathbb{Q}) \\
&\cong \tau'_{\theta,!}\,\mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\theta}\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}) \\
&\cong \mathrm{Sym}_{\boxtimes_\oplus}(\tau'_{\theta,!}\mathcal{BPS}^{\zeta}_{\widetilde{Q},\widetilde{W},\theta}\otimes\mathrm{H}_c(\mathrm{B}\mathbb{C}^*,\mathbb{Q})_{\mathrm{vir}}) \\
&\cong \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}^{\zeta}_{\Pi_Q,\theta}\otimes\mathrm{H}(\mathrm{B}\mathbb{C}^*,\mathbb{Q})^{\vee}),
\end{aligned}$$

giving isomorphism (1.4).

The construction of the PBW isomorphism is similar; via dimensional reduction and Lemma 6.3, there is an isomorphism

$$\mathcal{A}^{\mathcal{S},\zeta}_{\Pi_Q,\theta} \cong \mathcal{A}^{\widetilde{\mathcal{S}},\zeta}_{\widetilde{Q},\widetilde{W},\theta},$$

where $\widetilde{\mathcal{S}}$ is the Serre subcategory of $\mathbb{C}\widetilde{Q}$-modules for which the underlying $\mathbb{C}\overline{Q}$-module is in $\mathcal{S}$, defining a Hall algebra structure on $\mathcal{A}^{\mathcal{S},\zeta}_{\Pi_Q,\theta}$. Then the required PBW isomorphism is constructed from Theorem 3.6 and the isomorphisms

$$\mathrm{BPS}^{\widetilde{\mathcal{S}},\zeta}_{\widetilde{Q},\widetilde{W},\theta} = \mathrm{H}_c(\mathcal{M}(\widetilde{Q})^{\widetilde{\mathcal{S}},\zeta\text{-ss}}_\theta, \mathcal{BPS}^\zeta_{\widetilde{Q},\widetilde{W},\theta})^\vee \cong \mathrm{H}_c(\mathcal{M}(\overline{Q})^{\mathcal{S},\zeta\text{-ss}}_\theta, \mathcal{BPS}^\zeta_{\Pi_Q,\theta})^\vee \otimes \mathbb{L}^{-1/2},$$

following from isomorphism (7.2). □

7.3. Applications for Nakajima quiver varieties. We explain the special case of Theorem C which gives rise to Hausel's original formula for the Poincaré polynomials of Nakajima quiver varieties. In brief, we choose $\Pi_{Q_{\mathbf{f}}}$ to be the preprojective algebra for a framed quiver $Q_{\mathbf{f}}$, pick ζ to be the usual stability condition defining the Nakajima quiver variety, set $\mathcal{S} = \mathbb{C}Q_{\mathbf{f}}$-mod and specialise the Hodge series to the Poincaré series, to derive Hausel's result. For this set of choices, an analogue of equation (1.3) was demonstrated by Dimitri Wyss [47], working in the naive Grothendieck ring of exponential motives. We describe in a little more detail how our derivation runs.

Let Q be a quiver, and let $\mathcal{S} \subset \mathbb{C}\overline{Q}$-mod be a Serre subcategory. Let $\mathbf{f} \in \mathbb{N}^{Q_0}$ be a framing vector, assumed nonzero, and let $\mathcal{S}_{\mathbf{f}} \subset \Pi_{Q_{\mathbf{f}}}$-mod be the Serre subcategory consisting of those modules for which the underlying $\mathbb{C}\overline{Q}$-module is in $\mathcal{S}$. We let $\zeta = (i, \dots, i)$ be the degenerate stability condition on Q and define $\zeta^{(0)}$ as in Section 3.5. If X is an Artin stack, we define its Poincaré series via $\mathrm{P}(X, q) = \mathtt{h}(\mathrm{H}_c(X, \mathbb{Q}), 1, 1, q)$. Equating coefficients in (1.3) for which $\mathbf{d}_\infty = 1$, and specialising, we obtain, from (6.10),

$$\begin{aligned}\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} &\mathrm{P}(\mathfrak{M}(\Pi_{Q_{\mathbf{f}}})^{\mathcal{S}_{\mathbf{f}}}_{(1,\mathbf{d})}, q) q^{2((\mathbf{d},\mathbf{d})-\mathbf{f}\cdot\mathbf{d}+1)} x^{\mathbf{d}} \\ &= \Big(\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{P}(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, q) q^{2(\mathbf{d},\mathbf{d})} x^{\mathbf{d}}\Big) \\ &\quad \cdot \Big(\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{P}(\mathbf{M}(\mathbf{f},\mathbf{d})^{\mathcal{S}}, q) q^{2((\mathbf{d},\mathbf{d})-\mathbf{f}\cdot\mathbf{d}+1)} x^{\mathbf{d}} (q^2-1)^{-1}\Big),\end{aligned}$$

where $\mathbf{M}(\mathbf{f},\mathbf{d})^{\mathcal{S}}$ is the subvariety of the Nakajima quiver variety for the dimension vector $\mathbf{d}$ and framing vector $\mathbf{f}$ corresponding to those points for which the underlying $\overline{Q}$-representation is in $\mathcal{S}$. Putting $\mathcal{S} = \mathbb{C}\overline{Q}$-mod (or, equivalently, removing $\mathcal{S}$ from the above formulae) and using Hua's formula [21] to rewrite both sides as rational functions in q defined in terms of Kac polynomials, we recover [17, Theorem 5]. The advance that Theorem C gives us is an upgrade from an equality of generating series to an isomorphism in cohomology, i.e. it tells us that the above identity is induced by a graded isomorphism of (pure) Hodge structures

$$\begin{aligned}\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} &\mathrm{H}_c(\mathfrak{M}(\Pi_{Q_{\mathbf{f}}})^{\mathcal{S}_{\mathbf{f}}}_{(1,\mathbf{d})}, \mathbb{Q}) \otimes \mathbb{L}^{((\mathbf{d},\mathbf{d})-\mathbf{f}\cdot\mathbf{d}+1)} \\ &\cong \Big(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})}\Big) \\ &\quad \otimes \Big(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathbf{M}(\mathbf{f},\mathbf{d})^{\mathcal{S}}, \mathbb{Q}) \otimes \mathbb{L}^{((\mathbf{d},\mathbf{d})-\mathbf{f}\cdot\mathbf{d}+1)} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q})\Big)\end{aligned}$$

by taking Poincaré series of the two sides of the isomorphism.

8. Restricted Kac polynomials

8.1. Definition. Next we explain how Theorem D enables one to define and categorify the Kac polynomial $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2})$ associated to a quiver Q, a Serre subcategory $\mathcal{S} \subset \mathbb{C}\overline{Q}$, and a dimension vector $\mathbf{d}$. Furthermore, we explain a general mechanism for deducing positivity of such Kac polynomials from purity, and we prove Theorem E.

Defining $\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q}$ as $\mathrm{BPS}^{\mathcal{S},\zeta}_{\Pi_Q}$, for the degenerate stability condition $\zeta = (i, \dots, i)$ (equivalently, without any stability condition), the dual of isomorphism (1.5) yields

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \cong \mathrm{Sym}(\mathrm{BPS}^{\mathcal{S},\vee}_{\Pi_Q} \otimes \mathbb{L} \otimes \mathrm{H}_c(\mathrm{B}\mathbb{C}^*, \mathbb{Q})). \tag{8.1}$$

Isomorphism (8.1) can be restated as saying that $\mathrm{BPS}^{\mathcal{S},\vee}_{\Pi_Q}$ categorifies the *restricted Kac polynomials* $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2})$, defined by the plethystic logarithm (the inverse to the plethystic exponential)

$$q(q-1)^{-1} \sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2}) t^{\mathbf{d}} = \mathrm{Log}\Big(\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \chi_{\mathrm{wt}}\big(\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q}), q^{1/2}\big) q^{(\mathbf{d},\mathbf{d})} t^{\mathbf{d}}\Big).$$

Isomorphism (8.1) and the definition of Exp (see Section 4.3) imply

$$\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2}) = \chi_{\mathrm{wt}}(\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q,\mathbf{d}}, q^{-1/2}).$$

This is indeed a polynomial: despite its high-tech definition, $\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q,\mathbf{d}}$ is, after all, the hypercohomology of a bounded complex of mixed Hodge modules on an algebraic variety.

8.2. Positivity of Kac polynomials. A corollary of the existence of isomorphism (8.1) is that if $\bigoplus_{\mathbf{d}\in\mathbb{N}_{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{S}}_{\mathbf{d}}, \mathbb{Q})$ is pure, then so is $\mathrm{BPS}^{\mathcal{S}}_{\Pi_Q}$, and as a result, $\mathtt{a}^{\mathcal{S}}_{Q,\mathbf{d}}(q^{1/2})$ has only positive coefficients, when expressed as a polynomial in $-q^{1/2}$. This brings us to the special case of Theorem D that, along with Theorem A, implies the Kac positivity conjecture, first proved by Hausel, Letellier and Villegas in [19] via arithmetic Fourier analysis for *smooth* Nakajima quiver varieties. Namely, we set $\mathcal{S} = \mathbb{C}\overline{Q}$-mod, and we set $\zeta = (i, \dots, i)$ to be the degenerate stability condition. Then Theorem D states that there is an isomorphism

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \cong \mathrm{Sym}(\mathrm{BPS}^{\vee}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})^{\vee}),$$

while Theorem 3.4 states that there is an isomorphism

$$\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\Pi_Q)_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})} \cong \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}_c(\mathfrak{M}(\tilde{Q})_{\mathbf{d}}, \phi_{\mathfrak{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})_{\mathbf{d}}}(\mathbb{Q})).$$

On the other hand, by [32, Theorem 5.1], there is an equality

$$\begin{aligned}\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}} \chi_{\mathrm{wt}}\big(\mathrm{H}_c(\mathfrak{M}(\tilde{Q})_{\mathbf{d}}, \phi_{\mathfrak{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})_{\mathbf{d}}}(\mathbb{Q})), q^{1/2}\big) t^{\mathbf{d}} \\ = \mathrm{Exp}\Big(\sum_{\mathbf{d}\in\mathbb{N}^{Q_0}\setminus\{0\}} \mathtt{a}_{Q,\mathbf{d}}(q)(1-q^{-1})^{-1} t^{\mathbf{d}}\Big),\end{aligned}$$

where $\mathtt{a}_{Q,\mathbf{d}}(q)$ is Kac's original polynomial, from which we deduce that

$$\chi_{\mathrm{wt}}(\mathrm{BPS}_{\Pi_Q,\mathbf{d}}, q^{1/2}) = \mathtt{a}_{Q,\mathbf{d}}(q^{-1}).$$

From Corollary 4.4, we deduce that each

$$\mathrm{BPS}_{\Pi_Q,\mathbf{d}} \cong \mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathbb{L}^{1/2}$$

is pure, and so $\chi_{\mathrm{wt}}(\mathrm{BPS}_{\Pi_Q,\mathbf{d}}, q^{1/2})$ is a polynomial in $-q^{1/2}$ with positive coefficients. In particular, since $\mathtt{a}_{Q,\mathbf{d}}(q)$ is a polynomial in q, we have reproved the following theorem.

