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Absolutely Summing Terraced Matrices

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Abstract: Let $\alpha > 0$. By C_α we mean the terraced matrix defined by $c_{nk} = \frac{1}{n^\alpha}$ if $1 \le k \le n$ and 0 if $k > n$. In this paper, we show that a necessary and sufficient condition for the induced operator on l^p, to be p-summing, is $\alpha > 1$; $1 \le p < \infty$. When the more general terraced matrix B, defined by $b_{nk} = \beta_n$ if $1 \le k \le n$ and 0 if $k > n$, is considered, the necessary and sufficient condition turns out to be $\sum_n n^{\frac{q}{p^*}} \mid \beta_n \mid^q < \infty$ in the region $1/p + 1/q \le 1$.

Keywords: Operator, absolutely summing operators, Terraced Matrices

1 Introduction

In [8, theorem 1], it was shown that

$$||C_\alpha||_{l^2 \longrightarrow l^2} \le 1 + \max\left\{ \frac{\sqrt{n}}{(n+1)^{\alpha-\frac{1}{2}}} \, : \, n = 1, 2, \dots \right\}.$$

Corollary 8 in [1] gives a characterization for C_α to map l^p into l^q, this happens if and only if:

$$\begin{aligned}
&\alpha \ge 0 && (p = 1, q = \infty)\\
&\alpha \ge \frac{1}{p^*} && (p > 1, q = \infty)\\
&\alpha \ge \frac{1}{q} + \frac{1}{p^*} && (1 \le p \le q < \infty)\\
&\alpha > \frac{1}{q} + \frac{1}{p^*} && (p > q)
\end{aligned}$$

In [8, theorem 2], Rhaly, also computed the Hilbert-Schmidt norm of C_α , which is $\sum_n n^{-2\alpha+1}$. In other words, C_α is 2-summing on l^2 if $\alpha > 1$, (see [3, page 84] for instance). We have found it natural to investigate the problem in general; i.e. when is $C_\alpha : l^p \longrightarrow l^q$ r-summing. Unfortunately, we no longer have the luxury of the rich theory of Hilbert spaces, which enables us to compute the 2-summing norm of C_α exactly.
In section 2 we set up notation and terminology. Section 3 concentrates on the l^1-case and gives an application. In section 4 we obtain the main result of C_α on l^p. Section 5 is devoted to the more general situation about a terraced matrix B.

2 Notations and Definitions

By l^p we mean the space of complex-valued sequences x satisfying

$$||x||_p := (\sum \mid x_n \mid^p)^{1/p} < \infty; \; 1 \le p < \infty, \quad ||x||_\infty := sup| \, x_n \,|: 1 \le n; p = \infty$$

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If $1 \le p \le \infty$, we define the conjugate index p^* of p by $1/p + 1/p^\star = 1$.
$\| A \|_{p,q}$ denote the norm of the operator A from l^p into l^q. If X is a Banach space, its dual will be denoted by $X^\star$.
Let T be an operator from a normed space X into a normed space Y. Then we say that T is r–summing ($1 \le r < \infty$) if, for all natural numbers N and for all vectors $x_1, x_2, \ldots, x_N$ in X, there exists an absolute constant $k > 0$ such that

$$\left(\sum_{n=1}^{N} ||Tx_n||^r\right)^{\frac{1}{r}} \le k \sup \left\{ (\sum_{n=1}^{N} | \phi(x_n |^r)^{\frac{1}{r}} : \phi \in X^*, ||\phi|| \le 1 \right\}.$$

The infimum of all such k is the r–summing norm $\pi_r(T)$ of T. When $r = \infty$, the r–summing norm reduces to the ordinary operator norm.
An equivalent inequality to the above one, which we shall be using, is;

$$(\sum_{n=1}^{N} ||Tx_n||^{r})^{\frac{1}{r}} \le \pi_r(T) \sup \left\{ || \sum \lambda_n x_n || : \lambda \in l^{r^*}, ||\lambda|| \le 1 \right\}$$

see, for instance [3, page 35].

A Banach space X has the Dunford-Pettis property if and only if for any Banach space space Y, each weakly compact linear operator $T : X \longrightarrow Y$ is completely continuous; i.e., takes weakly convergent sequences in X into norm convergent sequences in Y.

Throughout this paper B will be used to denote, either the terraced matrix defined by $b_{nk} = \beta_n$ if $1 \le k \le n$ and 0 if $k > n$, or the operator induced by the matrix B itself. The matrix C_α is B with $\beta_n^{-\alpha}$; $\alpha > 0$.

