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Convergence of metrics under self-dual Weyl tensor and scalar curvature bounds

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Abstract. We establish a $C^{1,\alpha}$-compactness theorem for metrics with bounded self-dual Weyl tensor and scalar curvature. The main ingredient is the $C^{1,\alpha}$-harmonic radius estimate, where a blow up analysis as in [2] is used. Our result is motivated by, and may be applied to, the Calabi flow on complex surfaces.

1. Introduction

In Riemannian geometry, it is interesting to consider the convergence of metrics and the finiteness of differential structures in the presence of basic geometric data, such as curvature, diameter and volume. One of the most prominent early work is Cheeger's finiteness theorem in [10]. Intuitively, the idea behind the theory is that these manifolds can be constructed from a finite number of standard pieces, i.e. coordinate patches with certain uniform regularity. Now, it is called the fundamental Cheeger–Gromov convergence theory, see Theorem 3.3.

To get a better estimate in the regularity problem, we use harmonic coordinates, which goes back to DeTurck–Kazdan [13]. In [13], it was shown that, in general, optimal regularity of the metric is obtained in harmonic coordinates. Moreover, Jost and Karcher deduced an explicit bound for the $C^{1,\alpha}$-harmonic radius which depends only on lower volume, upper diameter and sectional curvature bounds [15, 16].

However, one of the other most important features of harmonic coordinates is that the metric is apparently controlled by the Ricci curvature. Roughly speaking, the Laplacian of the metric is just the Ricci curvature, which is a determined system of PDE, see (3.1); whereas assuming bounds on the full curvature corresponds to an overdetermined system. This was exploited by Anderson [2], i.e. the bound on the Ricci curvature and the lower bound on the injectivity radius together imply a lower bound for the harmonic radius.

Note that in Cheeger's finiteness theorem, one can bound the injectivity radius provided that one has lower volume bounds and bounded full curvature. Without the assumption on injectivity radius, relaxing it to a lower bound on volume, there are examples which show that degeneration may occur under a Ricci curvature bound, see Example 1.6. Pursuing these phenomena of thought, geometers had made a well-known conjecture [11]: if $|\mathrm{Ric}| \le n-1$, then the convergence is $C^{1,\alpha}$ off a singular set of Hausdorff codimension at least four.

One of the key steps in Andersen's proof is to establish the continuous property of the harmonic radius, regarded as a function, under the $C^{k,\alpha}$ $(k \ge 1)$ or $W^{k,p}$ $(k \ge 1)$ convergence, which in turn ultimately depends on some type of partial differential equations estimate. Besides the elliptic regularity theory, the main ingredient in Andersen's results is a blow up argument that relates the validity of rigidity theorems to the global behavior of complete, non-compact manifolds with critical metrics. The hypotheses on the manifolds, such as curvature and injectivity radius bounds, were basically only used in the characterization of the Euclidean space, i.e. the rigidity theorem. It is likely that some other weakened hypothesis is still sufficient to control the harmonic radius.

We proceed with another generalization of the result on metrics with bounded Ricci curvature to bounded scalar curvature, but at the expense of some hypotheses on the structure of metric, for example, by requiring an anti-self-dual or Kähler metric. The motivation here also comes from the Calabi flow. In estimating the long-time existence and convergence of the Calabi flow, one crucial estimate which is still lacking is the $C^{1,\alpha}$-Cheeger–Gromov compactness of Kähler metrics with bounded scalar curvature, see [7] and [12].

Theorem 1.1. *Let M^4 be a compact oriented 4-dimensional manifold, and let $\{g_i\}$ be a sequence metrics on M^4 with*

(1) *either bounded self-dual Weyl tensor, i.e. $|W^+(g_i)| \le \Lambda$, or (M, g_i) are Kähler metrics,*

(2) *bounded scalar curvature, i.e. $|S(g_i)| \le \Lambda$,*

(3) *unit volume, i.e. $|\mathrm{Vol}(M, g_i)| \equiv 1$,*

(4) *bounded Sobolev constant $C_S(g_i) \le C_S$, i.e.*

$$\left\{\int_M |v|^4 dv_{g_i}\right\}^{\frac{1}{2}} \le C_S \int_M |dv|^2_{g_i} dv_{g_i}, \quad \text{for all } v \in C_0^{0,1}(M). \tag{1.1}$$

Then there exist a subsequence $\{j\} \subset \{i\}$ such that (M, g_j) converges to a compact multi-fold (M_∞, g_∞) in the Gromov–Hausdorff topology. Moreover, on the regular set $M_\infty \backslash \{x_1, \dots, x_m\}$, the metric g_∞ is $C^{1,\alpha}$ and the convergence is in the $C^{1,\alpha}$-Cheeger–Gromov topology, while each of the singular points x_i has a neighborhood homeomorphic to a finite disjoint union of cones $C(S^3/\Gamma)$ with identifications of vertex, where Γ is finite subgroup of O(4).

Remark 1.2. The above theorem holds, without fixing topology, under the additional assumption that

$$\int_{M_i^4} |\mathrm{Rm}(g_i)|^2 \le \Lambda_1.$$

By the Gauss–Bonnet formula and the Hirzebruch signature formula [6], we then have

$$\int_{M^4} |\mathrm{Rm}|^2 = -24\pi^2\tau(M) - 8\pi^2\chi(M) + \int_M \frac{1}{12}S^2 + 4|W^+|^2. \tag{1.2}$$

If the scalar curvature S and self-dual Weyl tensor $|W^+|$ have uniformly bounded L^2-norms, then we have a prior L^2-bound on the curvature tensor Rm.

Remark 1.3. In fact, under the controlled topology, i.e. $b_1(M) \leq b_0$, the Sobolev constant bound can be released to a lower volume growth assumption, i.e. $\mathrm{Vol}(B(x,r)) \geq V_0 r^n$. Gang Tian and Jeff Viaclovsky's proof for critical metrics [28] also holds in our case, since the volume is continuous in the $C^{1,\alpha}$-topology.

Analogous convergence results also hold for metrics with bounded Bach tensor and a C^2-bound of the scalar curvature, which is actually easier to prove since we have the precise elliptic system (3.11).

Theorem 1.4. *Let $\{M_i, g_i\}$ be a sequence of Riemannian manifolds with*

(1) *either bounded Bach tensor, i.e. $|B(g_i)| \leq \Lambda$ or $\{M_i, g_i\}$ are Kähler,*

(2) *C^2-bounded scalar curvature, i.e. $|\nabla^2 S(g_i)| \leq \Lambda$,*

(3) *unit volume, i.e. $|\mathrm{Vol}(M_i, g_i)| \equiv 1$,*

(4) *bounded Sobolev constant C_S in* (1.1)*,*

(5) *bounded $L^{\frac{n}{2}}$-norm of curvature:*

$$\int_{M_i} |\mathrm{Rm}|^{\frac{n}{2}} \leq \Lambda_1.$$

Then there exist a subsequence $\{j\} \subset \{i\}$ such that (M_j, g_j) converges to a compact multi-fold (M_∞, g_∞) in the $C^{3,\alpha}$-topology outside finitely many singularities.

Example 1.5 ([23]). There exists a family of conformal flat metrics with constant scalar curvature g_t on $S^1 \times S^3$ $(t \geq 1)$ such that the diameter of $S^1 \times \{x\}$ goes to 0 as $t \to \infty$ for a point $x \in S^3$. For other points $y \neq x$, the diameter of the slice $S^1 \times \{y\}$ is bounded from below. The limit space is the quotient space of S^4 which identifies the north pole and the south pole.