Theorem 8.1 ([19]). *For a finite quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, the Kac polynomial $\mathtt{a}_{Q,\mathbf{d}}(q)$ has positive coefficients.*

8.3. Positivity of restricted Kac polynomials. For *new* positivity results, we turn to the examples of Serre subcategories appearing in the work of Bozec, Schiffmann and Vasserot – see Examples 7.1, 7.2 and 7.3 for the definitions. Setting $\mathcal{N}, \mathcal{SN}, \mathcal{SSN} \subset \mathbb{C}\overline{Q}$-mod to be the full subcategory of nilpotent, *-semi-nilpotent and *-strongly semi-nilpotent $\mathbb{C}\overline{Q}$-modules, respectively, we define

$$\mathtt{a}^{\sharp}_{Q,\mathbf{d}}(q^{1/2}) := \chi_{\mathrm{wt}}\big(\mathrm{H}_c(\mathcal{M}(\overline{Q})_{\mathbf{d}}, \iota'^{\sharp,*}_{\mathbf{d}}\mathcal{BPS}_{\Pi_Q,\mathbf{d}}), q^{1/2}\big)$$

for $\sharp = \mathcal{N}, \mathcal{SN}, \mathcal{SSN}$, where $\iota'^{\sharp}\colon \mathcal{M}(\overline{Q})^{\sharp}_{\mathbf{d}} \hookrightarrow \mathcal{M}(\overline{Q})_{\mathbf{d}}$ is the inclusion. In this way we obtain a new description of the nilpotent, semi-nilpotent and strongly semi-nilpotent Kac polynomials of [3]. By [3], the polynomials $\mathtt{a}^{\mathcal{SN}}_{Q,\mathbf{d}}(q)$ and $\mathtt{a}^{\mathcal{SSN}}_{Q,\mathbf{d}}(q)$ have an enumerative definition when q is a prime power: the former counts absolutely indecomposable $\mathbf{d}$-dimensional $\mathbb{F}_q Q$-modules such that each loop acts via a nilpotent operator, while the latter counts absolutely indecomposable $\mathbf{d}$-dimensional nilpotent $\mathbb{F}_q Q$-modules.

Theorem 8.2. *For a finite quiver Q, the Kac polynomials $\mathtt{a}^{\mathcal{SN}}_{Q,\mathbf{d}}(q)$ and $\mathtt{a}^{\mathcal{SSN}}_{Q,\mathbf{d}}(q)$ have positive coefficients.*

Proof. The proof proceeds exactly as in the above reproof of Theorem 8.1, using the results and proofs of [3,43] to deduce purity of $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{SN}}, \mathbb{Q})$ and $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\mathcal{SSN}}, \mathbb{Q})$. For example, one may extract this purity result as follows. Let $\sharp$ be either of the conditions $\mathcal{SN}$ or $\mathcal{SSN}$. By [43, Theorem 3.2.d], the Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_T(\mathrm{pt}, \mathbb{Q}) \otimes \mathrm{H}^q_{c,\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{\mathbf{d}}(0)^{\sharp}, \mathbb{Q})^{\vee} = \mathrm{H}^p_T(\mathrm{pt}, \mathbb{Q}) \otimes \mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\sharp}, \mathbb{Q})^{\vee}$$

converging to $\mathrm{H}_{c,\mathrm{GL}_{\mathbf{d}}\times T}(\mu^{-1}_{\mathbf{d}}(0)^{\sharp}, \mathbb{Q})^{\vee}$ degenerates at the second sheet (here T is an extra complex torus acting on all relevant varieties, and is a special case of one of the tori T^{τ} that we consider in Section 9.1). In particular, purity of $\mathrm{H}_c(\mathfrak{M}(\Pi_Q)^{\sharp}, \mathbb{Q})^{\vee}$ follows from purity of $\mathrm{H}_{c,\mathrm{GL}_{\mathbf{d}}\times T}(\mu^{-1}_{\mathbf{d}}(0)^{\sharp}, \mathbb{Q})^{\vee}$, which is [43, Theorem 3.2.b]. □

8.4. Verdier duality and nilpotent Kac polynomials. We finish our discussion of restricted Kac polynomials with a result relating $\mathtt{a}_{Q,\mathbf{d}}(q)$ with $\mathtt{a}^{\mathcal{N}}_{Q,\mathbf{d}}(q)$, providing a cohomological refinement of a Kac polynomial identity [3, Theorem 1.4], which in turn extended the main result of [41] from the case of a quiver without loops.

Proposition 8.3. *For a quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, there is an isomorphism $\mathrm{H}_c(\mathcal{M}(\overline{Q}), \mathcal{BPS}_{\Pi_Q,\mathbf{d}}) \cong \mathrm{H}_c(\mathcal{M}(\overline{Q})^{\mathcal{N}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}})^{\vee}$ providing a cohomological refinement of the identity $\mathtt{a}^{\mathcal{N}}_{Q,\mathbf{d}}(q) = \mathtt{a}_{Q,\mathbf{d}}(q^{-1})$.*

Proof. The torus $T = (\mathbb{C}^*)^2$ acts on $\mathcal{M}(\widetilde{Q})$ via the rescaling action

$$(z_1, z_2) \cdot \rho(b) = \begin{cases} z_1\rho(b) & \text{if } b \in Q_1, \\ z_2\rho(b) & \text{if } b^* \in Q_1, \\ (z_1 z_2)^{-1}\rho(b) & \text{if there exists } i \in Q_0 \text{ such that } b = \omega_i. \end{cases}$$

This action preserves $\mathrm{Tr}(\widetilde{W})$ so that $\mathcal{BPS}_{\widetilde{Q},\widetilde{W},\mathbf{d}}$ lifts to a T-equivariant mixed Hodge module on $\mathcal{M}(\widetilde{Q})_{\mathbf{d}}$, and so $\mathcal{BPS}_{\Pi_Q,\mathbf{d}}$ lifts to a T-equivariant MHM on $\mathcal{M}(\overline{Q})_{\mathbf{d}}$. By Theorem B, the MHM $\mathcal{BPS}_{\Pi_Q,\mathbf{d}}$ is Verdier self-dual so that there is an isomorphism

$$\mathrm{H}_c(\mathcal{M}(\overline{Q})_{\mathbf{d}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}}) \cong \mathrm{H}(\mathcal{M}(\overline{Q})_{\mathbf{d}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}})^{\vee}.$$

Since T contracts $\mathcal{M}(\overline{Q})_{\mathbf{d}}$ to the point $\mathcal{M}(\overline{Q})^{\mathcal{N}}_{\mathbf{d}}$, there are isomorphisms

$$\mathrm{H}(\mathcal{M}(\overline{Q})_{\mathbf{d}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}}) \cong \mathrm{H}(\mathcal{M}(\overline{Q})^{\mathcal{N}}_{\mathbf{d}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}}) \cong \mathrm{H}_c(\mathcal{M}(\overline{Q})^{\mathcal{N}}_{\mathbf{d}}, \mathcal{BPS}_{\Pi_Q,\mathbf{d}}).$$

Combining these isomorphisms gives the isomorphism in the proposition. The identity in the proposition then follows from the definitions. □

Remark 8.4. Combining Theorems 8.1 and 8.2 with Proposition 8.3, we conclude that all of the Kac polynomials $\mathtt{a}_{Q,\mathbf{d}}(q)$, $\mathtt{a}^{\mathcal{N}}_{Q,\mathbf{d}}(q)$, $\mathtt{a}^{\mathcal{SN}}_{Q,\mathbf{d}}(q)$ and $\mathtt{a}^{\mathcal{SSN}}_{Q,\mathbf{d}}(q)$ have positive coefficients.

9. Deformations of Hall algebras

9.1. Kontsevich–Soibelman CoHAs. In [28], a method was given for associating a cohomological Hall algebra (CoHA for short) to the data of an arbitrary QP (Q, W). The construction provides a mathematically rigorous approach to defining algebras of BPS states – see [16] for the physical motivation. We will work with a slight generalisation of the original definition, denoted $\mathcal{A}_{\tau,Q,W}$, incorporating extra parameters depending on a weight function τ.

Definition 9.1. If (Q, W) is a QP, then a W-invariant grading for Q is a function $\tau\colon Q_1 \to \mathbb{Z}^s$ such that every cyclic word appearing in W is homogeneous of weight zero.

Example 9.2. For $s = 0$, the function $\tau = 0\colon Q_1 \to \mathbb{Z}^0$ gives a W-invariant grading for any potential W, and we will recover below the original definition of Kontsevich and Soibelman, by considering this grading.

Example 9.3. For a quiver Q and $s = 2$, the weight function

$$\tau(a) = (1, 0), \quad \tau(a^*) = (0, 1) \quad \text{for all } a \in Q_1,$$
$$\tau(\omega_i) = (-1, -1) \qquad\qquad \text{for all } i \in Q_0$$

is a $\widetilde{W}$-invariant grading for the tripled quiver $\widetilde{Q}$.