3 The l^1- Case

We start with a general result that will be used repeatedly.

Proposition 1. *Let $1 \le p, q < \infty$ and $A = (a_{nk})$ be any matrix; then $\sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} | a_{nk} |^{p^*})^{\frac{q}{p^*}} < \infty$ implies that the induced operator $A : l^p \longrightarrow l^q$ is q–summing.*
Furthermore, $\pi_q^p(A) \le \sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} |a_{nk}|^{p^*})^{\frac{1}{p^*}}$

Proof. $||AX||_q^q = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{nk}x_k|^q$ for any $X \in l^p$.
Define f_n on l^p; $n \ge 1$, as follows:

$$f_n(X) = f_n((x_k)) := \sum_{k=1}^{\infty} a_{nk}x_k.$$

Then the f_n are linear functionals with $||f_n|| = (\sum_{k=1}^{\infty} |a_{nk}|^{p^*})^{\frac{1}{p^*}}$, $1 \le n$.

This means that there are functionals on l^p with the property

$$||AX||_q^q \le \sum_{k=1}^{\infty} |f_n(X)|^q \;\forall X \in l^p.$$

By invoking Proposition 3.2 of [5] we get

$$\pi_q(A) \le (\sum_{n=1}^{\infty} ||f_n||^q)^{\frac{1}{q}} = [\sum_{n=1}^{\infty}(\sum_{k=1}^{\infty} |a_{nk}|^{p^*})^{\frac{q}{p^*}}]^{\frac{1}{q}}$$

Specializing Proposition 1 to the operators we are interested in, we get the following corollary.

□

Corollary 1. *(a) If $B : l^p \longrightarrow l^q$ satisfying $\sum_{n=1}^{\infty} n^{\frac{q}{p^*}} |\beta_n|^q < \infty$, then B is q–summing*
(b) If $C_\alpha : l^p \longrightarrow l^p$ with $\alpha > 1$, then C_α is p–summing.

Proof. The result follows from Proposition 1 by replacing a_{nk} by β_n in the case of B, and by $n^{-\alpha}$ in the case of C_α. □

The remainder of this section will be devoted to the l^1–case. We think that this case is particularly important, because it gives us another reason for the way we proposed our conjecture about B at the end of the paper. Another reason for singling out this case lies in the fact that arguments to be seen later about necessity don't work for the l^1–case.

The following result gives a necessary condition for B to be 1-summing in the l^1-case.

Proposition 2. *If $B : l^1 \longrightarrow l^1$ is 1-summing, then $\sum_n |\beta_n| < \infty$*

Proof. Since B is 1-summing, for any $x_1, x_2, \cdots, x_N \in l^1$, we have
$\sum_{k=1}^{N} ||Bx_k|| \le \pi_1(B) \sup \left\{ || \sum_{k=1}^{N} \lambda_k x_k || : \lambda \in l^\infty, ||\lambda|| \le 1 \right\}$

Choose N vectors as:

$$x_k = \left(0, \cdots, 0, \frac{1}{k^\alpha}, 0, \cdots, 0, \frac{1}{(N+k)^\alpha}, 0, \cdots, 0, \frac{1}{(2N+k)^\alpha}, \cdots \right); \alpha > 1$$

With these vectors, and after some calculation, the above inequality reduces to

$$\sum_{n\ge1} |\beta_n| \left(1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha}\right) \le \pi_1(B) sup \left\{ \left\| \left(\frac{\lambda_n}{n^\alpha}\right)_{n\ge1} \right\| : \| \lambda \|_{l^\infty} \le 1 \right\}$$

Use the integral test and observe that the right hand side is finite. □

Now we are in a position to state our first main result.

Theorem 1. *(a) $B : l^1 \to l^1$ is 1-summing if and only if $\sum_{n=1}^{\infty} | \beta_n | < \infty$*
(b) $C_\alpha : l^1 \longrightarrow l^1$ is 1-summing if and only if $\alpha > 1$.

Proof. Part (a) follows from Corollary 1(a) with $p = q = 1$ and from Proposition 2.
Part (b) follows from Corollary 1(a) with $p = q = 1$, and from Proposition 2. □

Application. Let t be a sequence in (0,1) that converges to 0, and define the Abel matrix A_t by $a_{nk} = t_n(1-t_n)^k$. In his paper [4, theorem 1], Friday proved the Theorem "A_t is an $l - l$ matrix if and only if t is in l^1. Using Proposition 1, we obtain a stronger result.