Example 1.6 ([18, 20]). Let M_∞ be an orbifold given by the $\mathbb{Z}/2\mathbb{Z}$-quotient of the complex 2- torus $T^2 = \mathbb{C}^2/\mathbb{Z}^4$, where $-1 \in \mathbb{Z}/2\mathbb{Z}$ acts on T by

$$(z_1, z_2) \bmod \mathbb{Z}^4 \mapsto (-z_1, -z_2) \bmod \mathbb{Z}^4.$$

The flat metric on T descends to an orbifold metric g_∞ on M_∞, and it is an orbifold Ricci flat anti-self-dual metric. Moreover, M_∞ has a complex manifold structure (with singularities) since the $\mathbb{Z}/2\mathbb{Z}$-action is holomorphic. Let us take the minimal resolution $\pi : M \to M_\infty$. The singularities are sixteen simple singularities of type A_1. The minimal resolution M is called the Kummer surface and is an example of a K3 surface. Let $\mathcal{S} = \{x_1, \ldots, x_{16}\} \subset M_\infty$ be the singular set, and let $E_1, \ldots, E_{16}$ be the exceptional divisors in M. These are complex submanifolds of M biholomorphic to $\mathbb{C}P^1$ with the self-intersection number -2. By the solution of the Calabi conjecture, we have a unique Calabi–Yau metric on M in each Kähler class, which is automatically anti-self-dual. Take a Kähler class, and there exists a sequence of Ricci–flat Kähler metrics g_i, as follows:

(1) the volume of M with respect to g_i is equal to 1,

(2) the volume of the exceptional divisor E_k is equal to $\frac{1}{i}$ for $k = 1, \ldots, 16$.

It can be shown that the sequence g_i converges to $\pi^* g_\infty$ over $M \setminus \bigcup_{k=1}^{16} E_k$, but condition (2) forces the metric to become degenerate along E_k as $i \to \infty$, i.e. E_k collapses to a point, and the Riemannian curvature concentrate along $\bigcup_{k=1}^{16} E_k$. Moreover, the curvature concentrates so completely that the limit metric is a flat orbifold-metric.

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2. Self-dual Weyl tensor

Riemannian geometry in dimension 4 has some special features which are not present in any other dimension. In dimensions $n \geq 5$ the group $\mathrm{SO}(n)$ is simple, and the space of Weyl tensors is irreducible, whereas in dimension 4 we have $\mathrm{SO}(4) = \mathrm{SO}(3) \times_{\mathbb{Z}_2} \mathrm{SO}(3)$, and a decomposition of the space of Weyl tensors into two $\mathrm{SO}(4)$-irreducible components. The fact that $\mathrm{SO}(4)$ is not simple is reflected at the Lie algebra level in the decomposition of the bundle of 2-forms into self-dual and anti-self-dual parts under the Hodge star operator.

Let (M^4, g) be an oriented 4-dimensional Riemannian manifold. The Hodge star operator $*$ associated to g takes $\Lambda := \Lambda^2 TM$ to itself and satisfies $** = 1$,

$$\phi \wedge *\varphi = \langle \phi, \varphi \rangle d\,\mathrm{vol}_g,$$

where $\phi, \varphi \in \Lambda$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on Λ induced by g. Then Λ admits a decomposition of the form

$$\Lambda = \Lambda^+ \oplus \Lambda^-,$$

where $\Lambda^\pm$ is the ± 1-eigenspace of $*$. The curvature operator Rm, viewed as an endomorphism on Λ, has the following matrix expression [6]:

$$\mathrm{Rm} = \begin{bmatrix} W^+ + \frac{S}{12}\,\mathrm{Id}_{\Lambda^+} & \overset{\circ}{\mathrm{Ric}} \\ (\overset{\circ}{\mathrm{Ric}})^t & W^- + \frac{S}{12}\,\mathrm{Id}_{\Lambda^-} \end{bmatrix},$$

where $\overset{\circ}{\mathrm{Ric}} \in \mathrm{Hom}(\Lambda^+, \Lambda^-)$ is the trace free part of the Ricci curvature, S is the scalar curvature, and $W^\pm \in S_0^2(\Lambda^\pm)$ is the (anti-)self-dual part of the Weyl tensor, where S_0^2 denote traceless symmetric endomorphisms. If we denote the projection operator by

$$P_\pm := \frac{1}{2}(1 \pm *) : \Lambda \to \Lambda^\pm,$$

then

$$W^\pm = P_\pm \circ \mathrm{Rm} \circ P_\pm - \frac{S}{12}\,\mathrm{Id}_{\Lambda^\pm}.$$

Definition 2.1. Let (M^4, g) be a compact 4-dimensional Riemannian manifold. The metric g is called anti-self-dual if $W^+ = 0$ ($\Longleftrightarrow *W = -W$).

We note that reversing the orientation transfers the self-dual part to the anti-self-dual part. For anti-self-dual metrics, if we reverse the orientation, then $W^-(g) = 0$ and g is said to be self-dual ($W^-(\mathbb{C}P^2, J, \omega_{\mathrm{FS}}) = 0$, i.e. $(\overline{\mathbb{C}P^2}, g_{\mathrm{FS}})$ is anti-self-dual).

There are many interesting examples of anti-self-dual metrics. First, the (anti-)self-duality of the metric is a conformally invariant property. In particular, the conformal flat metrics will be anti-self-dual. Second, a large number of anti-self-dual metrics are Kähler metrics with zero scalar curvature, since the self-dual part of the Weyl tensor is given by ([6, Proposition 16.62])

$$W^+ = \operatorname{diag}\left(\frac{S}{6}, -\frac{S}{12}, -\frac{S}{12}\right).$$

In particular, a K3 surface with a Calabi–Yau metric is anti-self-dual. This also follows from the Gauss–Bonnet formula and the Hirzebruch signature formula, since with the canonical orientation $\tau = -16$ and $\chi = 24$.

Definition 2.2 (Bach tensor). Let (M^n, g) be a compact Riemannian manifold. The Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}\mathrm{Ric}^{kl} W_{ikjl}.$$

The Bach tensor is a symmetric, trace free and divergence free 2-tensor. Using the Bianchi identities, we may rewrite the Bach flat (Kähler) metric with constant scalar curvature as an elliptic system [26]:

$$\begin{cases} \Delta\mathrm{Rm} = L(\nabla^2\mathrm{Ric}) + \mathrm{Rm} * \mathrm{Rm}, \\ \Delta\mathrm{Ric} = \mathrm{Rm} * \mathrm{Ric}. \end{cases} \tag{2.1}$$

In dimension 4, the Bach tensor arises as the Euler–Lagrange equations of the L^2-norm of the Weyl curvature tensor. Using the Bianchi identity, the Bach tensor can also be written as

$$B_{ij} = 2\nabla^k\nabla^l W^+_{ikjl} + \mathrm{Ric}^{kl} W^+_{ikjl},$$

thus an anti-self-dual metric will be Bach flat. As we have mentioned earlier, the Kähler metric on a complex surface with zero scalar curvature will be anti-self-dual, so it is also Bach flat.

Let M be a closed oriented smooth manifold. A smooth Riemannian metric g on M is a smooth section of the bundle S^2T^*M of positive definite symmetric 2-tensors. The space $\mathcal{M}$ of all Riemannian metrics on M is a convex open cone in $\Gamma(S^2T^*M)$. A Riemannian metric is given locally by functions, so we can define the (Sobolev) norm on $\mathcal{M}$ with respect to some fixed metric, and the prescribed curvature condition can be viewed as a partial differential equation on $\mathcal{M}$.

With this viewpoint, we want to use an a priori estimate of elliptic equations to study the convergence theory for the metric with bounded self-dual Weyl tensor. Note that the self-dual Weyl tensor $W^+(g)$ is equivariant under the action of the diffeomorphism transformation and conformal changes, but if we fix the gauge, the prescribed self-dual Weyl tensor does form an elliptic system. In fact, it is now a standard technique when studying geometric problems, such as compactness in geometric calculus of variations (Yang–Mills instanton, Einstein metric), DeTurck trick for existence theory of geometric (Ricci) flow, etc. For anti-self-dual structures, it is well known that the local structure of moduli spaces is controlled by an elliptic deformation complex [17, 25]. For the reader's convenience, we would like make it more clear from a PDE point of view (2.2). As a consequence, we get the crucial estimate for this paper, i.e. an a priori L^p-estimate (2.6) for the metric with bounded self-dual Weyl tensor and scalar curvature.