From now on, we will only consider the case in which our quiver with potential is $(\widetilde{Q}, \widetilde{W})$ for some quiver Q. Given a grading $\tau\colon Q_1 \to \mathbb{Z}^s$, define $T^{\tau} := \mathrm{Hom}(\mathbb{Z}^s, \mathbb{C}^*)$. Given a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we form the extended gauge group $\mathrm{GL}^{\tau}_{\mathbf{d}} := \mathrm{GL}_{\mathbf{d}} \times T^{\tau}$. The group $\mathrm{GL}^{\tau}_{\mathbf{d}}$ acts on $X(Q)_{\mathbf{d}}$ via $((\{g_i\}_{i\in Q_0}, \upsilon) \cdot \rho)(a) = \upsilon(\tau(a))g_{t(a)}\rho(a)g^{-1}_{s(a)}$ extending the action of $\mathrm{GL}_{\mathbf{d}}$ on $X(Q)_{\mathbf{d}}$. Similarly, if $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{Q_0}$, we define $\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''} := \mathrm{GL}_{\mathbf{d}',\mathbf{d}''} \times T^{\tau}$, the parabolic gauge group, acting on $X(Q)_{\mathbf{d}',\mathbf{d}''}$ via the same formula, and $\mathrm{GL}^{\tau}_{\mathbf{d}'\times\mathbf{d}''} := \mathrm{GL}_{\mathbf{d}'} \times \mathrm{GL}_{\mathbf{d}''} \times T^{\tau}$, acting on $X(Q)_{\mathbf{d}'} \times X(Q)_{\mathbf{d}''}$ via

$$((\{g'_i\}_{i\in Q_0}, \{g''_i\}_{i\in Q_0}, \upsilon) \cdot (\rho', \rho''))(a) = \upsilon(\tau(a))(g'_{t(a)}\rho'(a)g'^{-1}_{s(a)}, g''_{t(a)}\rho''(a)g''^{-1}_{s(a)}).$$

For fixed $\upsilon \in T^{\tau}$, the action of υ on the category of $\mathbb{C}Q$-modules is functorial and preserves dimension vectors. It follows that if $\zeta \in \mathbb{H}_{+}^{Q_0}$ is a stability condition, the spaces $X(Q)^{\zeta\text{-ss}}_{\mathbf{d}}$ and $X(Q)^{\zeta\text{-st}}_{\mathbf{d}}$ are preserved by $\mathrm{GL}^{\tau}_{\mathbf{d}}$. We define the stack ${}^{\tau}\mathfrak{M}(Q)^{\zeta\text{-ss}}_{\mathbf{d}} := X(Q)^{\zeta\text{-ss}}_{\mathbf{d}}/\mathrm{GL}^{\tau}_{\mathbf{d}}$.

For the remaining part of the section, we will only consider the degenerate stability condition $\zeta = (i, \dots, i)$, and so we drop ζ from our notation. We denote by $\mathrm{Dim}^{\tau}\colon {}^{\tau}\mathfrak{M}(\widetilde{Q}) \to \mathbb{N}^{Q_0}$ the map taking a $\widetilde{Q}$-representation to its dimension vector.

Assume that the grading $\tau\colon \widetilde{Q}_1 \to \mathbb{Z}^s$ is $\widetilde{W}$-invariant. The function $\mathrm{Tr}(\widetilde{W})$ induces a function $\mathfrak{T}\mathrm{r}(\widetilde{W})$ on ${}^{\tau}\mathfrak{M}(\widetilde{Q})$. Let $\mathcal{S}$ be a Serre subcategory of the category of $\mathbb{C}\widetilde{Q}$-modules, which we assume to be invariant under the action of T^{τ}, with induced morphism

$$\iota\colon {}^{\tau}\mathfrak{M}(\widetilde{Q})^{\mathcal{S}} \hookrightarrow {}^{\tau}\mathfrak{M}(\widetilde{Q}).$$

We define

$$\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W}} := \mathrm{Dim}^{\tau}_{*}\,\iota^{!}\phi_{\mathfrak{T}\mathrm{r}(\widetilde{W})}\mathcal{IC}_{{}^{\tau}\mathfrak{M}(Q)}(\mathbb{Q}) \otimes \mathbb{L}^{-\dim(T^{\tau})/2} \in \mathcal{D}^{\geq}(\mathrm{MHM}(\mathbb{N}^{Q_0})),$$

the underlying cohomologically graded mixed Hodge module or, equivalently, $\mathbb{N}^{Q_0}$-graded mixed Hodge structure, of $\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W}}$. We endow $\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W}}$ with the structure of an algebra object in the category of complexes of mixed Hodge modules on $\mathbb{N}^{Q_0}$. In order to achieve this, as in Section 3.2, a little care has to be taken to approximate morphisms of stacks by morphisms of varieties so that we can apply Saito's theory of mixed Hodge modules to these morphisms. We spell this out in detail.

We define

$$V_{\mathbf{d},N} = \Big(\bigoplus_{i\in Q_0}\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{\mathbf{d}_i})\Big), \quad V_{\tau,\mathbf{d},N} = \Big(\bigoplus_{i\in Q_0}\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{\mathbf{d}_i})\Big) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathfrak{t}^{\tau}).$$

We let $\mathrm{GL}^{\tau}_{\mathbf{d}}$ act on $V_{\tau,\mathbf{d},N}$ via the product of the natural action of $\mathrm{GL}_{\mathbf{d}}$ on the first component, and the action of T^{τ} on $\mathfrak{t}^{\tau}$ given by the embedding $(\mathbb{C}^*)^s \subset \mathbb{C}^s = \mathfrak{t}^{\tau}$, and componentwise multiplication. We define $U_{\tau,\mathbf{d},N} \subset V_{\tau,\mathbf{d},N}$ to be the subset consisting of those

$$(\{g_i\}_{i\in Q_0}, f) \in V_{\tau,\mathbf{d},N}$$

such that each g_i is surjective, and f is too. Then $\mathrm{GL}^{\tau}_{\mathbf{d}}$ acts freely on $U_{\tau,\mathbf{d},N}$.

We break the multiplication into two parts. Fix a pair of dimension vectors $\mathbf{d}', \mathbf{d}''$ and set $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. We write $\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''} := \mathrm{GL}_{\mathbf{d}'} \times \mathrm{GL}_{\mathbf{d}''}$. We embed $\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}$ and $\mathrm{GL}_{\mathbf{d}',\mathbf{d}''}$ into $\mathrm{GL}_{\mathbf{d}}$ as a Q_0-indexed product of Levi or parabolic subgroups, respectively. We define $\mathrm{GL}^{\tau}_{\mathbf{d}}$, $\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''}$ and $\mathrm{GL}^{\tau}_{\mathbf{d}'\times\mathbf{d}''}$ as the product of T^{τ} with $\mathrm{GL}_{\mathbf{d}}$, $\mathrm{GL}_{\mathbf{d}',\mathbf{d}''}$ and $\mathrm{GL}_{\mathbf{d}'\times\mathbf{d}''}$, respectively.

For G an algebraic group with a fixed embedding $G \subset \mathrm{GL}^{\tau}_{\mathbf{d}}$, we define a functor on G-equivariant varieties X by $A_N(X, G) := X \times_G U_{\tau,\mathbf{d},N}$. If $f\colon X \to Y$ is a G-invariant morphism, we denote by $f_N\colon A_N(X) \to Y$ the induced morphism. For $\iota\colon Y \hookrightarrow X$ a G-invariant subvariety, then, as discussed in Section 3.2, for fixed i, the mixed Hodge structure

$$\mathcal{H}^i((Y/G \to \mathrm{pt})_*\iota^!\phi_f\mathbb{Q}_{X/G})$$

is defined as

$$\mathcal{H}^i\big((A_N(Y,G)\to \mathrm{pt})_* A_N(\iota,G)^! \phi_{f_N}\mathbb{Q}_{A_N(X,G)}\big),$$

for $N \gg 0$ depending on i. Consider the commutative diagram

$$\begin{array}{ccc} & & A_N(X(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''},\mathrm{GL}^{\tau}_{\mathbf{d}'\times\mathbf{d}''}) \\ & \swarrow^{q_2} & \downarrow^{q_1} \\ A_N(X(\widetilde{Q})_{\mathbf{d}'}\times X(\widetilde{Q})_{\mathbf{d}''},\mathrm{GL}^{\tau}_{\mathbf{d}'\times\mathbf{d}''}) & & A_N(X(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''},\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''}) \\ \downarrow^{(\mathrm{Dim}^{\tau}\times\mathrm{Dim}^{\tau})_N} & & \downarrow^{\mathrm{Dim}^{\tau,\circ}_N} \\ \mathbb{N}^{Q_0}\times\mathbb{N}^{Q_0} & \xrightarrow{\quad + \quad} & \mathbb{N}^{Q_0}, \end{array} \tag{9.1}$$

where q_1 and q_2 are the natural affine fibrations, inducing isomorphisms

$$\begin{aligned}\alpha_{\mathbf{d}',\mathbf{d}''}\colon +_*(\mathrm{Dim}^{\tau}\times\mathrm{Dim}^{\tau})_*(\iota^!\phi_{\mathfrak{Tr}(\widetilde{W})\boxtimes\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\tau\mathfrak{M}(\widetilde{Q})_{\mathbf{d}'}\times_{\mathrm{B}T^{\tau}}\mathfrak{M}(\widetilde{Q})_{\mathbf{d}''}}(\mathbb{Q})) \\ \to \mathrm{Dim}^{\tau,\circ}_*\,\iota^!\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\tau\mathfrak{M}(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''}}(\mathbb{Q})\otimes\mathbb{L}^{-(\mathbf{d}',\mathbf{d}'')_{\widetilde{Q}}/2}.\end{aligned}$$

Consider the composition of proper maps

$$A_N(X(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''},\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''})\xrightarrow{r_N}A_N(X(\widetilde{Q})_{\mathbf{d}},\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''})\xrightarrow{s_N}A_N(X(\widetilde{Q})_{\mathbf{d}},\mathrm{GL}^{\tau}_{\mathbf{d}}),$$

where r_N is induced by the inclusion $X(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''}\hookrightarrow X(\widetilde{Q})_{\mathbf{d}}$ and s_N is induced by the inclusion $\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''}\hookrightarrow \mathrm{GL}^{\tau}_{\mathbf{d}}$. Since r_N and s_N are proper, there is a natural morphism

$$s_{N,*}r_{N,*}\mathbb{Q}_{A_N(X(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''},\mathrm{GL}^{\tau}_{\mathbf{d}',\mathbf{d}''})}\otimes\mathbb{L}^{-(\mathbf{d}',\mathbf{d}'')_{\widetilde{Q}}}\to\mathbb{Q}_{A_N(X(\widetilde{Q})_{\mathbf{d}},\mathrm{GL}^{\tau}_{\mathbf{d}})}. \tag{9.2}$$