Proposition 3. *A_t is 1-summing on l^1 if and only if $t \in l^1$.*

Proof. By Proposition 1,

$$\pi_1(A_t) \le \sum_{n=1}^{\infty} sup \left\{ | (t_n(1-t_n)^k |: k \ge 1 \right\} \le \sum | t_n | = ||t||_{l^1}.$$

This proves that the condition is sufficient. The condition is necessary since 1-summing implies boundedness. □

4 The C_α on l^p Case

To complete the characterization of C_α on l^p, we need to investigate a necessary condition for C_α to be p–summing on l^p; $1 < p < \infty$. The following lemma will be needed for our argument.

Lemma 1. *Let $1 \le p < \infty$. Then $C_1 : l^p \to l^p$ does not have the Dunford-Pettis property.*

Proof. Define a sequence of vectors $E_n \in l^p$ as follows: $E_n = (e_1 + \cdots + e_n)/n^{1/p}$ where e_i is the standard unit vector with 1 in the ith place and 0 elsewhere. To see that E_n converges weakly, we must show that $f(E_n) \to 0$ for every $f = (f_j)_{j=1}^{\infty}$ of l^{p^*}. But this is equivalent to proving that $n^{-1/p} \sum_{j=1}^{\infty} f_j \to 0$ for every $f \in l^{p^*}$. so let $\epsilon > 0$, and $f \in l^{p^*}$. Then there exists a positive integer N such that $\sum_{j=N}^{\infty} \mid f_j \mid^{p^*} < \frac{\epsilon}{2}$ and $(\frac{1}{N})^{1/p} < \frac{\epsilon}{2}$. Letting $n = N^2$and using Hölder's inequality, we have

$$\begin{aligned}\mid \frac{1}{n^{1/p}} \sum_{j=1}^{n} f_j \mid \leq \mid \frac{1}{n^{1/p}} \sum_{j=1}^{N-1} f_j \mid + \frac{1}{n^{1/p}} \sum_{j=N}^{n} f_j \mid &\leq \frac{1}{n^{1/p}} \left(\sum_{j=1}^{N-1} \mid f_j \mid^{p^\star} \right)^{1/p^\star} \left(\sum_{j=1}^{N-1} 1^p \right)^{1/p} + \frac{1}{n^{1/p}} \left(\sum_{j=N}^{n} \mid f_j \mid^{p^\star} \right)^{1/p^\star} \left(\sum_{j=N}^{n} 1^p \right)^{1/p} \\ &< N^{-1/p} + \left(\frac{n-N}{n} \right)^{1/p} \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

On the other hand,

$$||C_1 E_n||_p^p = \left\| \left(\frac{1}{n^{1/p}}, \cdots, \frac{1}{n^{1/p}}, \frac{n}{n+1} . \frac{1}{n^{1/p}}, \frac{n}{n+2} \frac{1}{n^{1/p}}, \cdots \right) \right\|_p^p \geq 1;$$

i.e. $\| C_1 E_n \|_p$ does not converge to 0. Thus C_1 does not have the Dunford-Pettis property on l^p.

□

Corollary 2. *$C_1 : l^p \to l^p$ is not r-summing for any $1 \leq r < \infty$.*

Proof. This result follows from Lemma 1 and from the fact that says: if $T : X \to Y$ is p-summing, then T must have the Dunford-Pettis property (see [7, page 343]).

□

Now we state and prove a necessary condition for $C_\alpha : l^p \to l^p$ to be p-summing.

Proposition 4. *Let $1 \leq p < \infty$. If $C\alpha : l^p \to l^p$ is p-summing, then $\alpha > 1$.*

Proof. Suppose, on the contrary, that $\alpha \leq 1$. We split the proof into two cases: $\alpha < 1$ and $\alpha = 1$. For the case $\alpha < 1$, C_α does not even map l^p to l^p, see [1, Corollary 8]. For the case $\alpha = 1$, from Corollary 7 C_1 is not p-summing. Thus α must be larger than 1. □

Now we can state the following main result.

Theorem 2. *Let $1 \leq p < \infty$. Then $C_\alpha : l^p \to l^p$ is p-summing if and only if $\alpha > 1$.*

Proof. The proof follows from Corollary 1(a) and proposition 4. □

Kwapien's result on the equivalence of π_r (see [6, page 109] gives the following Corollary that sounds a little restrictive.