Theorem 2.3. *Let M be a closed oriented smooth 4-manifold. We consider the following map (equation):*

$$L : \mathcal{M} \to S_0^2\Lambda^+ \oplus C^\infty(M) \oplus TM, \quad g \mapsto (W^+(g), S(g), \tau_{g,\bar{g}}(\mathrm{id})), \tag{2.2}$$

where $\tau_{\tau_{g,\bar{g}}}(\mathrm{id}) := \mathrm{tr}_g \nabla^{g\otimes\bar{g}} d(\mathrm{id})$ is the tension field of the identity map $\mathrm{id} : (M, g) \to (M, \bar{g})$, and $\bar{g}$ is some fixed background metric. Then the principal symbol $\sigma(L)$ of the linearized operator of L at (x, g) is injective, which is given as follows: for any $\xi \in T_xM$ and $h \in S_x^2TM$,

$$\sigma(L)(x,\xi)(h) = \left(h_{\Lambda^+}, -\mathrm{tr}(h)|\xi|^2 + h(\xi,\xi), h(\xi,\cdot) - \frac{1}{2}\mathrm{tr}\, h \cdot \xi\right), \tag{2.3}$$

where h_{Λ^+} is defined by (2.5). In particular, equation (2.2) for the metric is an elliptic system of partial differential equations of mixed order. Consequently, the L^p-theory holds:

$$\|g\|_{W^{2,p}} \le C(\|g\|_{L^p} + \|W^+(g)\|_{L^p} + \|S(g)\|_{L^p} + \|\tau_{g,\bar{g}}\|_{W^{1,p}}).$$

Proof. Recall that the self-dual part of Weyl tensor W^+ is defined by

$$W^+ = P_+ \circ \mathrm{Rm} \circ P_+ - \frac{S}{12}\mathrm{Id}_{\Lambda^+}.$$

Then

$$\begin{aligned} \delta W_g^+(h) &= P_+ \circ \delta\mathrm{Rm}_g(h) \circ P_+ + \delta P_{+g}(h) \circ \mathrm{Rm} \circ P_+ + P_+ \circ \mathrm{Rm} \circ \delta P_{+g}(h) \\ &\quad - \frac{1}{12}\delta S_g(h)\,\mathrm{Id}_{\Lambda^+} - \frac{S}{12}\delta\,\mathrm{Id}_{\Lambda_g^+}(h) \\ &= P_+ \circ \delta\mathrm{Rm}_g(h) \circ P_+ - \frac{1}{12}\delta S_g(h)\,\mathrm{Id}_{\Lambda^+} + \text{lower order terms}. \end{aligned}$$

Now we need the following geometric quantities from [6] or [29] under deformation of a metric. The first variation of curvature is

$$\begin{aligned} \delta\,\mathrm{Rm}_g(h)(X,Y,Z,U) &= \frac{1}{2}[h(R(X,Y)Z,U) - h(R(X,Y)U,Z)] \\ &\quad + \frac{1}{2}[\nabla^2_{Y,Z}h(X,U) + \nabla^2_{X,U}h(Y,Z) \\ &\quad - \nabla^2_{X,Z}h(Y,U) - \nabla^2_{Y,U}h(X,Z)]. \end{aligned}$$

Recall one can calculate the principal symbol for a kth order differential operator L in the following way [29]: Given (x,ξ), and a smooth function f with $df = \xi$,

$$\sigma(L)(x,\xi)h = \lim_{s\to\infty} s^{-k}e^{-sf}L(e^{sf}h)(x).$$

Then the principal symbol for the second order linear operator $\delta\mathrm{Rm}_g$ is given by

$$\begin{aligned} &\sigma(\delta\mathrm{Rm}_g)(x,\xi)(h)(X,Y,Z,U) \\ &= \frac{1}{2}[\langle Y,\xi\rangle\langle Z,\xi\rangle h(X,U) + \langle X,\xi\rangle\langle U,\xi\rangle h(Y,Z) \\ &\quad - \langle X,\xi\rangle\langle Z,\xi\rangle h(Y,U) - \langle Y,\xi\rangle\langle U,\xi\rangle h(X,Z)] \\ &= \frac{1}{2}h(X\wedge Y(\xi), Z\wedge U(\xi)), \end{aligned} \tag{2.4}$$

where

$$X \wedge Y(\xi) := \langle X,\xi\rangle Y - \langle Y,\xi\rangle X.$$

Note that the scalar curvature term $\delta S_g(h)\,\mathrm{Id}_{\Lambda^+}$ does not contribute to the principal symbol of the operator of δW_g^+, since δW_g^+ is traceless. Therefore, combining with the computation (2.4), by taking the traceless symmetric part, the principal symbol of the operator of δW_g^+ is given by

$$\begin{aligned}\sigma(\delta W_g^+)(x,\xi)(h) &= \sigma(P_+ \circ \delta\mathrm{Rm}_g \circ P_+)(x,\xi)(h)\\ &\qquad - \frac{1}{3}\,\mathrm{tr}_{\Lambda^+}(\sigma(P_+ \circ \delta\mathrm{Rm}_g \circ P_+)(x,\xi)(h))g_{\Lambda^+}\\ &= h_{\Lambda^+},\end{aligned}$$

where h_{Λ^+} is defined as follows: for $e, e' \in \Lambda^+(x)$, and $\{e_i\}$ an orthonormal basis of $\Lambda^+(x)$, we have

$$h_{\Lambda^+}(e,e') = \frac{1}{2}h(e(\xi), e'(\xi)) - \frac{1}{6}\sum_{i=1}^{3} h(e_i(\xi), e_i(\xi))g_x(e,e'). \tag{2.5}$$

The linearization of the scalar curvature map is

$$\delta S_g(h) = \Delta\,\mathrm{tr}\,h + \delta^2 h - g(\mathrm{Ric}, h).$$

Then its symbol is

$$\sigma(\delta S_g)(x,\xi)(h) = -g(\xi,\xi)\,\mathrm{tr}\,h + h(\xi,\xi).$$

The linearization of the tension field map from [29] is given by

$$\delta\tau_{g,\bar g}(\mathrm{id})(h) = \langle \nabla d(\mathrm{id}), h\rangle - \left\langle \delta h + \frac{1}{2}d(\mathrm{tr}\,h), d(\mathrm{id})\right\rangle,$$

therefore

$$\sigma(\delta\tau_{g,\bar g(\mathrm{id})}g)(x,\xi)(h) = h(\xi,\cdot) - \frac{1}{2}\,\mathrm{tr}\,h\xi.$$

Combining the above three symbolic computations gives (2.3).

Now we will verify that the symbol $\sigma(L)$ is injective, i.e. for all $\xi \neq 0$, $\sigma(L)(x,\xi)(h) = 0$ implies that $h = 0$. It is easy to check that $\sigma(L)(x,\xi)(h) = 0$ implies $h_{\Lambda^+} = 0$, $h(\xi,\xi) = 0$, and $\mathrm{tr}\,h = 0$.

