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Research Article

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Remarks on a result of Chen-Cheng

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Abstract: In their seminal work, Chen and Cheng proved *a priori* estimates for the constant scalar curvature metrics on compact Kähler manifolds. They also prove $C^{3,\alpha}$-estimate for the potential of the Kähler metrics under boundedness assumption on the scalar curvature and the entropy. The goal of this article is to replace the uniform boundedness of the scalar curvature to the L^p-boundedness of the scalar curvature.

Keywords: Calabi flow, Chen-Cheng estimate, scalar curvature

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1 Introduction

A fundamental theorem in the realm of complex analysis is the uniformization theorem. One of the implications of the uniformization theorem is that every compact Riemann surface admits a metric with consistent Gaussian curvature. This principle can be extended in numerous ways to manifolds of higher dimensions. Within complex geometry, the aspiration is to discover canonical metrics on a Kähler manifold, those that align with the complex structure and exhibit curvature with specified characteristics. Kähler-Einstein metrics, constant scalar curvature Kähler metrics, and extremal metrics are prime examples of such metrics.

The existence of Kähler-Einstein metrics on compact complex manifolds was proved by Yau for manifolds with a trivial canonical class [13,14]. In the case of negative first Chern classes, both Aubin and Yau independently affirmed the existence of Kähler-Einstein metrics [1,13,14]. However, the scenario is most challenging for Fano manifolds, where the first Chern class is positive, and there exist known barriers to the realization of Kähler-Einstein metrics. As conjectured by Yau, these barriers should all correlate with the stability of the manifolds.

The challenge concerning Fano manifolds was eventually overcome by Chen et al. [4–6] and Tian [8] a few years back. Regarding cscK metrics, the Yau-Tian-Donaldson conjecture proposes that the presence of such metrics corresponds to a form of stability. The cscK metrics scenario is notably more intricate than that of Kähler-Einstein metrics, primarily because the constant scalar curvature equation is a fourth-order fully nonlinear elliptic partial differential equation (PDE), while our understanding of fourth-order nonlinear PDEs is still limited. In contrast, the Kähler-Einstein equation is a second-order fully nonlinear elliptic PDE, a field that has been extensively explored over the years.

Progress in the constant scalar curvature equation had been stagnant until the recent breakthrough of Chen and Cheng [2,3], who established *a priori* estimates for cscK equations, providing significant insights that the Kähler potential and all its derivatives of a cscK metric can be controlled in terms of the relative entropy.

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Let M be a Kähler manifold of dimension n and ω be its Kähler form. For any Kähler potential φ, define $\omega_\varphi = \omega + \sqrt{-1}\ \partial\bar{\partial}\varphi$. We consider the equations

$$\omega_\varphi^n = (\omega + \sqrt{-1}\ \partial\bar{\partial}\varphi)^n = e^F\omega^n, \quad \sup_M \varphi = 0, \quad \Delta_{\omega_\varphi} F = -R + \mathrm{tr}_{\omega_\varphi}\eta, \tag{1}$$

where R is the scalar curvature of the metric ω_φ, and η is a fixed smooth $(1,1)$-form. The prototype of η is the Ricci curvature $\mathrm{Ric}(\omega)$ of ω.

In their papers [2,3], Chen and Cheng proved the following:

Theorem 1.1. [2, 3] *For any $p \geq 1$, there exists a constant C that depends on $n, p, \omega, \eta, ||R||_\infty$, and $\int_M e^F\sqrt{1+F^2}\,\omega^n$ such that $||F||_{W^{2,p}}, ||\varphi||_{W^{4,p}} \leq C$. In particular, F and φ are uniformly bounded in $C^{1,\alpha}$ and $C^{3,\alpha}$, respectively, for any $\alpha \in (0,1)$.*

With some modifications to the argument in [2], we slightly generalize the proceeding theorem. Namely, we replace the uniform bound on the scalar curvature with the L^p-bound for some $p > 0$.

Let $\Phi(t) = \sqrt{1+t^2}$. Define A_F and $A_{R,p}$ by

$$A_F^n = \int_M e^F\Phi(F)\omega^n, \quad A_{R,p}^n = \int_M e^F\Phi(R)^p\omega^n$$

for $p > 0$. A_F gives an upper bound for the entropy

$$\int_M Fe^F\omega^n \leq A_F,$$

and $A_{R,p}$ gives an upper bound for the L^p-norm of R with respect to ω_φ

$$\left(\int_M |R|^p\omega_\varphi^n\right)^{1/p} \leq A_{R,p}^{n/p}.$$

The main results of this article are the following theorems.