Corollary 3. *Let $1 \leq p \leq 2$. Then $C_\alpha : l^p \to l^p$ is r-summing if and only if $\alpha > 1$; $1 \leq r < \infty$*

5 The General Terraced Matrix B

We shall investigate a necessary condition, as only a sufficient condition was considered in Section 3. We start with the region $1/p + 1/q \leq 1$.

Theorem 3. *Let $1 < p \leq \infty$, $1 \leq q < \infty$ and $1/p + 1/q \leq 1$. Then B is q-summing implies that $\sum_{n=1}^{\infty} n^{\frac{q}{p^\star}} \mid \beta_n \mid^q < \infty$.*

Proof. Since B is q-summing,

$$\left(\sum_{n=1}^{N} \| Bx_n \|_q^q\right)^{1/q} \leq \pi_q(B)\, sup \left\{ \left\| \sum_{n=1}^{N} \lambda_n x_n \right\|_p : \lambda \in l^{q^\star}, \| \lambda \| \leq 1 \right\}$$

Define the x_n as follows:

$$\begin{aligned} x_1 &= (1, 0, 0, 0, \cdots) \\ x_2 &= \left(\frac{1}{2^\alpha}, \frac{1}{2^\alpha}, 0, 0, \cdots\right) \\ x_3 &= \left(\frac{1}{3^\alpha}, \frac{1}{3^\alpha}, \frac{1}{3^\alpha}, 0, \cdots\right) \end{aligned}$$

and so on, where $\alpha = 1/p + 1/q$.
Then the right hand side of the above inequality becomes

$$sup \left\{ \| (\lambda_1, 0, \cdots + (\frac{\lambda_2}{2^\alpha}, \frac{\lambda_2}{2^\alpha}, 0, \cdots) + \cdots + (\frac{\lambda_N}{N^\alpha}, \cdots, \frac{\lambda_N}{N^\alpha}, 0, \cdots) \| : \lambda \in l^{q^\star}, \| \lambda \| \leq 1 \right\}$$

$$= sup \left\{ \| (\sum_{m=1}^{N} \frac{\lambda_m}{m^\alpha}, \sum_{m=2}^{N} \frac{\lambda_m}{m^\alpha}, \cdots, \sum_{m=N}^{N} \frac{\lambda_m}{m^\alpha}, 0, \cdots) \|_p : \lambda \in l^{q^\star}, \| \lambda \| \leq 1 \right\}$$

$$= sup \left\{ (\sum_{n=1}^{N} | \sum_{m=n}^{N} \frac{\lambda_m}{m^\alpha} |^p)^{1/p} : \lambda \in l^{q^\star}, \| \lambda \| \leq 1 \right\}$$

This could be thought of as $\|A\|$, where A is the operator from $l^{q^\star}$ into l^p induced by the matrix

$$\begin{pmatrix} 1 & \frac{1}{2^\alpha} & \frac{1}{3^\alpha} & \cdots \\ 0 & \frac{1}{2^\alpha} & \frac{1}{3^\alpha} & \cdots \\ 0 & 0 & \frac{1}{3^\alpha} & \cdots \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

But $||A||_{q^\star,p} = ||A^T||_{p^\star,q}$ which is bounded when $1 \leq p^\star \leq q < \infty$ (see [1, Corollary 8]).
To estimate the left hand side of the above inequality:

$$\sum_{n=1}^{N} \|BX_n\|_q^q = \sum_{n=1}^{N} \left\| \left(\frac{\beta_1}{n^\alpha}, \frac{2\beta_2}{n^\alpha}, \cdots, \frac{(n-1)\beta_{n-1}}{n^\alpha}, \frac{n\beta_n}{n^\alpha}, \frac{n\beta_{n+1}}{n^\alpha}, \cdots \right) \right\|_q^q$$

Collecting the terms that contain β_n we get $| \beta_n |^q \,(1 + 2^{q-\alpha q} + \cdots + n^{q-\alpha q} + \cdots)$, which is greater than or equal to $K \mid \beta_n \mid^q n^{q-\alpha q+1}$ where K is some constant.
It remains to check whether the following is true, and, if yes, whether $| \beta_n |^q n^{q-\alpha q+1} \geq | \beta_n |^q n^{q/p^\star}$
This reduces to checking the inequality $q - \alpha q + 1 \geq q/p^\star$, which is true under our choice of α; i.e. with $\alpha = 1/p + 1/q$. □

The following theorem investigates a necessary condition in the case $1/p + 1/q > 1$