First choose an orthonormal basis X_1, X_2, X_3, X_4 for T_xM. Then

$$\begin{aligned}e_1 &= \frac{\sqrt{2}}{2}(X_1\wedge X_2 + X_3 \wedge X_4),\\ e_2 &= \frac{\sqrt{2}}{2}(X_1\wedge X_3 + X_4 \wedge X_2),\\ e_3 &= \frac{\sqrt{2}}{2}(X_1\wedge X_4 + X_2 \wedge X_3)\end{aligned}$$

gives an orthonormal bases of Λ_x^+. For any $\xi = \xi^i X_i$ with $|\xi| = 1$, we have

$$\begin{bmatrix} e_1(\xi)\\ e_2(\xi)\\ e_3(\xi)\\ \frac{\sqrt{2}}{2}\xi\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} \xi^2 & -\xi^1 & \xi^4 & -\xi^3\\ \xi^3 & -\xi^4 & -\xi^1 & \xi^2\\ \xi^4 & \xi^3 & -\xi^2 & -\xi^1\\ \xi^1 & \xi^2 & \xi^3 & \xi^4\end{bmatrix}\begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} := \frac{\sqrt{2}}{2}UX.$$

If $h_{\Lambda^+} = 0$ and $h(\xi,\xi) = 0$, from (2.5), it is equivalent to

$$\begin{aligned}\begin{bmatrix}(h(e_i(\xi), e_j(\xi)) & 0\\ 0 & \frac{1}{2}h(\xi,\xi)\end{bmatrix} &= \frac{1}{2}UhU^T\\ &= \frac{1}{3}\sum_i h(e_i(\xi), e_i(\xi))\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Since $U \in \mathrm{O}(4)$, we have

$$h = \frac{2}{3}\sum_i h(e_i(\xi), e_i(\xi))U^T\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}U.$$

Furthermore, if $\operatorname{tr} h = 0$, then

$$0 = \operatorname{tr} h = 2\sum_i h(e_i(\xi), e_i(\xi)),$$

and consequently, $h = 0$.

Since the symbol $\sigma(L)$ is injective, by comparing the dimension ($\dim = 10$), we conclude that equation (2.2) is an elliptic system of partial differential equations with mixed order. For an elliptic system (of mixed order), we have an a priori estimate, for example, the Schauder or L^p-theory [19]. Alternatively, if we replace the tension field $\tau_{g,\bar{g}}(\mathrm{id})$ in equation (2.2) by $\mathscr{L}_{\tau_{g,\bar{g}}(\mathrm{id})} g$, then it will be an overdetermined elliptic system of second order PDEs. In fact, the a priori estimate (L^p-theory) which holds for differential operators between vector bundles is equivalent to the injectivity of the symbol, while the 'solubility criteria' holds if the symbol is surjective (see [21, Theorem 19.25] or [14, Theorem A.8]).

In local harmonic coordinates, since the metric is $C^{1,\alpha}$-close to the Euclidean metric, we can if necessary make the harmonic norm C small enough so that the quasi-linear operator L is uniformly elliptic with $C^{1,\alpha}$-continuous leading terms. By L^p-theory, we have the estimate

$$\begin{aligned}\|g\|_{W^{2,p}(B_r)} &\le C(\|g\|_{L^p(B_{2r})} + \|W^+(g), S(g)\|_{L^p(B_{2r})}\\ &\qquad + \|\tau_{g,\bar{g}}(\mathrm{id})\|_{W^{1,p}(B_{2r})})\\ &= C(\|g\|_{L^p(B_{2r})} + \|W^+(g)\|_{L^p(B_{2r})} + \|S(g)\|_{L^p(B_{2r})}).\end{aligned} \tag{2.6}$$

The last step holds, since $\tau_{\tau_{g,\bar{g}}}(\mathrm{id}) = 0$ if we identify the geodesic ball with the Euclidean ball under harmonic coordinates. □

3. Convergence theory

To generalize the convergence theory with the assumption on sectional curvature to a weaker curvature hypothesis, it is convenient to use the concept of harmonic radius, which was introduced and developed in [2, 4, 22].

Definition 3.1 (Harmonic radius). Let (M, g) be a Riemannian manifold. Fixing any $m \in \mathbb{N}$, and given any $x \in M$, there is a radius $r = r(x)$ for which we can find a harmonic coordinate system

$$\Phi = (x_1, \ldots, x_n) : B(x,r) \subset (M, g) \to \mathbb{R}^n$$

such that

(1) the coordinate function $x_i : B(x,r) \subset (M,g) \to \mathbb{R}$ is harmonic, for $i = 1,\dots,n$,

(2) the metric tensor $g_{ij} := g(\nabla x_i, \nabla x_j)$ is $C^{m,\alpha}$-bounded on $B(x,r)$, i.e.

$$e^{-C}\delta_{ij} \le g_{ij} \le e^{C}\delta_{ij} \quad \text{(as bilinear forms)},$$

and

$$\sum_{1\le|\beta|\le m} r^{|\beta|}\sup|\partial^\beta g_{ij}| + \sum_{|\beta|=m} r^{m+\alpha}[\partial^\beta g_{ij}]_\alpha \le C,$$

for some fixed constant $C \ge 0$, where the norms are taken with respect to the coordinates $(\{x_i\}_i^n)$ on $B(x,r)$. We say that a point $x \in (M,g)$ admits harmonic coordinates with bounded $C^{m,\alpha}$-norm on the scale r, which we denote by

$$\|x \in (M,g)\|_{C^{m,\alpha},r} \le C.$$

Moreover, we let $r_h(x)$ be the $C^{m,\alpha}$-harmonic radius at x, which is defined as the radius of the largest geodesic ball about x which admits $C^{m,\alpha}$-harmonic coordinates, i.e.

$$r_h(x) = \sup\{r > 0 : \|x \in (M,g)\|_{C^{m,\alpha},r} \le C\}.$$

The $C^{m,\alpha}$-harmonic radius of (M,g) is defined by

$$r_h(M) = \inf_{x\in M} r_h(x).$$

In harmonic coordinates on Riemannian manifolds, the Ricci tensor has a particularly nice formula [22]:

$$\Delta g_{ij} = -2\,\mathrm{Ric}_{ij} + Q_{ij}(g,\partial g), \tag{3.1}$$

here Q is a polynomial in the matrix g, quadratic in ∂g.

Definition 3.2. For a compact metric space X, define the covering number of the geodesic ball on the scale ϵ as follows:

$$\mathrm{Cov}(\epsilon) = \min\left\{N \in \mathbb{N} : \exists\{x_i\}_{i=1}^N \subset X,\ \bigcup_{i=1}^N B(x_i,\epsilon) = X,\ B\left(x_i,\frac{\epsilon}{2}\right) \text{ mutually disjoint}\right\}.$$

Now, let us state the fundamental convergence theorem:

Theorem 3.3 (Fundamental convergence theorem [2, 22]). *For given $n \ge 2$, $C \ge 0$, $N > 0$, $\alpha \in (0,1]$ and $r_0 > 0$, consider the class $\mathcal{M}(n,C,N,r_0)$ of n-dimensional Riemannian manifolds*

$$\left\{(M,g) : \|(M,g)\|_{C^{m,\alpha},r_0} \le C \text{ and } \mathrm{Cov}\left(\frac{r_0}{10}\right) \le N\right\}.$$

Then $\mathcal{M}(n,C,N,r_0)$ is compact in the $C^{m,\beta}$-Cheeger–Gromov topology for all $\beta < \alpha$. Moreover, the theorem is also valid for bounded domains in Riemannian manifolds, as well as for pointed complete Riemannian manifolds, provided one works on compact subsets.

Theorem 3.4 (Volume growth and ϵ-rigidity [9, 26, 28]). *Let (M, g) be a complete Riemannian manifold or Riemannian multi-fold with finite point singularities, and g be a critical metric (for example, Bach flat with zero scalar curvature). Assume that (M^n, g) satisfies the Sobolev inequality,*

$$\left\{\int_M |v|^{\frac{2n}{n-2}} dv\right\}^{\frac{n-2}{n}} \leq C_S \int_M |dv|^2 dv, \quad \text{for all } v \in C_0^{0,1}(M);$$

and the curvature has bounded $L^{\frac{n}{2}}$-norm,

$$\left\{\int_M |\mathrm{Rm}|^{\frac{n}{2}} dv\right\}^{\frac{n}{2}} \leq \epsilon.$$

Then (M^n, g) has finitely many ends, which is ALE of order 2. Moreover, the volume has at most Euclidean volume growth,

$$\mathrm{Vol}(B(p, r)) \leq V r^n$$

for some positive constant $V = V(n, C_S, \epsilon)$. Moreover, if (M, g) is smooth and $\epsilon = \epsilon_0$ is small enough, which depends on n and the Sobolev constant C_S, then (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, g_E)$.