Theorem 1.2. *For any $p > n$, there exists a constant C that depends on n, p, ω, A_F, and $A_{R,p}$ such that $||F||_\infty \leq C$ and $||\varphi||_\infty \leq C$.*

Theorem 1.3. *Let $n = \dim M$. Then, there exist $p_n > 2n$ that depends only on n such that $||F||_{W^{2,p_n}} \leq C$ and $||\varphi||_{W^{4,p_n}} \leq C$ for a constant C depending on n, ω, η, A_F, and A_{R,p_n}.*

Moreover, for any $p \geq p_n$, there exists a constant C_p that depend on n, ω, η, A_F, and $A_{R,p}$ such that $||F||_{W^{2,p}} \leq C_p$ and $||\varphi||_{W^{4,p}} \leq C_p$.

Note that in Theorem 1.3, $W^{2,p}$ and $W^{4,p}$ are optimal regularity for φ and F, respectively, because of (1) and the fact that R is L^p for some $p > 0$.

Theorem 1.3 gives *a priori* $C^{3,\alpha}$ and $C^{1,\alpha}$ estimate for φ and F, respectively for some $\alpha = \alpha(p,n) \in (0,1)$ by Sobolev embedding theorem.

This article is organized as follows. In Section 2, we prove Theorem 1.2. Our argument does not use the Alexandrov maximum principle and the cut-off function as in Chen and Cheng [2,3]. Instead, we use Kołodziej's theorem to prove the boundedness of the auxiliary functions. We then prove the result using the classical maximum principle.

In Section 3, we prove that there is an L^p-estimate of $n + \Delta\varphi$. The C^2 estimate is obtained in Section 4 using Moser iteration. The arguments in Sections 3 and 4 are essentially the same as those in [3].

Throughout this article, we shall use $\int_M f$ to denote $\int_M f\omega^n$, where ω is the background metric of the manifold. We use $||f||_p$ to denote the L^p-norm of function f with respect to the background metric ω.

2 Proof of Theorem 1.2

The section's main goal is to prove a uniform estimate for φ and F. This section's constant C depends on $n = \dim M$, ω, and η, which may differ line by line.

Lemma 2.1. *Let $h : M \to \mathbb{R}$ be a positive function and φ and v be Kähler potentials such that*

$$(\omega + \sqrt{-1}\ \partial\bar\partial\varphi)^n = e^F\omega^n,$$

$$(\omega + \sqrt{-1}\ \partial\bar\partial v)^n = e^F h^n\omega^n.$$

Then, $\Delta_\varphi v \geq nh - \mathrm{tr}_{\omega_\varphi}(\omega)$. Here, $\omega_\varphi = \omega + \sqrt{-1}\ \partial\bar\partial\varphi$ and Δ_φ is the Laplacian with respect to the metric ω_φ.

Proof. This follows by applying the AM-GM inequality to $\mathrm{tr}_{\omega_\varphi}(\omega + \sqrt{-1}\ \partial\bar\partial v)$. □

Let $\alpha = \alpha(M, \omega)$ be the α-invariant of (M, ω). By definition, for any smooth function $\varphi : M \to \mathbb{R}$ such that $\omega + \sqrt{-1}\ \partial\bar\partial\varphi > 0$, we have

$$\int_M e^{-\frac{1}{2}\alpha(\varphi - \sup_M \varphi)}\omega^n \leq C$$

for some $C > 0$ independent to φ.

Theorem 2.1. *For any $p > n$, there exists $\delta_0 = \delta_0$ depending on $n, p, \omega, \eta, A_F, ||R||_p$ such that for any $\delta < \delta_0$, we have*

$$\int_M e^{(1+\delta)F} \leq C,$$

where $C = C(n, p, \delta_0, \omega, \eta, A_F, ||R||_p)$.

Proof. For a fixed $p > n$, we define functions ψ and ρ as the solutions of the following:

$$(\omega + \sqrt{-1}\ \partial\bar\partial\psi)^n = A_F^{-n}e^F\Phi(F)\omega^n = A_F^{-n}\Phi(F)\omega_\varphi^n, \quad \sup_M \psi = 0; \tag{2}$$

$$(\omega + \sqrt{-1}\ \partial\bar\partial\rho)^n = A_{R,p}^{-n}e^F\Phi(R)^p\omega^n = A_{R,p}^{-n}\Phi(R)^p\omega_\varphi^n, \quad \sup_M \rho = 0. \tag{3}$$