Applying $\mathrm{Dim}^{\tau}_{N,*}\,\phi_{\mathrm{Tr}(\widetilde{W})}$ and letting $N\mapsto\infty$, morphism (9.2) induces the morphism

$$\begin{aligned}\beta_{\mathbf{d}',\mathbf{d}''}\colon \mathrm{Dim}^{\tau,\circ}_*\,\iota^!\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\tau\mathfrak{M}(\widetilde{Q})_{\mathbf{d}',\mathbf{d}''}}(\mathbb{Q})\otimes\mathbb{L}^{-(\mathbf{d}',\mathbf{d}'')_{\widetilde{Q}}/2} \\ \to \mathrm{Dim}^{\tau}_*\,\iota^!\phi_{\mathfrak{Tr}(\widetilde{W})}\mathcal{IC}_{\tau\mathfrak{M}(\widetilde{Q})_{\mathbf{d}}}(\mathbb{Q}).\end{aligned}$$

Defining

$$m_{\mathbf{d}',\mathbf{d}''}=(\beta_{\mathbf{d}',\mathbf{d}''}\otimes\mathbb{L}^{-\dim(T^{\tau})/2})\circ(\alpha_{\mathbf{d}',\mathbf{d}''}\otimes\mathbb{L}^{-\dim(T^{\tau})/2})\circ\mathtt{TS},$$

where $\mathtt{TS}$ is the Thom–Sebastiani isomorphism [40], gives the multiplication

$$m\colon \mathcal{A}^{\mathcal{S}}_{\tau,\widetilde{Q},\widetilde{W}}\otimes_{\mathrm{H}_{T^{\tau}}}\mathcal{A}^{\mathcal{S}}_{\tau,\widetilde{Q},\widetilde{W}}\to\mathcal{A}^{\mathcal{S}}_{\tau,\widetilde{Q},\widetilde{W}}.$$

We write $\mathcal{A}^{\mathcal{S}}_{\widetilde{Q},\widetilde{W}}$ for the special case in which T^{τ} is the zero-dimensional torus (as in Example 9.2). In this case, the above multiplication is exactly the multiplication defined by Kontsevich and Soibelman in [28]. The proof that, for general T^{τ}, the multiplication is associative is standard, and is in particular unchanged from the proof given in [28, Section 7], to which we refer for fuller details.

9.2. Degeneration. The extra equivariant parameters arising from the torus action on $\mathfrak{M}(\widetilde{Q})$ are not considered in the original paper [28], but were introduced, for the particular cohomological Hall algebras we are considering, in [36,48,49]. In general, such extra parameters are of most interest when they provide a geometric deformation of the original algebra, i.e.

when they provide a flat family of algebras over $\mathrm{Spec}(\mathrm{H}_T)$, such that the specialisation at the central fibre is our original algebra, which in this case is $\mathcal{A}_{\widetilde{Q},\widetilde{W}}$. For T^τ the torus associated to a $\widetilde{W}$-invariant grading of $\widetilde{Q}$, this is precisely the result we prove in this section.

Let $\tau\colon \widetilde{Q}_1 \to \mathbb{Z}^s$ be a $\widetilde{W}$-invariant grading, with associated torus T. Let $\upsilon\colon \mathbb{Z}^s \to \mathbb{Z}^{s'}$ be a surjective morphism of groups, inducing the inclusion of tori $T' \hookrightarrow T$, where T' is the torus associated to the $\widetilde{W}$-invariant grading $\tau' = \upsilon \circ \tau$. Write $s'' = s - s'$. Then picking a splitting of υ, i.e. an extension of υ to an isomorphism $\mathbb{Z}^s \to \mathbb{Z}^{s'} \oplus \mathbb{Z}^{s''}$, induces an isomorphism

$$\mathrm{H}_T \cong \mathrm{H}_{T'} \otimes \mathrm{H}_{T^\chi}, \tag{9.3}$$

where $\chi\colon \widetilde{Q}_1 \to \mathbb{Z}^{s''}$ is induced by the splitting. The splitting of υ induces a splitting

$$\mathfrak{t} \cong \mathfrak{t}' \oplus \mathfrak{t}^\chi.$$

We define $Y_{\tau,\mathbf{d},N} := X(\widetilde{Q})_{\mathbf{d}} \times_{\mathrm{GL}^\tau_{\mathbf{d}}} U_{\tau,\mathbf{d},N}$ and consider the natural maps

$$\upsilon_{\mathbf{d},N}\colon Y_{\tau,\mathbf{d},N} \to \mathrm{Hom}^{\mathrm{surj}}(\mathbb{C}^N, \mathfrak{t}^\chi)/T^\chi =: S_{\chi,N}$$

defined by the morphism

$$\mathrm{Hom}^{\mathrm{surj}}(\mathbb{C}^N, \mathfrak{t}) \to \mathrm{Hom}^{\mathrm{surj}}(\mathbb{C}^N, \mathfrak{t}^\chi), \quad f \mapsto \pi_{\mathfrak{t}^\chi} \circ f.$$

The function $\mathrm{Tr}(\widetilde{W})$ induces functions $\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d},N}\colon Y_{\tau,\mathbf{d},N} \to \mathbb{C}$.

Lemma 9.4. *The space $S_{\chi,N}$ is $s''(N-1)$-dimensional and simply connected, and $\mathrm{H}(S_{\chi,N}, \mathbb{Q})$ is pure.*

Proof. By choosing a splitting $\mathfrak{t}^\chi = \mathbb{C}^{\oplus s''}$ and considering the entries of a morphism $f \in S_{\chi,N}$ one by one, we obtain a sequence of morphisms

$$S_{\chi,N} = H_{s''} \xrightarrow{l_{s''-1}} H_{s''-1} \xrightarrow{l_{s''-2}} \cdots \xrightarrow{l_1} H_1 = \mathbb{P}^{N-1},$$

where l_e is an $(\mathbb{A}^N \setminus \mathbb{A}^e)/\mathbb{C}^*$-fibration, with $\mathbb{C}^*$ acting on $\mathbb{A}^N$ via scaling. All the claims follow from this description. □

Each of the maps $\upsilon_{\mathbf{d},N}$ is a fibre bundle with fibre $Y_{\tau',\mathbf{d},N}$. Picking $i\colon \Upsilon \hookrightarrow S_{\chi,N}$ the inclusion of a sufficiently small open ball (in the analytic topology) contained in the base, we may write

$$\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d},N}|_{\upsilon^{-1}_{\mathbf{d},N}(\Upsilon)}\colon \Upsilon \times_{\upsilon_{\mathbf{d},N}} Y_{\tau,\mathbf{d},N} \cong \Upsilon \times Y_{\tau',\mathbf{d},N} \to \mathbb{C}$$

as $\mathrm{Tr}(\widetilde{W})_{\tau',\mathbf{d},N} \circ \pi$, where $\pi\colon \upsilon^{-1}_{\mathbf{d},N}(\Upsilon) \to Y_{\tau',\mathbf{d},N}$ is the projection, and so we deduce that the mixed Hodge modules $\mathcal{H}^q(\upsilon_{\mathbf{d},N,*}\phi_{\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d},N}}\mathbb{Q}_{Y_{\tau,\mathbf{d},N}})$ are locally trivial in the analytic topology, with fibre given by $\mathrm{H}^q(Y_{\tau',\mathbf{d},N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau',\mathbf{d},N}}\mathbb{Q}_{Y_{\tau',\mathbf{d},N}})$, and are furthermore globally trivial by the rigidity theorem [44, Theorem 4.20], since the base of $\upsilon_{\mathbf{d},N}$ is simply connected.

The Leray spectral sequence $E^{\bullet,\bullet}_{\upsilon,\mathbf{d},N,\bullet}$ converging to

$$\mathrm{H}(Y_{\tau,\mathbf{d},N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d},N}}\mathcal{IC}_{Y_{\tau,\mathbf{d},N}}(\mathbb{Q}) \otimes \mathbb{L}^{(\dim(V_{\tau,\mathbf{d},N})-s)/2})$$

therefore satisfies

$$E^{p,q}_{\upsilon,\mathbf{d},N,2} = \mathrm{H}^p(S_{\chi,N}, \mathbb{Q}) \otimes \mathrm{H}^q(Y_{\tau',\mathbf{d},N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau',\mathbf{d},N}}\mathcal{IC}_{Y_{\tau',\mathbf{d},N}}(\mathbb{Q}) \otimes \mathbb{L}^{(\dim(V_{\tau',\mathbf{d},N})-s')/2}). \tag{9.4}$$

Set

$$\heartsuit^{(\prime)} = (\dim(U_{\tau^{(\prime)},\mathbf{d},N}) - s^{(\prime)}N - (\mathbf{d}',\mathbf{d}'')_{\widetilde{Q}})/2, \quad \spadesuit^{(\prime)} = (\dim(U_{\tau^{(\prime)},\mathbf{d},N}) - s^{(\prime)}N)/2.$$

In similar fashion, we obtain spectral sequences $E^{\bullet,\bullet}_{\upsilon,N,\mathbf{d}',\mathbf{d}'',\bullet}$ and $E^{\bullet,\bullet}_{\upsilon,N,\mathbf{d}'\times\mathbf{d}'',\bullet}$ satisfying

$$E^{p,q}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,2} = \mathrm{H}^p(S_{\chi,N},\mathbb{Q}) \otimes \mathrm{H}^q(Y_{\tau',\mathbf{d}',\mathbf{d}'',N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau',\mathbf{d}',\mathbf{d}'',N}} \mathcal{IC}_{Y_{\tau',\mathbf{d}',\mathbf{d}'',N}}(\mathbb{Q}) \otimes \mathbb{L}^{\heartsuit'}),$$
$$E^{p,q}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,2} = \mathrm{H}^p(S_{\chi,N},\mathbb{Q}) \otimes \mathrm{H}^q(Y_{\tau',\mathbf{d}'\times\mathbf{d}'',N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau',\mathbf{d}'\times\mathbf{d}'',N}} \mathcal{IC}_{Y_{\tau',\mathbf{d}'\times\mathbf{d}'',N}}(\mathbb{Q}) \otimes \mathbb{L}^{\spadesuit'})$$

converging to

$$\mathrm{H}(Y_{\tau,\mathbf{d}',\mathbf{d}'',N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d}',\mathbf{d}'',N}} \mathcal{IC}_{Y_{\tau,\mathbf{d}',\mathbf{d}'',N}}(\mathbb{Q}) \otimes \mathbb{L}^{\heartsuit}),$$
$$\mathrm{H}(Y_{\tau,\mathbf{d}'\times\mathbf{d}'',N}, \phi_{\mathrm{Tr}(\widetilde{W})_{\tau,\mathbf{d}'\times\mathbf{d}'',N}} \mathcal{IC}_{Y_{\tau,\mathbf{d}'\times\mathbf{d}'',N}}(\mathbb{Q}) \otimes \mathbb{L}^{\spadesuit}),$$

respectively. As in the construction of $\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W}}$, we obtain a commutative diagram of morphisms of spectral sequences, with vertical morphisms provided by restriction morphisms in cohomology,

$$\begin{array}{ccccc} E^{\bullet,\bullet}_{\upsilon,\mathbf{d},N+1,\bullet} & \longleftarrow & E^{\bullet,\bullet}_{\upsilon,\mathbf{d}',\mathbf{d}'',N+1,\bullet} & \longleftarrow & E^{\bullet,\bullet}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N+1,\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ E^{\bullet,\bullet}_{\upsilon,\mathbf{d},N,\bullet} & \longleftarrow & E^{\bullet,\bullet}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,\bullet} & \longleftarrow & E^{\bullet,\bullet}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,\bullet}. \end{array} \tag{9.5}$$