Lemma 3.5. *Let $B(r) := B(x_0, r)$ be a geodesic ball in a compact oriented Riemannian 4-manifold (M^4, g), where g has bounded self-dual Weyl tensor (or is Kähler) and bounded scalar curvature, $|W^+| + |S| \leq \Lambda$. Then there exist two positive constants $\epsilon_0 = \epsilon(C_S)$ and $\kappa_0 = \kappa_0(C_S, \Lambda)$ such that if*

$$\left\{\int_{B(x_0,2r)} |\mathrm{Rm}|^2 dv\right\}^{\frac{1}{2}} \leq \epsilon_0, \tag{3.2}$$

then for all $x \in B(x_0, r)$, the $C^{1,\alpha}$-harmonic radius $r_h(x)$, with respect to some fixed $\alpha \in (0, 1)$ and $C > 0$, satisfies

$$\frac{r_h(x)}{dist(x, \partial B(x_0, r))} \geq \kappa_0 > 0. \tag{3.3}$$

Proof. On a fixed smooth Riemannian manifold (M, g), it is clear that the harmonic radius $r_h(x)$ is positive, i.e. (3.3) holds, but κ_0 depends on (M, g) and x. Thus, we must show that κ_0 depends only on the hypothesis prescribed in the lemma.

We argue by contradiction, which is similar to the blow up analysis for the Ricci curvature case as in [2]. If (3.3) is false, then there is a sequence of Riemannian 4-manifolds $\{(M_i, g_i)\}$ with the bounds in the lemma, and points $x_i \in B_i(r) \subset (M_i, g_i)$ such that

$$\frac{r_h(x_i)}{\mathrm{dist}(x_i, \partial B_i)} \to 0, \quad \text{as } i \to \infty. \tag{3.4}$$

We may assume, without loss of generality, that the points x_i realize the minimum of the left side of (3.3), and

$$\|x_i \in (M_i, g_i)\|_{C^{1,\alpha}, r_h(x_i)} \in \left[\frac{C}{2}, C\right].$$

By scaling theses metrics suitably, namely, $\bar{g}_i = r_h(x_i)^{-2} g_i$, then

(1) $\bar{r}_h(x_i) = 1$ and $\bar{r}_h(x)$ is bounded below on balls of finite distance to x_i, which follows from the scale invariant property of harmonic norm [22],

$$\|x \in (M, \lambda^2 g)\|_{C^{m,\alpha},\lambda r} = \|x \in (M, g)\|_{C^{m,\alpha},r},$$

(2) $\mathrm{dist}_{\bar{g}_i}(x_i, \partial B_i) \to \infty$, since the ratio in (3.3) is scale invariant,

(3) $|W^+(\bar{g}_i)| + |S(\bar{g}_i)| \le r_h^2(x_i)\Lambda \to 0$, and the curvature has ϵ-small L^2-norm

$$C_S \left\{ \int_{B(x, \frac{2r}{r_h(x_i)})} |\mathrm{Rm}(\bar{g}_i)|^2 d\bar{v}_i \right\}^{\frac{1}{2}} \le \epsilon_0,$$

with respect to the metric $\bar{g}_i$.

We now consider the sequence of pointed Riemannian manifolds

$$\left\{ \left(B_i\left(x_i, \frac{r}{r_h(x_i)} \right), x_i, \bar{g}_i \right) \subset (M_i, x_i, \bar{g}_i) \right\}.$$

By the Fundamental Convergence Theorem 3.3, the sequence is subconvergent, in the pointed $C^{1,\beta}$-topology (for all $\beta < \alpha$), uniformly on compact subsets, to a complete $C^{1,\alpha}$-Riemannian manifold $(N, \bar{x}, h)$.

Claim 1. *The convergence is actually better, namely in the $C^{1,\alpha}$-topology, where α is given by the hypothesis of the lemma.*

We can even prove more than we need, i.e. the convergence is in the $C^{1,\alpha} \cap W^{2,p}$-topology, for any $\alpha < 1$ and $1 < p < \infty$. By the Sobolev Embedding Theorem, $W^{2,p} \subset C^{1,\alpha}$ if $p > n$, so it suffices to prove the convergence is in the $W^{2,p}$-topology. To see this, by Theorem 2.3, we know that the prescribed self-dual Weyl tensor and scalar curvature equation (2.2) is an elliptic system of partial differential equations of second order under harmonic coordinates:

$$\begin{cases} W^+(g) = L(g^{-1}\partial\partial g) + Q_1(\partial g, \partial g) \in L^\infty, \\ S(g) = -\frac{1}{2} g^{ij} g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} g_{ij} + Q_2(\partial g, \partial g) \in L^\infty, \end{cases} \tag{3.5}$$

where L denotes linear combination, and Q is a quadratic term in the first order derivatives of g. For an a priori estimate, since $\|g_{ij} - \delta_{ij}\|_{C^{1,\alpha}} < C$, if necessary we can make C small, so that the above system actually can be viewed as a uniform linear elliptic system of g_{ij} with $C^{1,\alpha}$ coefficients. By the a priori estimate (2.6), the L^p-theory for elliptic systems gives a uniform bound on $\|g\|_{W^{2,p}}$ for any $1 < p < \infty$,

$$\|g\|_{W^{2,p}} \le C(\|g\|_{L^p} + \|Q(\partial g, \partial g)\|_{L^p} + \|W^+(g)\|_{L^p} + \|S(g)\|_{L^p}) \le C.$$

As a consequence, the convergence is in the $C^{1,\alpha} \cap W^{2,p}$-topology, for any $\alpha < 1$, $1 < p < \infty$.

More precisely, the geodesic balls $(B_i(\frac{r}{r_h(x_i)}), x_i, \bar{g}_i)$ are covered by harmonic coordinates that converge in the $C^{2,\alpha}$-topology to the harmonic coordinates on limit space N, and the metric coefficients $\bar{g}_i$ converge in the $C^{1,\alpha}$-topology to h.

Since the $C^{1,\alpha}$-norm and the harmonic radius are continuous with respect to $C^{1,\alpha}$- or $W^{2,p}$-convergence [2, 22], we get

$$r_h(\bar{x}) = 1, \quad \|\bar{x} \in (N,h)\|_{C^{1,\alpha}, r_h(\bar{x})} \geq \frac{C}{2} > 0. \tag{3.6}$$

Claim 2. *The manifold (N,h) is a smooth Riemannian manifold and isometric to the Euclidean space $(\mathbb{R}^4, g_E)$.*

Since the convergence is in the $C^{1,\alpha} \cap W^{2,p}$-topology, we can conclude that the limit metric h is a weak $C^{1,\alpha} \cap W^{2,p}$-solution of the elliptic system

$$\begin{cases} W^+(g) = L(g^{-1}\partial\partial g) + Q_1(\partial g, \partial g) = 0, \\ S(g) = -\frac{1}{2} g^{ij} g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} g_{ij} + Q_2(\partial g, \partial g) = 0, \end{cases} \tag{3.7}$$

namely, the anti-self-dual or Kähler metric with zero scalar curvature is a second order quasi-linear elliptic system of the metric modulo diffeomorphisms by Theorem 2.3. With the a priori estimate (2.6) and a standard bootstrap argument, and also the Sobolev Embedding Theorem, we conclude that the metric h is actually a smooth (in fact, analytic) Riemannian metric with

$$C_S \left\{ \int_N |\mathrm{Rm}(h)|^2 dv_h \right\}^{\frac{1}{2}} \leq \epsilon_0.$$

If $\epsilon = \epsilon_0$ is sufficiently small (which will depend only on the Sobolev constant), by the ϵ-Rigidity Theorem 3.4, we conclude that $\mathrm{Rm}(h) \equiv 0$, i.e. N is flat. On the other hand, the bounded Sobolev constant implies Euclidean volume growth. Consequently, (N,h) is isometric to the Euclidean space $(\mathbb{R}^n, g_E)$.