Theorem 4. *Let* $1/p + 1/q > 1$ *and* $\epsilon > 0$. *Then* $\sum_{n\geq 1} n^{\frac{q}{p^\star}-\epsilon_2} \mid \beta_n \mid^q < \infty$ *if* $B : l^p \to l^q$ *is q-summing, where* $\epsilon_2 = k\,\epsilon$ *with k being a constant.*

Proof. . We arque as in Theorem 3, except that we take the vectors x_n of l^p as follows:

$$\begin{aligned} x_1 &= (a_1, 0, \cdots) \\ x_2 &= x_3 = (0, a_2, 0, \cdots) \\ x_4 &= x_5 = x_6 = (0, a_3, 0, \cdots) \end{aligned}$$

and so on, where the $a^{'}s$ will be determined later.
Starting with the right hand side of the q-summing inequality, we have

$$sup\{||\sum \lambda_n x_n|| : \lambda \in l^{q^\star}, ||\lambda|| \le 1\} = sup\{||a_1\lambda_1, a_2(\lambda_2+\lambda_3), a_3(\lambda_4+\lambda_5+\lambda_6), \cdots || : \lambda \in l^{q^\star}, ||\lambda|| \le 1\} = ||A||_{q^\star,p},$$

where A is the matrix

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & a_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

But $||A|| = (\sum_{n=1}^{\infty} ||A_n||_{q^\star,p}^r)^{1/r}$; for example, see Proposition 3 of [2], where A_n is the $1 \times n$ matrix $\begin{pmatrix} a_n & a_n & \cdots & a_n \end{pmatrix}$ and $1/r = 1/p - 1/q\star$.

$$\begin{aligned} ||A||_{q^\star,p} &= sup\{||A_n y|| : y \in l^{q^\star}, ||y|| \le 1\} \\ &= sup\{|a_n(y_1+\cdots+y_n)| : y \in l^{q^\star}, ||y|| \le 1\} \\ &= |a_n|\, n^{\frac{1}{q}}. \end{aligned}$$

Therefore

$$||A||^r = \sum \left(n^{\frac{1}{q}} \mid a_n \mid\right)^{\frac{pq}{p+q-pq}},$$

which is finite if

$$\left(n^{\frac{1}{q}} \mid a_n \mid\right) \frac{pq}{p+q-pq} = O\left(\frac{1}{n^{1+\epsilon}}\right),$$

or if $n \mid a_n \mid^q = O\left(1/n^{1+q/p-q+\epsilon_1}\right)$ where $\epsilon_1 = (1+q/p-q)\,\epsilon$
So, we may define a_n as $1/n^{2/q+1/p-1+\epsilon_2}$ where $\epsilon_2 = k\epsilon_1$ and k is a positive number.
On the other hand, calculating the left hand side, we get

$$\begin{aligned} \sum ||Bx_n||_q^q &= \sum \mid \beta_n \mid^q (\mid a_1^q + 2 \mid a_2 \mid^q + \cdots + n \mid a_n \mid^q) \\ &= \sum \mid \beta_n \mid^q \left(\sum_{m=1}^{n} m.\frac{1}{m^{2+\frac{q}{p}-1+\epsilon_2}}\right) \\ &\approx \sum n^{\frac{q}{p^\star}-\epsilon_2} \mid \beta_n \mid^q. \end{aligned}$$

□

We now are in a position to give the main result about B.

Theorem 5. *Let $1 < p \le \infty$, $1 \le q < \infty$, and $\epsilon > 0$, and B maps l^p into l^q. Then*
(i) B is q-summing if and only if $\sum n^{q/p^\star} \mid \beta_n \mid^q < \infty$; $\frac{1}{p} + \frac{1}{q} \le 1$,
(ii) There exist constants K_1 and K_2 such that

$$K_1 \sum n^{\frac{q}{p^\star}-\epsilon_2} \le \pi_q(B) \le K_2 \sum n^{\frac{q}{p^\star}} \mid \beta_n \mid^q; 1/p + 1/q \le 1.$$

Proof. The proof of (i) follows from Corollary 1(a) and Theorem 3, and the proof of (ii) follows from Corollary 1(a) and Theorem 4. □

Note. It would be desirable to determine, whether the ϵ in Theorem 12 is really needed or not, but we have not been able to do this despite our trials in both directions. One may conjecture that the ϵ is not really needed. This is not only because this is the case in the region $1/p + 1/q \le 1$, but also, and more importantly, because the ϵ in the l^1-case was not needed (as was shown in section 3), which is obviously in the region $1/p+1/q > 1$.

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