Each of the spectral sequences $E^{\bullet,\bullet}_{\upsilon,\mathbf{d},N,\bullet}$, $E^{\bullet,\bullet}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,\bullet}$, $E^{\bullet,\bullet}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,\bullet}$ is a first quadrant spectral sequence, and each of the limits

$$\lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d},N,2}, \quad \lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,2}, \quad \text{and} \quad \lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,2}$$

exists as in Section 3.2. We claim the following commutativity of limits:

$$\begin{aligned} \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}} &\cong \lim_{N\mapsto\infty}\lim_{l\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d},N,l} \\ &\cong \lim_{l\mapsto\infty}\lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d},N,l}, \\ \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}',\mathbf{d}''} &\cong \lim_{N\mapsto\infty}\lim_{l\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,l} \\ &\cong \lim_{l\mapsto\infty}\lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}',\mathbf{d}'',N,l}, \\ \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}'} \otimes_{\mathrm{H}_T} \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}''} &\cong \lim_{N\mapsto\infty}\lim_{l\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,l} \\ &\cong \lim_{l\mapsto\infty}\lim_{N\mapsto\infty} E^{p,q}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',N,l}, \end{aligned} \tag{9.6}$$

using the shorthand $\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}',\mathbf{d}''} \cong \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}'} \otimes_{\mathrm{H}_T} \mathcal{A}_{\tau,\widetilde{Q},\widetilde{W},\mathbf{d}''} \otimes \mathbb{L}^{-(\mathbf{d}',\mathbf{d}'')_{\widetilde{Q}}/2}$.

The argument for all three statements is the same: fixing p and q, the limit $E^{p,q}_{\upsilon,\mathbf{d},N,\infty}$ depends only on a finite portion of $E^{p,q}_{\upsilon,\mathbf{d},N,s}$, which therefore stabilises for sufficiently large $N = N_{p,q}$. The (p,q)-term of both the second and third expression of (9.6) are then given by $E^{p,q}_{\upsilon,\mathbf{d},N_{p,q},\infty}$.

We may define the cohomological Hall algebra multiplication on $\mathcal{A}_{\tau,\widetilde{Q},\widetilde{W}}$ via the commutative diagram obtained from (9.1) or as the morphism induced in the double limit by the composition of the horizontal morphisms in (9.5). Via the morphism $E^{\bullet,\bullet}_{\upsilon,\mathbf{d},\infty,2} \to E^{0,\bullet}_{\mathbf{d},\infty,2}$ to the degenerate spectral sequence concentrated on the first nontrivial column, and the analogous morphisms for the spectral sequences $E^{\bullet;\bullet}_{\upsilon,\mathbf{d}',\mathbf{d}'',\infty,\bullet}$ and $E^{\bullet;\bullet}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',\infty,\bullet}$, we obtain a commutative diagram of double limits

$$\begin{array}{ccccc}
\lim_{l\mapsto\infty} E^{\bullet;\bullet}_{\upsilon,\mathbf{d},\infty,l} & \longrightarrow & \lim_{l\mapsto\infty} E^{\bullet;\bullet}_{\upsilon,\mathbf{d}',\mathbf{d}'',\infty,l} & \longrightarrow & \lim_{l\mapsto\infty} E^{\bullet;\bullet}_{\upsilon,\mathbf{d}'\times\mathbf{d}'',\infty,l} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_{\tau',\tilde{Q},\tilde{W},\mathbf{d}} & \longrightarrow & \mathcal{A}_{\tau',\tilde{Q},\tilde{W},\mathbf{d}',\mathbf{d}''} & \longrightarrow & \mathcal{A}_{\tau',\tilde{Q},\tilde{W},\mathbf{d}'} \otimes_{\mathrm{H}'_T} \mathcal{A}_{\tau',\tilde{Q},\tilde{W},\mathbf{d}''},
\end{array}$$

providing a lift of the natural morphism

$$\mathcal{A}_{\tau,\tilde{Q},\tilde{W}} \otimes_{\mathrm{H}_T} \mathrm{H}_{T'} \to \mathcal{A}_{\tau',\tilde{Q},\tilde{W}} \tag{9.7}$$

in cohomology to a morphism in the category of algebra objects in the category of complexes of mixed Hodge structures.

Theorem 9.5. *Let*

$$\tilde{Q}_1 \xrightarrow{\tau} \mathbb{Z}^s \xrightarrow{\upsilon} \mathbb{Z}^{s'}, \qquad \tau' = \upsilon\circ\tau$$

be as above a specialisation of a $\tilde{W}$-invariant weighting of $\tilde{Q}$. Then there is an isomorphism of mixed Hodge structures $\mathcal{A}_{\tau,\tilde{Q},\tilde{W}} \cong \mathcal{A}_{\tau',\tilde{Q},\tilde{W}} \otimes_{\mathbb{Q}} \mathrm{H}_{T^\chi}$ with T^χ as in (9.3)*. Furthermore, morphism* (9.7) *is an isomorphism, and both sides of this isomorphism are pure.*

Proof. First we consider the special case $s'=0$, $\tau'=0$. Then, by Theorem A and Lemma 9.4, the right-hand side of (9.4) is a pure Hodge structure, and so the spectral sequence $E^{\bullet;\bullet}_{\upsilon,d,\infty,\bullet}$ degenerates at the second sheet, and the existence of the required isomorphism follows, along with the fact that (9.7) is an isomorphism. As a consequence, $A_{\tau,\tilde{Q},\tilde{W}}$ is pure for all τ. So it follows that, for general υ, the right-hand side of (9.4) is pure, and the general case follows via the same argument as the special case. □

Let the torus $T=(\mathbb{C}^*)^s$ act on $\mathfrak{M}(\tilde{Q})_{\mathbf{d}}$ via the weight function $\tau\colon \tilde{Q}_1 \to \mathbb{Z}^s$. Then, for each $\mathbf{d}\in\mathbb{N}^{Q_0}$, ignoring the overall Tate twist, via Theorem 3.4, there is an isomorphism in Borel–Moore homology

$$\Psi_{\tau,Q,\mathbf{d}}\colon \mathrm{H}^{\mathrm{BM}}({}^{\tau}\mathfrak{M}(\tilde{Q})_{\mathbf{d}}, \phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{Q,\mathbf{d}}(0),\mathbb{Q}) =: \mathcal{A}_{\tau,\Pi_Q,\mathbf{d}}.$$

The domain of $\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \Psi_{\tau,Q,\mathbf{d}}$ carries the Kontsevich–Soibelman cohomological Hall algebra product recalled above, while the target carries the Schiffmann–Vasserot product [42, Section 4]. By [36, Corollary 4.5] or [49], the modified morphism

$$\Psi'_{\tau,Q} := \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} (-1)^{\sum_{i\in Q_0}\binom{\mathbf{d}_i}{2}} \Psi_{\tau,Q,\mathbf{d}}\colon \mathcal{A}_{\tau,\tilde{Q},\tilde{W}} \to \mathcal{A}_{\tau,\Pi_Q} \tag{9.8}$$

is an isomorphism of algebras. Since $\Psi_{\tau,Q,\mathbf{d}}$ is a morphism of H_T-modules, we deduce the following corollary of Theorem 9.5.

Corollary 9.6. *Let $\mathfrak{m}$ be the maximal homogeneous ideal in H_T. Then $\mathcal{A}_{\tau,\Pi_Q}$ is free as an H_T-module, and the natural morphism of algebras*

$$\Xi\colon \mathcal{A}_{\tau,\Pi_Q} \otimes_{\mathrm{H}_T} (\mathrm{H}_T/\mathfrak{m}) \to \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}_{\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{Q,\mathbf{d}}(0),\mathbb{Q})$$

is an isomorphism.