It is obvious that the Euclidean space admits global harmonic coordinates, i.e.

$$r_h(x) = \infty, \quad \|x \in (\mathbb{R}^n, g_E)\|_{C^{1,\alpha}, r} = 0, \quad \text{for all } r > 0.$$

However, this violates (3.6). □

Now we can prove the Main Theorem 1.1, which is an immediate consequence of the Main Lemma 3.5 on the harmonic radius estimate and the Fundamental Convergence Theorem 3.3.

Theorem 3.6. *Let $\{(M_i, g_i)\}$ be a sequence of Riemannian 4-manifolds, which satisfy the hypotheses of Theorem 1.1 and which also have uniformly bounded L^2-curvature. Then there exists a subsequence $\{j\} \subset \{i\}$ such that (M_j, g_j) converges to a compact metric space (M_∞, g_∞) in the $C^{1,\alpha}$-topology outside the finite singular set $\mathcal{S} = \{x_1, \ldots, x_m\}$.*

Proof. As in the case of bounded Ricci curvature or Bach flat metric with constant scalar curvature, take $\epsilon = \epsilon_0$ in Theorem 3.4, and consider the sets

$$\mathcal{R}_i(r) = \left\{ x \in M_i : \int_{B(x,2r)} |\mathrm{Rm}(g_i)|^2 < \epsilon_0^2 \right\},$$

$$\mathcal{S}_i(r) = \left\{ x \in M_i : \int_{B(x,2r)} |\mathrm{Rm}(g_i)|^2 \geq \epsilon_0^2 \right\}.$$

Then $M_i = \mathcal{R}_i(r) \cup \mathcal{S}_i(r)$, and also $\mathcal{R}_i(r_1) \subset \mathcal{R}_i(r_2)$, $\mathcal{S}_i(r_1) \supset \mathcal{S}_i(r_2)$, for any $r_1 > r_2$.

For all $x \in \mathcal{R}_i(r)$, by the Main Lemma 3.5, we have the estimate on the $C^{1,\alpha}$-harmonic radius

$$r_h(x) \geq \kappa_0 r,$$

where $\kappa_0 = C(C_S, \Lambda)$. On the other hand, the uniform Sobolev constant implies noncollapsing, namely, $\mathrm{Vol}(B(x,r)) \geq C(C_S) r^4$. Then the covering number (see Definition 3.2) on any compact subset of $\mathcal{R}_i(r)$ on the scale $\inf_{x \in \mathcal{R}_i(r)} r_h(x) \geq \kappa_0 r$ can be bounded by

$$\mathrm{Cov}\left(\frac{1}{10}\kappa_0 r\right) \leq \frac{\mathrm{Vol}(M_i)}{\mathrm{Vol}(B(x, \frac{1}{10}\kappa_0 r))} \leq \frac{C(C_S, \Lambda)}{r^4}.$$

Applying now the Fundamental Convergence Theorem 3.3, the sequence $(\mathcal{R}_i(r), g_i)$ is $C^{1,\alpha}$-subconvergent to a $C^{1,\alpha}$-(open) Riemannian manifold $(\mathcal{R}_\infty(r), g_\infty)$ on the compact set.

To construct the limit space, we will be brief since it is quite standard, see for example [1, 5, 24], and also [3, 27].

We now choose a sequence $\{r_j\} \to 0$ with $r_{j+1} < \frac{1}{2} r_j$, and repeat the above construction by choosing a subsequence, which we still denote by $\{j\}$. Since $\mathcal{R}_i(r_j) \subset \mathcal{R}_i(r_{j+1})$, we have a sequence of limit spaces with natural inclusions

$$\mathcal{R}_\infty(r_j) \subset \mathcal{R}_\infty(r_{j+1}) \subset \cdots \subset \mathcal{R}_\infty := \mathrm{dir.\,lim}\, \mathcal{R}_\infty(r_j).$$

By the $C^{1,\alpha}$-convergence, $(\mathcal{R}_\infty, g_\infty)$ is a $C^{1,\alpha}$-(open) Riemannian manifold, and there are $C^{2,\alpha}$-smooth embeddings $F_i : (\mathcal{R}_\infty, g_\infty) \to (M_i, g_i)$ such that $F_i^* g_i \to g_\infty$ in the $C^{1,\alpha}$-topology on any compact set of $\mathcal{R}_\infty$.

Letting $\{B(x_k^i, \frac{r}{4})\}_{k \in \mathbb{N}}$, $r < \frac{1}{4}\rho_0$, be a collection of a maximal family of disjoint balls in M_i, where ρ_0 is the Euclidean volume growth scale in Theorem 4.1, we have

$$M_i \subset \bigcup_k B(x_k^i, r).$$

There is a uniform bound, independent of i, on the number of points $\{x_k^i \in \mathcal{S}_i(r)\}$, which follows from

$$m \leq \sum_{i=1}^{m} \epsilon_0^{-2} \int_{B(x_k^i, 2r)} |\mathrm{Rm}(g_i)|^2 \leq C \epsilon_0^{-2} \int_{M_i} |\mathrm{Rm}(g_i)|^2, \tag{3.8}$$

where

$$C = \sup_{x \in M_i} \frac{\mathrm{Vol}(B(x, \frac{9r}{4}))}{\mathrm{Vol}(B(x, \frac{r}{4}))} \leq C(C_S, \Lambda).$$

The last inequality holds because we have an upper bound on the volume growth (4.1).

Let $\{r_j\}$ be as above. Without loss of generality, we will assume m is fixed, i.e. the number of mutually disjoint balls which are centered in $\mathcal{S}_i(r_j)$ and have radius $\frac{r_j}{4}$ is independent of i and j. As a consequence, every point of $\mathcal{S}_i(r_j)$ is contained in a ball of diameter no greater than $m r_j$. Hence, most of the volume (M_i, g_i) is contained in $\mathcal{R}(r_j)$. Using the embedding $F_i^j : (\mathcal{R}_\infty(r_j), g_\infty) \to (\mathcal{R}_i(r_j), g_i)$, we see that for any fixed j, and i sufficiently large, arbitrarily large compact subsets of $\mathcal{R}_\infty \setminus \mathcal{R}_\infty(r_j)$ are almost isometrically embedded into m disjoint balls of radius r_j. Letting $j \to \infty$, it follows that the boundary components shrink to points with respect to g_∞. In other words, one can add finite points $\mathcal{S}_\infty = \{x_1, \ldots, x_m\}$

to $\mathcal{R}_\infty$ such that $M_\infty := \mathcal{R}_\infty \cup \mathcal{S}_\infty$ is complete with respect to the length structure g_∞, i.e. the Riemannian metric has a C^0-extension across the singularity. Moreover, since we showed that the curvature concentration part shrinks off, (M_i, g_i) is subconvergent to M_∞ in the Gromov–Hausdorff topology, and the volume of the geodesic ball (which may contain singularities) is continuous with respect to the $C^{1,\alpha}$-convergence (off finitely many singularities).

We now examine the topological structure near the singularity by essentially studying the tangent cones at the singularity. Fix $p \in \mathcal{S}_\infty \subset M_\infty$, let $r(x) = \mathrm{dist}(x, p)$ and denote the annulus around p by

$$A(r_1, r_2) = \{x \in M_\infty : r_1 < r(x) < r_2\}, \quad r_1 < r_2 < \mathrm{dist}(p, \mathcal{S}_\infty \setminus \{p\}).$$

By the $C^{1,\alpha} \cap W^{2,p}$-convergence, recalling the $C^{2,\alpha}$-smooth embedding

$$F_i : (\mathcal{R}_\infty, g_\infty) \to (M_i, g_i),$$

the curvature will converge in the L^p sense, and then

$$\int_{F_i(\mathcal{R}_\infty)} |\mathrm{Rm}(g_i)|^2 < \infty, \quad \text{for all } i.$$

In particular, for ϵ_0 in Theorem 3.4, there exists a radius $r_0 > 0$ such that

$$\int_{F_i(A(0,r_0))} |\mathrm{Rm}(g_i)|^2 \le \epsilon_0^2, \quad \text{for all } i. \tag{3.9}$$

Now we do blow up analysis on M_∞; it is equivalent to do blow up analysis on the sequence. Namely, given any sequence $r_j \to 0$, $j \to \infty$, the metric annulus

$$\left(F_i\left(A\left(\frac{r_j}{j}, jr_j\right)\right), \frac{1}{r_j^2} g_i\right)$$

(by taking diagonal sequence) sub-converges to a $C^{1,\alpha} \cap W^{2,p}$-annulus $(A_\infty(0,\infty), g_\infty)$, where g_∞ is a weak solution of the anti-self-dual with zero scalar curvature equation. With the regularity theory of elliptic equation, it follows that $(A_\infty(0,\infty), g_\infty)$ is smooth. On the other hand, by (3.9), we know

$$\int_{A(0,\infty)} |\mathrm{Rm}(g_\infty)|^2 \le \epsilon_0^2. \tag{3.10}$$

Applying Theorem 3.4 again, we conclude that each component of $(A_\infty(0,\infty), g_\infty)$ is isometric to the Euclidean cone on a space form S^3/Γ for some finite subgroup of O(4).