10. Shuffle algebras, torsion-freeness and noncommutativity

10.1. Definition. Fix a weight function $\tau\colon \widetilde{Q}_1 \to \mathbb{Z}^s$, and set $T = T^\tau = \mathrm{Hom}(\mathbb{Z}^s,\mathbb{C}^*)$. We recall the shuffle algebra description of the cohomological Hall algebra

$$\mathcal{A}_{\tau,\widetilde{Q}} = \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}({}^{\tau}\mathfrak{M}(\widetilde{Q})_{\mathbf{d}},\mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_{\widetilde{Q}}/2}.$$

Set $\Bbbk := \mathrm{H}_T$. Since $X(\widetilde{Q})_{\mathbf{d}}$ is equivariantly contractible, there is an isomorphism in cohomology

$$\mathrm{H}_{T\times\mathrm{GL}_{\mathbf{d}}}(X(\widetilde{Q})_{\mathbf{d}},\mathbb{Q}) \cong \Bbbk[x_{i,n} \mid i\in Q_0,\ 1\le n\le \mathbf{d}_i]^{\mathfrak{S}_{\mathbf{d}}}.$$

Here $\mathfrak{S}_{\mathbf{d}} = \prod_{i\in Q_0}\mathfrak{S}_{\mathbf{d}_i}$ is the product of symmetric groups, with $\mathfrak{S}_{\mathbf{d}_i}$ acting by permuting the variables $x_{i,1},\dots,x_{i,\mathbf{d}_i}$. For $\mathbf{d}'+\mathbf{d}''=\mathbf{d}$, we define $\mathrm{Sh}_{\mathbf{d}',\mathbf{d}''}\subset\mathfrak{S}_{\mathbf{d}}$ to be the subset of permutations $(\sigma_i)_{i\in Q_0}$ such that, for each $i\in Q_0$, we have inequalities $\sigma_i(1)<\sigma_i(2)<\dots<\sigma_i(\mathbf{d}'_i)$ and $\sigma_i(\mathbf{d}'_i+1)<\dots<\sigma_i(\mathbf{d}_i)$. We fix generators $t_1,\dots,t_s$ of H_T, with t_i corresponding to the generator of the equivariant cohomology of $\mathrm{Hom}(\mathbb{Z}_i,\mathbb{C}^*)$, where $\mathbb{Z}_i$ is the ith copy of $\mathbb{Z}$ inside $\mathbb{Z}^s$. For $a\in\widetilde{Q}_1$, define $E_a(z) = z+\sum_{i\le s}\tau(a)_i t_i$. We use $\star$ to denote the multiplication in the CoHA $\mathcal{A}_{\tau,\widetilde{Q}}$. Then it is shown, as in [28, Section 1],

$$\begin{aligned}
&f(x_{1,1},\dots,x_{r,\mathbf{d}'_r}) \star g(x_{1,1},\dots,g_{r,\mathbf{d}''_r})\\
&\quad= \sum_{\sigma\in\mathrm{Sh}_{\mathbf{d}',\mathbf{d}''}} \sigma\Bigg(f(x_{1,1},\dots,x_{r,\mathbf{d}'_r}) g(x_{1,\mathbf{d}'_1+1},x_{1,\mathbf{d}'_1+2},\dots,x_{1,\mathbf{d}_1},x_{2,\mathbf{d}'_2+1},\dots,x_{r,\mathbf{d}_r})\\
&\qquad\qquad \cdot \prod_{a\in\widetilde{Q}_1}\Big(\prod_{\substack{1\le m\le \mathbf{d}'_{s(a)}\\ \mathbf{d}'_{t(a)}<n\le\mathbf{d}_{t(a)}}} E_a(x_{t(a),n}-x_{s(a),m})\Big)\prod_{i\in Q_0}\Big(\prod_{\substack{1\le m\le\mathbf{d}'_i\\ \mathbf{d}'_i<n\le\mathbf{d}_i}}(x_{i,n}-x_{i,m})^{-1}\Big)\Bigg).
\end{aligned}$$

Let $z\colon Z\hookrightarrow X(\widetilde{Q})_{\mathbf{d}} =: X$ be the subvariety cut out by the matrix-valued equation

$$\sum_{a\in Q_1}[a,a^*] = 0.$$

Then, since $Z\subset \mathrm{Tr}(\widetilde{W})^{-1}(0)$, there is a (dual) restriction map $z_*z^!\mathbb{Q}_X \to \phi_{\mathrm{Tr}(\widetilde{W})}\mathbb{Q}_X$, inducing the morphism α in the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^{T\times\mathrm{GL}_{\mathbf{d}}}(X(\widetilde{Q})_{\mathbf{d}},\phi_{\mathrm{Tr}(\widetilde{W})}\mathbb{Q})\otimes\mathbb{L}^{(\mathbf{d},\mathbf{d})_{\widetilde{Q}}/2} & & \\
\uparrow{\scriptstyle\alpha} & \overset{\Phi_{\mathbf{d}}}{\searrow} & \\
\mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(Z,\mathbb{Q})\otimes\mathbb{L}^{s-(\mathbf{d},\mathbf{d})_{\widetilde{Q}}/2} & \xrightarrow{\ \beta\ } & \mathrm{H}^{T\times\mathrm{GL}_{\mathbf{d}}}(X(\widetilde{Q})_{\mathbf{d}},\mathbb{Q})\otimes\mathbb{L}^{(\mathbf{d},\mathbf{d})_{\widetilde{Q}}/2}\\
\downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\epsilon}\\
\mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{Q,\mathbf{d}}(0),\mathbb{Q})\otimes\mathbb{L}^{s-(\mathbf{d},\mathbf{d})_Q} & \xrightarrow{\ \gamma\ } & \mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(X(\overline{Q})_{\mathbf{d}},\mathbb{Q})\otimes\mathbb{L}^{s-(\mathbf{d},\mathbf{d})_Q}.
\end{array}$$

The morphisms κ, ϵ are isomorphisms because they are induced by affine fibrations, while α is an isomorphism by Theorem 3.4. Then we define $\Phi_{\mathbf{d}} = \beta\alpha^{-1}$. In particular, $\Phi_{\mathbf{d}}$ is injective if and only if γ is.

Proposition 10.1. *The morphism*

$$\Phi\colon \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{T\times \mathrm{GL}_{\mathbf{d}}}(X(\tilde{Q})_{\mathbf{d}}, \phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_{\tilde{Q}}/2} \to \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{T\times \mathrm{GL}_{\mathbf{d}}}(X(\tilde{Q})_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d},\mathbf{d})_{\tilde{Q}}/2}$$

is an algebra morphism, where the domain and target are given the KS cohomological Hall algebra structure.

Proof. For $a \in \tilde{Q}_1$, we define

$$E_a^{\mathrm{tw}}(z) = \begin{cases} E_a(z) & \text{if } a \neq \omega_i \text{ for } i \in Q_0, \\ -E_a(z) & \text{if } a = \omega_i \text{ for } i \in Q_0. \end{cases}$$

The shuffle algebra $\mathcal{A}^{\mathrm{tw}}_{\tau,\tilde{Q}}$ is defined to have the same underlying graded vector space as $\mathcal{A}_{\tau,\tilde{Q}}$, with shuffle multiplication defined as above, but with all instances of $E_a(\bullet)$ replaced by $E_a^{\mathrm{tw}}(\bullet)$. Let

$$F\colon \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{Q,\mathbf{d}}(0), \mathbb{Q}) \otimes \mathbb{L}^{s-(\mathbf{d},\mathbf{d})_Q} \to \mathcal{A}^{\mathrm{tw}}_{\tau,\tilde{Q}}$$

be the morphism defined by taking the sum of $\epsilon^{-1}\gamma$ over all $\mathbf{d}$. By [42], F is an algebra homomorphism. We define an isomorphism $\Gamma\colon \mathcal{A}^{\mathrm{tw}}_{\tau,\tilde{Q}} \to \mathcal{A}_{\tau,\tilde{Q}}$ by setting

$$\Gamma_{\mathbf{d}} = (-1)^{\sum_{i\in Q_0}\binom{\mathbf{d}_i}{2}} \cdot \mathrm{id}_{\mathcal{A}_{\tau,\tilde{Q},\mathbf{d}}}.$$

We define $\Psi'_{\tau,Q}$ as in (9.8). Then it follows that $\Phi = \Gamma \circ F \circ \Psi'_{\tau,Q}$ is a composition of algebra morphisms. □

10.2. Torsion-freeness. The algebra $\mathcal{A}_{\tau,\Pi_Q}$ is still not wholly understood, despite intensive study. On the other hand, in their work on the AGT conjectures [42, Section 4.3], Schiffmann and Vasserot conjectured that, for τ as defined in Example 9.3, the morphism Φ (or equivalently, the morphism F) is in fact an embedding of algebras, making the cohomological Hall algebra $\mathcal{A}_{\tau,\Pi_{Q_{\mathrm{Jor}}}}$ much more manageable. We prove this conjecture in the case of a *general* quiver Q, for any sufficiently large T.

Theorem 10.2. *Let Q be a finite quiver, and let $\mathbf{d} \in \mathbb{N}^{Q_0}$ be a dimension vector. Let $\tau\colon \tilde{Q}_1 \to \mathbb{Z}^s$ be a $\tilde{W}$-invariant grading such that the grading of Example* 9.3 *is a specialisation of τ. The $\mathrm{H}_{T\times\mathrm{GL}_{\mathbf{d}}}$-module $\mathrm{H}^{\mathrm{BM}}_{T\times\mathrm{GL}_{\mathbf{d}}}(\mu^{-1}_{Q,\mathbf{d}}(0), \mathbb{Q})$ is torsion-free, and the natural map F to the shuffle algebra $\mathcal{A}^{\mathrm{tw}}_{\tau,\tilde{Q}}$ is an inclusion of algebras.*

Proof. The passage from torsion-freeness to all of the other statements of the theorem is as explained in [42], so we focus on torsion-freeness. The proof for this is a modification of [43, Proposition 4.6]; the original statement of this result in [43], and its proof, require modification, which we indicate.

Firstly, by assumption, T contains two one-dimensional tori $\mathbb{C}_1^*$ and $\mathbb{C}_2^*$, where $\mathbb{C}_1^*$ acts on arrows a, a^*, ω_i, for $a \in Q_1$ and $i \in Q_0$ with weights $1, -1, 0$, respectively, and $\mathbb{C}_2^*$ acts with weights $1, 0, -1$, respectively. Let $\Bbbk_i = \mathrm{H}_{\mathbb{C}_i^*}$, let $I_i \subset \mathrm{H}_{T\times \mathrm{GL}_{\mathbf{d}}}$ be the ideal of functions vanishing on $\mathrm{Lie}(\mathbb{C}_i^*) \subset \mathrm{Lie}(T \times \mathrm{GL}_{\mathbf{d}})$, and let K_i be the fraction field of $\Bbbk_i$. Considering $X(Q)_{\mathbf{d}}$ as a subvariety of $\mu_{Q,\mathbf{d}}^{-1}(0)$ via the extension by zero map, $X(Q)_{\mathbf{d}}$ contains the fixed locus of the $\mathbb{C}_1^*$-action on $\mu_{Q,\mathbf{d}}^{-1}(0)$, and so the pushforward

$$(\mathrm{H}_{T\times \mathrm{GL}_{\mathbf{d}}})_{I_1} \cong \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(X(Q)_{\mathbf{d}}, \mathbb{Q})_{I_1} \to \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q})_{I_1}$$

is an isomorphism by [15, Theorem 6.2]. It is thus enough to prove that $\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q})$ is S_1-torsion-free for $S_1 = \mathrm{H}_{T\times \mathrm{GL}_{\mathbf{d}}} \setminus I_1$. The $\Bbbk_2$-module $\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q})$ is free by Theorem 9.5, and so the morphism

$$\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q}) \to \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q}) \otimes_{\Bbbk_2} K_2 \tag{10.1}$$

is an embedding. Therefore, it is sufficient to show that the right-hand side of (10.1) has no S_1-torsion. By two applications of dimensional reduction (see Theorem 3.4, and the discussion before the proof of Lemma 4.3), we have the $\Bbbk_2$-linear isomorphisms (leaving out Tate twists/shifts in cohomological degree)

$$\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mu_{Q,\mathbf{d}}^{-1}(0), \mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}({}^{\tau}\mathfrak{M}(\tilde{Q})_{\mathbf{d}}, \phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mathcal{C}_{\mathbf{d}}, \mathbb{Q}),$$

where $\mathcal{C}_{\mathbf{d}} \subset X(Q^+)_{\mathbf{d}}$ is the subspace of Q^+-modules such that the linear maps assigned to the loops ω_i define a $\mathbb{C}Q$-module endomorphism. We define $\mathcal{N}_{\mathbf{d}} = \mathcal{C}_{\mathbf{d}} \cap X(Q^+)_{\mathbf{d}}^{\omega\text{-nilp}}$. Since the torus $\mathbb{C}_2^*$ acts by scaling the loops ω_i, the natural map

$$\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mathcal{N}_{\mathbf{d}}, \mathbb{Q}) \otimes_{\Bbbk_2} K_2 \to \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mathcal{C}_{\mathbf{d}}, \mathbb{Q}) \otimes_{\Bbbk_2} K_2$$

is an isomorphism. So it is sufficient to show that $\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_{\mathbf{d}}}(\mathcal{N}_{\mathbf{d}}, \mathbb{Q})$ has no S_1-torsion.

Set $\mathbb{Q}[t_1] = \mathrm{H}_{\mathbb{C}_1^*}$ and $A = \mathrm{H}_{\mathbb{C}_2^*\times \mathrm{GL}_{\mathbf{d}}}$, so there is a natural isomorphism $\mathrm{H}_{T\times \mathrm{GL}_{\mathbf{d}}} \cong A[t_1]$. Elements of S_1 are written $p(t_1) + \sum_{i\geq 0} a_i t_1^i$, where $a_i \in A$ and $0 \neq p(t_1) \in \mathbb{Q}[t_1]$. We consider the following stratification of the space $\mathcal{N}_{\mathbf{d}}$ by Jordan types: if $\overline{\pi} = (\pi^{(i)})_{i\in Q_0}$ is a tuple of partitions, with each $\pi^{(i)}$ a partition of $\mathbf{d}_i$, the stratum $\mathcal{N}_{\overline{\pi}} \subset \mathcal{N}_{\mathbf{d}}$ is the space for which the Jordan normal form of the operator assigned to ω_i has blocks with sizes given by $\pi^{(i)}$. The space $\mathcal{N}_{\overline{\pi}}$ can be $T \times \mathrm{GL}_{\mathbf{d}}$-equivariantly contracted onto the subspace $\mathcal{N}'_{\overline{\pi}}$ for which all arrows $a \in Q_1$ act via the zero matrix, and the $T \times \mathrm{GL}_{\mathbf{d}}$-action is transitive on this subspace. So if $\rho \in \mathcal{N}'_{\overline{\pi}}$ has stabiliser group H, which we may decompose $H = \mathbb{C}_1^* \times H'$ since $\mathbb{C}_1^*$ acts trivially, there are isomorphisms $\mathrm{H}^{\mathrm{BM}}_T(\mathcal{N}_{\overline{\pi}}, \mathbb{Q}) \cong \mathrm{H}_H(\mathrm{pt}, \mathbb{Q}) \cong \mathrm{H}_{H'}(\mathrm{pt}, \mathbb{Q})[t_1]$. This module has no S_1-torsion, and the claim that $\mathrm{H}^{\mathrm{BM}}(\mathcal{N}_{\mathbf{d}}, \mathbb{Q})$ has no S_1-torsion follows from the long exact sequences in compactly supported cohomology induced by the stratification of $\mathcal{N}_{\mathbf{d}}$. □

The same proof works with $\mu_{Q,\mathbf{d}}^{-1}(0)$ replaced by $\mu_{Q,\mathbf{d}}^{-1}(0)^{\sharp}$ for $\sharp$ any of $\mathcal{SN}, \mathcal{SSN}, \mathcal{N}$.

Remark 10.3. It is possible for the strata $\mathcal{N}_{\overline{\pi}}$ to have S_2-torsion, so we cannot substitute I_2 for I_1 in the above proof, and merely insist on the inclusion $\mathbb{C}_2^* \subset T$. Indeed, we show in Section 10.4 that this (stronger) version of the statement of Theorem 10.2 with (weaker) assumptions is false.

10.3. Noncommutativity. Theorem 10.2 enables explicit calculations inside $\mathcal{A}_{\tau,\Pi_Q}$. Furthermore, although (as we have seen in Remark 10.3, and will see further, with Proposition 10.7) it is important that we work equivariantly with respect to a sufficiently large torus T in Theorem 10.2, we will demonstrate in this section how Theorem 10.2 enables us to perform concrete calculations for *trivial* T, i.e. in the undeformed preprojective CoHA $\mathcal{A}_{\Pi_Q}$.

We use explicit calculations in the algebra $\mathcal{A}_{\tau,\widetilde{Q_{\mathrm{Jor}}}}$ to show that $\mathcal{A}_{\widetilde{Q_{\mathrm{Jor}}},\widetilde{W}} \cong \mathcal{A}_{\Pi_{Q_{\mathrm{Jor}}}}$ is noncommutative[6]. Recall that, by Theorem 3.6, for an arbitrary (symmetric) quiver Q with potential W, there is a PBW isomorphism

$$\mathrm{Sym}(\mathrm{BPS}_{Q,W} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}}) \cong \mathcal{A}_{Q,W}.$$

By Theorem 5.1, there is an isomorphism of cohomologically graded vector spaces

$$\mathrm{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\widetilde{W},d} \cong \mathbb{Q}[3]$$

so that, for each $d \geq 1$ and $e \geq 0$, there is an element $\alpha_d^{(e)}$, of cohomological degree $2e-2$, well defined up to scalar, defined to be the image of $1 \otimes u^e$ under the embedding

$$\mathrm{BPS}_{\widetilde{Q_{\mathrm{Jor}}},\widetilde{W},d} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}} \hookrightarrow \mathcal{A}_{\widetilde{Q_{\mathrm{Jor}}},\widetilde{W}}.$$

Lemma 10.4. *The commutator* $[\alpha_1^{(1)}, \alpha_1^{(0)}]$ *is nonzero, so* $\mathcal{A}_{\Pi_{Q_{\mathrm{Jor}}}}$ *is noncommutative.*

Proof. Set $Q = Q_{\mathrm{Jor}}$. Pick τ as in Example 9.3, with associated torus $T \cong \mathbb{C}_1^* \times \mathbb{C}_2^*$ in the notation of the proof of Theorem 10.2. By Theorem 10.2, the morphism $\iota\colon \mathcal{A}_{\tau,\Pi_Q} \to \mathcal{A}_{\tau,\tilde{Q}}$ is an inclusion of algebras. Write $\mathcal{A}' \subset \mathcal{A}_{\tau,\tilde{Q}}$ for the image of this inclusion. Then, by Corollary 9.6, there is an isomorphism of algebras $\mathcal{A}_{\Pi_Q} \cong \mathcal{A}'/(t_1,t_2)\cdot\mathcal{A}'$. We write

$$\mathcal{A}_{\tau,\Pi_Q,1} \cong \mathcal{A}_{\Pi_Q,1} \otimes \mathrm{H}_T$$

and define $\tilde{\alpha}_1^{(e)} = \alpha_1^{(e)} \otimes 1$. Then $\iota(\tilde{\alpha}_1^{(e)}) = x_1^e \in \mathbb{Q}[x_1,t_1,t_2]$.

First we calculate the commutator in $\mathcal{A}_{\tau,\tilde{Q}}$,

$$\begin{aligned}[x_1, x_1^0] = {} & (x_1-x_2)(x_2-x_1+t_1)(x_2-x_1+t_2)(x_2-x_1-t_1-t_2)/(x_2-x_1)\\ & + (x_2-x_1)(x_1-x_2+t_1)(x_1-x_2+t_2)(x_1-x_2-t_1-t_2)/(x_1-x_2)\\ = {} & -2t_1t_2(t_1+t_2).\end{aligned}$$

This element has cohomological degree -2. We claim that the unique nonzero element of cohomological degree less than -2 in $\iota(\mathcal{A}')$ is $x_1^0 \star x_1^0$ (up to scalar). Firstly, $x_1^0 \star x_1^0$ has cohomological degree -4 since x_1^0 has cohomological degree -2. Secondly, it is indeed nonzero, as we calculate below. Finally, it follows from e.g. Corollary 5.3 that the stack $\mathcal{C}_2 \cong \mathfrak{Coh}_d(\mathbb{A}^2)$ of pairs of commuting 2×2 matrices has a unique irreducible component of (complex) dimension greater than 1, and that component has dimension 2. Equivalently, $\mathcal{A}_{\tau,\Pi_Q}$ is concentrated in cohomological degrees at least -4, and in degree -4 is one-dimensional. Now we calculate

$$\begin{aligned}x_1^0 \star x_1^0 = {} & (x_2-x_1+t_1)(x_2-x_1+t_2)(x_2-x_1-t_1-t_2)/(x_2-x_1)\\ & + (x_1-x_2+t_1)(x_1-x_2+t_2)(x_1-x_2-t_1-t_2)/(x_1-x_2)\\ = {} & 2(x_1-x_2)^2 - 2(t_1^2+t_1t_2+t_2^2).\end{aligned}$$

[6] Equivalently, since the entire algebra lives in even cohomological degrees, we show that it is not *super*commutative.