If one has lower Ricci curvature, then the limit orbifold is irreducible, which is proved in [1] by means of the Cheeger–Gromoll Splitting Theorem. In our case, there may be more than one cone associated to one singularity. If we perform a standard bubble analysis, one can estimate the precise bound on the end of associated ALE space, which in turn implies a bound on the number of cones at each singular point, depending only on $\|\mathrm{Ric}_-\|_{L^2}$, C_S, see [8] and [28]. This also gives an alternative way to show the limit orbifold is irreducible if one has lower bound on the Ricci curvature. For a Kähler metric, only irreducible singular points can occur in limit, i.e. the singularities are orbifold points, see more details in [27].

It follows that the neighborhoods of each singular point is homomorphic to finite cones on spherical spaces forms. □

Remark 3.7. For the proof of Theorem 1.4, the argument is similar. In fact, it is much easier to estimate the harmonic radius as in the Main Lemma 3.5.

Using the Bianchi identity, the Laplacian of the Ricci curvature is related to the Bach tensor (Kähler) and scalar curvature [26], so we have a coupled system:

$$\begin{cases} \Delta \mathrm{Ric} = 2B + \frac{1}{3}\,\mathrm{Hess}\, S + \mathrm{Rm} * \mathrm{Ric}, \\ \quad \Delta g = Q(\partial g, \partial g) - 2\,\mathrm{Ric}\,. \end{cases} \tag{3.11}$$

Under the $C^{3,\alpha}$-harmonic coordinates, we have the improved estimate

$$\|g\|_{C^{3,\alpha'}} < C, \quad \text{for all } 0 < \alpha' < 1,$$
$$\|g\|_{W^{4,p}} < C, \quad \text{for all } 1 < p < \infty.$$

Moreover, the blow up limit will be flat since the ϵ-Rigidity Theorem 3.4 holds for Bach flat (Kähler) metrics with zero scalar curvature. The left argument is similar and will be omitted here.

4. Volume growth near singularity

We have already seen that the volume growth plays a crucial role in understanding the structure near the singular set, see (3.8). By a lack of the volume comparison, we must find an alternative approach to bound the volume growth on a fixed scale, i.e. for some $\rho > 0$, there exists a constant $V_1 > 0$ such that $\mathrm{Vol}(B(p,r)) \le V_1 r^n$, for all $r < \rho$. For Bach flat metrics with constant scalar curvature, Gang Tian and Jeff Viaclovsky concluded that the volume does bound on all scale, and the bound depends only on the Sobolev constant and L^2-norm of curvature [26–28]. In fact, if we check their paper carefully, we will find that the argument also holds in our case, where we work in $C^{1,\alpha}$-category in place of the C^{∞}-category. The difficulty is caused by the concentration of curvature. If we do blow up analysis carefully, as in the Einstein case [20], there will bubble out some non-flat ALE space (tree) which will satisfy stronger geometric conditions, and consequently, we can bound the volume growth.

Theorem 4.1. *Let (M^4, g) be a compact oriented 4-manifold with bounded self-dual Weyl tensor and scalar curvature, i.e. $|W^+(g)| + |S(g)| \le \Lambda$, bounded Sobolev constant C_S, and also finite L^2-curvature, i.e. $\|\mathrm{Rm}\|_{L^2} < \Lambda_1$. For some $\rho_0 > 0$, there exists a constant $V_1 > \omega_4$, depending only on Λ, Λ_1, C_S, such that*

$$\mathrm{Vol}(B(x,r)) \le V_1 r^4 \tag{4.1}$$

for all $x \in M$ and $0 < r < \rho_0$.

Proof. The theorem can be established by using the same bubble procedure as Gang Tian and Jeff Viaclovsky did in [28]. For the reader's convenience, we will copy down their argument with some slight modifications to give a detailed argument in our case.

In the first place, if the curvature do not concentrate too much, i.e. for $\rho > 0$,

$$\int_{B(x,2\rho)} |\mathrm{Rm}(g)|^2 < \epsilon_0^2,$$

the volume growth will be controlled. In fact, by Lemma 3.5, the harmonic radius of $B(x,\rho)$ is bounded below, namely, there is a uniform constant κ_0 such that, for all $y \in B(x,\rho)$,

$$r_h(y) > \kappa_0 \rho, \quad \|y \in (M,g)\|_{C^{1,\alpha},\kappa_0\rho} < C,$$

and consequently,

$$\operatorname{Vol}(B(x,r)) \leq e^{2C}\omega_4 r^4, \quad \text{for all } r \leq \rho. \tag{4.2}$$

For any metric (M,g), define the maximal volume ratio on the scale ρ as

$$\mathrm{MV}(g,\rho) = \max_{x\in M,\, 0<r<\rho} \frac{\operatorname{Vol}(B(x,r))}{r^4}.$$

Note that for any compact smooth Riemannian 4-manifold (M,g),

$$\lim_{\rho\to 0} \mathrm{MV}(g,\rho) = \omega_4,$$

where ω_4 is the volume of the unit ball in $\mathbb{R}^4$.

In this paper, we consider the maximal volume ratio on finite scales rather than on all scales. On the one hand, the local non-inflated volume is enough to shrink the singular set to a point. On the other hand, one will see, lacking ϵ-regularity, we cannot prove the Euclidean volume growth on the large scale by volume comparison [28].

We argue by contradiction. If the theorem is not true, then for any sequence $\rho_j \to 0$ with $\rho_{j+1} < \frac{1}{2}\rho_j$, if we fix j, there exists a sequence of metrics $(M_{i,j}, g_{i,j})$ which satisfy the hypotheses of the theorem, but $\mathrm{MV}(g_{i,j},\rho_j) \to \infty$. By passing to a diagonal subsequence, for any sequence $\rho_i \to 0$ with $\rho_{i+1} < \frac{1}{2}\rho_i$, there exists a sequence of metrics (M_i, g_i) which satisfy the hypotheses of the theorem, but

$$\mathrm{MV}(g_i,\rho_i) \to \infty, \quad \text{as } i \to \infty. \tag{4.3}$$

For this sequence, we can extract a subsequence (which for simplicity we continue to denote by the index i), and $r_i < \rho_i$ such that

$$2e^{2C} = \frac{\operatorname{Vol}(B(x_i,r_i))}{r_i^4} = \max_{x\in M_i,\, r\leq r_i} \frac{\operatorname{Vol}(B(x,r))}{r^4}, \tag{4.4}$$

where e^{2C} comes from (4.2). We furthermore assume x_i is chosen so that r_i is minimal, that is, the smallest radius such that

$$\operatorname{Vol}(B_{g_i}(x,r)) \leq 2e^{2C} r^4, \quad \text{for all } x \in M_i \text{ and } r \leq r_i.$$

Note that the inequality

$$\int_{B(x_i,2r_i)} |\mathrm{Rm}(g_i)|^2 \geq \epsilon_0^2 \tag{4.5}$$

must hold, each ball with larger volume growth (singularity) takes at least ϵ_0 of L^2-curvature. Otherwise, by the estimate (4.2), we would have

$$\operatorname{Vol}(B_{g_i}(x_i,r_i)) \leq e^{2C} r_i^4,$$

which violates the choice of r_i in (4.4).