Thus

$$[x_1, x_1^0] \notin (t_1, t_2) \cdot (x_1^0 \star x_1^0) \quad \text{and} \quad [x_1, x_1^0] \notin (t_1, t_2) \cdot \mathcal{A}'.$$

It follows that $[\alpha_1^{(0)}, \alpha_1^{(1)}] \neq 0$. □

Corollary 10.5. *There exists a nonzero scalar $\lambda \in \mathbb{Q}$ such that $[\alpha_1^{(1)}, \alpha_1^{(0)}] = \lambda \alpha_2^{(0)}$.*

Proof. In [10], it is shown that, for a general (symmetric) QP (Q', W'), the CoHA $\mathcal{A}_{\tilde{Q},\tilde{W}}$ is a filtered algebra, for the perverse filtration defined by setting

$$P^n \mathcal{A}_{Q',W'} = \mathrm{H}(\mathcal{M}(Q'), \tau^{\leq n} \mathrm{JH}_*^{\zeta} \phi_{\mathfrak{Tr}(W')} \mathcal{IC}_{\mathfrak{M}(Q')}(\mathbb{Q})),$$

and the associated graded algebra is supercommutative. In particular, $[\alpha_1^{(1)}, \alpha_1^{(0)}] \in P^3 \mathcal{A}_{\tilde{Q},\tilde{W}}$ since $\alpha_d^{(i)} \in P^{2i+1} \mathcal{A}_{\tilde{Q},\tilde{W}}$; here we have used commutativity of the associated graded object, along with the calculation $3 + 1 - 1 = 3$.

Via (3.1), for general (symmetric) quiver Q' with potential W', $P^3 \mathcal{A}_{Q',W'}$ is spanned by

$$(\mathrm{BPS}_{Q',W'}[-1]) \star (\mathrm{BPS}_{Q',W'}[-1]), \quad \mathrm{BPS}_{Q',W'}[-1] \quad \text{and} \quad \mathrm{BPS}_{Q',W'}[-3].$$

So $P^3 \mathcal{A}_{\widetilde{Q_{\mathrm{Jor}}},\tilde{W},2}$ is spanned by $\alpha_1^{(0)} \star \alpha_1^{(0)}$, $\alpha_2^{(0)}$, $\alpha_2^{(1)}$, which have cohomological degrees -4, -2, 0, respectively. The cohomological degree of $[\alpha_1^{(1)}, \alpha_1^{(0)}]$ is -2. By Lemma 10.4, $[\alpha_1^{(1)}, \alpha_1^{(0)}] \neq 0$, and the result follows. □

A version of the following result is to be found in [24]. Using ideas from the proof of Corollary 10.5, we give an alternative proof.

Proposition 10.6. *The $\mathbb{Q}$-vector space $\hat{\mathfrak{g}} \subset \mathcal{A}_{\Pi_{Q_{\mathrm{Jor}}}}$ spanned by the elements $\alpha_i^{(n)}$ is closed under the commutator Lie bracket, and there is an isomorphism* $\mathrm{U}(\hat{\mathfrak{g}}) \cong \mathcal{A}_{\Pi_{Q_{\mathrm{Jor}}}}$.

Proof. Set $Q = Q_{\mathrm{Jor}}$. The final statement follows from the first statement and the PBW theorem (Theorem D) for $\mathcal{A}_{\Pi_Q}$. For the first statement, we consider the perverse filtration from the proof of Corollary 10.5. Then $P^l \mathcal{A}_{\tilde{Q},\tilde{W}}$ has a basis given by monomials $\overline{\alpha} = \alpha_{i_1}^{(n_1)} \cdots \alpha_{i_s}^{(n_s)}$ with $\sum_{a=1}^{s}(1 + 2n_a) \leq l$. On the other hand, the cohomological degree of such an element is given by $|\overline{\alpha}| = \sum_{a=1}^{s}(2n_a - 2)$. Set $\beta = [\alpha_i^{(e)}, \alpha_j^{(f)}]$. As in the proof of Corollary 10.5, we have

$$\beta \in P^{2e+2f+1} \mathcal{A}_{\Pi_Q}, \quad |\beta| = 2e + 2f - 4.$$

So β can be written as a linear combination of elements $\overline{\alpha}$ with

$$|\beta| = |\overline{\alpha}| \quad \text{and} \quad \sum_{a=1}^{s}(1 + 2n_a) \leq 2e + 2f + 1.$$

But then $\sum_{a=1}^{s}(2n_a - 2) = 2e + 2f - 4$ since $|\alpha| = |\overline{\beta}|$, and so $3s \leq 5$. So we find $s = 1$, as required. □

The same proof(s) demonstrate that $\mathcal{A}_{\Pi_{Q_{\mathrm{Jor}}}}^{\mathcal{SN}}$ is a universal enveloping algebra for some $\hat{\mathfrak{g}}^{\mathcal{SN}}$ satisfying $\hat{\mathfrak{g}}^{\mathcal{SN}} \cong \hat{\mathfrak{g}}[-2]$ (as graded vector spaces).

10.4. Torsion. We show that torsion-freeness of $\mathcal{A}_{\tau,\Pi_Q}$ can fail if we relax the conditions on τ in Theorem 10.2.

Proposition 10.7. *Let $T = \mathbb{C}^*$ be any one of the tori $\mathbb{C}^*_1$, $\mathbb{C}^*_2$, $\mathbb{C}^*_3$ acting with weights $(1,-1,0)$ or $(1,0,-1)$ or $(0,1,-1)$ on the three arrows a, a^*, ω of $\widetilde{Q_{\mathrm{Jor}}}$. Let $\sharp$ be one of $\emptyset, \mathcal{N}, \mathcal{SN}, \mathcal{SSN}$, chosen so that the pushforward morphism $\Phi_1\colon \mathcal{A}^{\sharp}_{\tau,\Pi_{Q_{\mathrm{Jor}}},1} \to \mathcal{A}_{\tau,\Pi_{Q_{\mathrm{Jor}}},1}$ is injective. Then the $\mathrm{H}_{T\times \mathrm{GL}_2(\mathbb{C})}$-module*

$$\mathcal{A}^{\sharp}_{\tau,\Pi_{Q_{\mathrm{Jor}}},2} = \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_2(\mathbb{C})}(\mu^{-1}_{Q_{\mathrm{Jor}},2}(0)^{\sharp}, \mathbb{Q})$$

is not torsion-free. The natural map $\Phi\colon \mathcal{A}^{\sharp}_{\tau,\Pi_{Q_{\mathrm{Jor}}}} \to \mathcal{A}_{\tau,\widetilde{Q_{\mathrm{Jor}}}}$ from the cohomological Hall algebra to the shuffle algebra is not injective.

For $\sharp = \emptyset, \mathcal{N}, \mathcal{SN}, \mathcal{SSN}$, we have that $\mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})^{\sharp}_1 = \mathbb{A}^i$ for $i = 3, 1, 2, 2$, respectively, and the condition on Φ_1 is just that the equivariant Euler class E of the normal bundle of the inclusion $\mathbb{A}^i \hookrightarrow \mathbb{A}^3 = \mathcal{M}(\widetilde{Q_{\mathrm{Jor}}})_1$ is nonzero.

Proof of Proposition 10.7. Set $Q = Q_{\mathrm{Jor}}$. Torsion-freeness is equivalent to the statement that the morphism

$$\mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_2(\mathbb{C})}(\mu^{-1}_{Q,2}(0)^{\sharp}, \mathbb{Q}) \to \mathrm{H}^{\mathrm{BM}}_{T\times \mathrm{GL}_2(\mathbb{C})}(X(\overline{Q})_2, \mathbb{Q})$$

is injective. We start with the $\sharp = \emptyset$ case. The commutator map

$$[\cdot,\cdot]\colon \mathcal{A}_{\tau,\Pi_Q,1} \otimes \mathcal{A}_{\tau,\Pi_Q,1} \to \mathcal{A}_{\tau,\Pi_Q,2}$$

is nonzero since, by Lemma 10.4, it is nonzero after tensoring with $(\mathrm{H}_T \,/\, \mathrm{H}^{\geq 2}_T)$. Writing

$$\mathcal{A}_{\tau,\Pi_Q,1} \cong \mathrm{H}(\mathbb{A}^3, \mathbb{Q}) \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q}) \otimes \mathrm{H}(\mathrm{B}T, \mathbb{Q}) \cong \mathbb{Q} \otimes \mathbb{Q}[u] \otimes \mathbb{Q}[t],$$

we have seen that $[1,u] \neq 0$. For $T = \mathbb{C}^*_l$ with $l = 1,2,3$, the shuffle algebra $\mathcal{A}_{\tau,\tilde{Q}}$ is commutative; e.g. for the $\mathbb{C}^*_2$ case, from the equalities $E_{a^*}(z) = z$ and $E_a(z) = -E_{\omega}(-z)$ for a the unique arrow in Q, it follows that $\mathcal{A}_{\tau,\tilde{Q}}$ is commutative. So $\Phi_2(\mathrm{Im}([\cdot,\cdot])) = 0$ and Φ_2 is not injective, proving both parts of the proposition.

Now let $\sharp \neq \emptyset$ and $E \neq 0$. Then the image of the algebra homomorphism

$$\mathcal{A}^{\sharp}_{\tau,\Pi_{Q_{\mathrm{Jor}}}} \to \mathcal{A}_{\tau,\Pi_{Q_{\mathrm{Jor}}}},$$

restricted to $\mathcal{A}^{\sharp}_{\tau,\Pi_{Q_{\mathrm{Jor}}},1}$, contains the elements E, Eu. Then (in $\mathcal{A}_{\tau,\Pi_{Q_{\mathrm{Jor}}},2}$) we have

$$[E, Eu] = E^2[1,u] \neq 0$$

since $\mathcal{A}_{\tau,\Pi_{Q_{\mathrm{Jor}}},2}$ is free as a $\mathbb{Q}[t]$-module. In particular, $[\cdot,\cdot]\colon \mathcal{A}^{\sharp}_{\tau,\Pi_Q,1} \otimes \mathcal{A}^{\sharp}_{\tau,\Pi_Q,1} \to \mathcal{A}^{\sharp}_{\tau,\Pi_Q,2}$ is nonzero, and the proof continues as in the $\sharp = \emptyset$ case. □

Acknowledgement. I would like to thank Sasha Minets, Tristan Bozec, Olivier Schiffmann, Eric Vasserot, Davesh Maulik and Victor Ginzburg for illuminating conversations that contributed greatly to the paper. In particular, the idea for the proof of the crucial "support lemma" (Lemma 4.1) came from seeing Victor Ginzburg talk about the results of [13] at the Warwick EPSRC symposium "Derived Algebraic Geometry, with a focus on derived symplectic techniques", and the final section of the paper benefitted greatly from Sasha Minets' careful reading of an earlier draft.

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Eingegangen 20. April 2022, in revidierter Fassung 1. September 2023