Now, we consider the rescaled metric $\tilde{g}_i = r_i^{-2} g_i$, so that $B_{g_i}(x_i, r_i) = B_{\tilde{g}_i}(x_i, 1)$. From the choice of x_i and r_i, the rescaled metrics $\tilde{g}_i$ have bounded volume ratio, in all of unit size.

From the Main Theorem 1.1, there exists a subsequence which converges on compact subsets to a complete length space $(M_\infty, g_\infty, x_\infty)$ in the $C^{1,\alpha}$-topology off finitely many singularities, where $(M_\infty, g_\infty, x_\infty)$ is a multi-fold, and g_∞ is a smooth anti-self-dual metric with zero scalar curvature. Further, from Theorem 3.4 for the multi-fold-case, see [28, Proposition 4.3 and Claim 4.4, p. 14], there exists a constant A_1 such that

$$\text{(4.6)} \qquad \operatorname{Vol}(B_{g_\infty}(x_\infty, r)) \le A_1 r^4, \quad \text{for all } r > 0.$$

We have seen that if $r_i \to 0$, then the blow up limit will be a smooth multi-fold with critical metric, which is crucial to conclude that (4.6) holds. This is the main reason that we consider the maximal volume ratio on the finite scale.

We next return to the (sub)sequence (M_i, g_i) and extract another subsequence (which for simplicity we continue to denote by the index i) so that

$$\text{(4.7)} \qquad 2600 A_1 = \frac{\operatorname{Vol}(B(x_i', r_i'))}{r_i'^4} = \max_{x \in M_i,\, r \le r_i'} \frac{\operatorname{Vol}(B(x, r))}{r^4}.$$

Again, we assume that x_i' is chosen so that r_i' is minimal, that is, the smallest radius for which

$$\operatorname{Vol}(B_{g_i}(x, r)) \le 2600 A_1 r^4, \quad \text{for all } x \in M_i \text{ and } r \le r_i'.$$

Clearly, $r_i < r_i' < \rho_i \to 0$.

Arguing as above, we repeat the rescaled limit construction, but now with scaled metric $g_i' = r_i'^{-2} g_i$, and base point x_i'. We find a limiting multi-fold $(M_\infty', g_\infty', x_\infty')$, and a constant $A_2 \ge 2600 A_1$ so that

$$\operatorname{Vol}(B_{g_\infty'}(x_\infty', r)) \le A_2 r^4, \quad \text{for all } r > 0.$$

For the same reason as in (4.5), we must have

$$\int_{B_{g_i}(x_i', 2r_i')} |\operatorname{Rm}(g_i)|^2 \ge \epsilon_0^2.$$

Since the L^2-curvature is finite, and each larger volume growth ball (singularity) takes at least ϵ_0 of L^2-curvature, it is reasonable to hope that the bubbling process will end in a finite number of steps. But we need to be a little careful, as in the Einstein case [20], there could be some overlap if any singular point lies in a ball centered at other singular point.

So we next consider the ratio r_i'/r_i.

- Case (i): there exists a subsequence (which we continue to index with i) satisfying $r_i' < C r_i$ for some constant C.
- Case (ii): we have
$$\lim_{i \to \infty} \frac{r_i'}{r_i} = \infty.$$

In Case (i) we proceed as follows: We claim that for i sufficiently large, the balls $B(x_i, 2r_i)$ (from the first subsequence but also occurring in the second) and $B(x_i', 2r_i')$ (from the second) must be disjoint because of the choice in (4.7). To see this, if $B(x_i, 2r_i) \cap B(x_i', 2r_i') \neq \emptyset$, then $B(x_i', 2r_i') \subset B(x_i, 6r_i')$. Then (4.6) and (4.7) imply that

$$\begin{aligned} 2600A_1(r_i')^4 &= \mathrm{Vol}(B(x_i', r_i')) \\ &< \mathrm{Vol}(B(x_i', 2r_i')) \\ &< \mathrm{Vol}(B(x_i, 6r_i')) \\ &\leq 2A_1(6(r_i')^4) \\ &= 2592(r_i')^4, \end{aligned}$$

which is a contradiction (note the last inequality is true for i sufficiently large since the volume is continuous in the $C^{1,\alpha}$-topology (even with finite singularity), and (4.6) is valid only in Case (i)).

In Case (ii), if the balls $B(x_i, 2r_i)$ (from the first subsequence) and $B(x_i', 2r_i')$ (from the second) are disjoint for all i sufficiently large, then we proceed to the next step. Otherwise, we consider the scaled metric $g_i' = (r_i')^{-2} g_i$ with basepoint x_i'; then

$$\mathrm{Vol}(B(x_i', 1)) = 2600A_1.$$

As above, we have a limiting smooth multi-fold $(M_\infty', g_\infty', x_\infty')$ satisfying

$$\mathrm{Vol}(B(x_\infty', 1)) = 2600A_1.$$

Since the metric is anti-self-dual with constant curvature, by the choice of A_1, we concluded that

$$\int_{B_{g_\infty'}(x_\infty, 2)} |\mathrm{Rm}(g_\infty')|^2 > \epsilon_0^2.$$

There is now a singular point of convergence corresponding to the balls $B(x_i, r_i)$ in the first subsequence. But since we are in Case (ii) with $\lim_{i\to\infty} r_i'/r_i = \infty$, in the g_i'-metric, these balls must shrink to a point in M_∞'. The only possibility is that the original sequence satisfies

$$\int_{B_{g_i}(x_i', 2r_i')} |\mathrm{Rm}(g_i)|^2 > 2\epsilon_0^2,$$

for all i sufficiently large.

We repeat the above procedure, considering the possible Cases (i) and (ii) at each step. At the kth step, we can always account for at least $k\,\epsilon_0$ of L^2-curvature. The process must terminate in finitely many steps from the bound $\|\mathrm{Rm}(g_i)\|_{L^2} < \Lambda_1$. This contradicts (4.3), which finishes the proof. □

We note that it may happen that (M_∞, g_∞) is a smooth Riemannian manifold, but the convergence is not in the $C^{1,\alpha}$-topology. In fact, the curvature concentration part, corresponding to some nontrivial 2- cycles in M, may shrink off in the limit. The topology 'decreases' and the singularity takes away a certain quantity of energy of curvature.

Proposition 4.2. *Let (M, g_i) satisfy the hypotheses of Theorem* 1.1. *Then*

$$\lim_{i\to\infty} \int_M |\mathrm{Rm}(g_i)|^2 \geq \int_{\mathcal{R}\subset M_\infty} |\mathrm{Rm}(g_\infty)|^2,$$

with inequality if and only if M_∞ is a $C^{1,\alpha}$-manifold diffeomorphic to M, and the convergence is in the $C^{1,\alpha}$-Cheeger–Gromov topology.

Proof. It is a straightforward consequence of the bubble analysis. Since the convergence is in $C^{1,\alpha} \cap W^{2,p}$, the measure $|\mathrm{Rm}(g_i)|^2 dv_{g_i}$ converges to

$$|\mathrm{Rm}(g_\infty)|^2 dv_{g_\infty} + \sum_{x_i \in \mathcal{S}_\infty} a_i \delta_{x_i}$$

in the L^p-sense, where δ_{x_i} is the Dirac measure supported at x_i, and a_k is given by

$$a_k = \sum_{(N_k^*, h_k^*)} \int_{(N_k^*, h_k^*)} |\mathrm{Rm}(h_k^*)|^2.$$

Here (N_k^*, h_k^*) is the bubble tree associated to the singular point x_k, see the clear description of the bubble tree in [20]. The equality implies there is no curvature concentration occurring and thus no singularities in the limit. □

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