Let $0 < \varepsilon \leq 1$ and $u = F + \varepsilon\psi + \varepsilon\rho - \lambda\varphi = v - \lambda\varphi$, where $v = F + \varepsilon\psi + \varepsilon\rho$ and $\lambda = |\eta|_\omega + 2$. Let $\delta > 0$. Then, by Lemma 2.1, we have

$$\begin{aligned} e^{-\delta u}\Delta_\varphi(e^{\delta u}) &\geq \delta\Delta_\varphi u \\ &\geq \delta(-R + \mathrm{tr}_{\omega_\varphi}\eta) + \varepsilon\delta(nA_F^{-1}\Phi(F)^{\frac{1}{n}} - \mathrm{tr}_{\omega_\varphi}\omega) + \varepsilon\delta(nA_{R,p}^{-1}\Phi(R)^{\frac{p}{n}} - \mathrm{tr}_{\omega_\varphi}\omega) - n\delta\lambda + \delta\lambda\ \mathrm{tr}_{\omega_\varphi}\omega \\ &= \delta(-R + \varepsilon nA_F^{-1}\Phi(F)^{\frac{1}{n}} + \varepsilon nA_{R,p}^{-1}\Phi(R)^{\frac{p}{n}} - \lambda n) + \delta(\mathrm{tr}_{\omega_\varphi}\eta - 2\varepsilon\ \mathrm{tr}_{\omega_\varphi}\omega + \lambda\ \mathrm{tr}_{\omega_\varphi}\omega) \\ &\geq \delta(-R + \varepsilon nA_F^{-1}\Phi(F)^{\frac{1}{n}} + \varepsilon nA_{R,p}^{-1}\Phi(R)^{\frac{p}{n}} - \lambda n). \end{aligned} \tag{4}$$

The last inequality holds since $\varepsilon \leq 1$.

Let

$$\delta_0 = \lambda^{-1}\min(\alpha, 1),$$

where $\alpha = \alpha(M, [\omega])$ is the α-invariant of M. We choose $0 < \delta < \frac{1}{2}\delta_0$. Fixing δ, we choose $\varepsilon > 0$ small so that

$$2(1 + \delta)\cdot\,\varepsilon < \min(\alpha, 1).$$

Let

$$\hat\Phi(F) = \varepsilon nA_F^{-1}\Phi(F)^{1/n}.$$

Then

$$\varepsilon A_{R,p}^{-1}\Phi(R)^{\frac{p}{n}} - R \geq -C(\varepsilon, p, A_{R,p}),$$

since $A_{R,p}$ is bounded and $p > n$. Therefore, (4) implies that

$$\Delta_\varphi e^{\delta u} \geq \delta e^{\delta u}(\hat{\Phi}(F) - C) \tag{5}$$

for some constant $C > 0$. As a result, we have

$$\int_M e^{\delta u}(\hat{\Phi}(F) - C)\omega_\varphi^n \leq 0.$$

We let

$$E_1 = \{x | \hat{\Phi}(F) - C \geq 1\};$$
$$E_2 = \{x | \hat{\Phi}(F) - C < 1\}.$$

On E_2, F is bounded, say $F \leq C$. Thus, we have

$$\int_{E_1} e^{\delta u + F} \leq \int_{E_1} e^{\delta u}(\hat{\Phi}(F) - C)\omega_\varphi^n \leq -\int_{E_2} e^{\delta u}(\hat{\Phi}(F) - C)\omega_\varphi^n.$$

Since $\hat{\Phi}(F)$ is nonnegative, and on E_2, we have $u \leq C - \lambda\varphi$, we have

$$\int_{E_1} e^{\delta u + F} \leq C\int_{E_2} e^{-\delta\lambda\varphi} \leq C\int_M e^{-\delta\lambda\varphi} \leq C,$$

since $\delta\lambda$ is less than half of the α-invariant. By definition of u, we have

$$\int_{E_1} e^{(1+\delta)F + \varepsilon\delta(\psi+\rho)} \leq \int_{E_1} e^{\delta u + F} \leq C.$$

Since

$$\omega + \sqrt{-1}\partial\bar{\partial}\frac{\psi + \rho}{2} > 0,$$

using the Hölder inequality, we have

$$\begin{aligned}\int_{E_1} e^{(1+\delta/2)F} &= \int_{E_1} e^{(1+\delta/2)F + \frac{(1+\delta/2)}{1+\delta}\varepsilon\delta(\psi+\rho)} \cdot e^{-\frac{(1+\delta/2)}{1+\delta}\varepsilon\delta(\psi+\rho)} \\ &\leq \left(\int_{E_1} e^{(1+\delta)F + \varepsilon\delta(\psi+\rho)}\right)^{\frac{1+\delta/2}{1+\delta}} \cdot \left(\int_{E_1} e^{-\frac{1+\delta/2}{\delta/2}\varepsilon\delta(\psi+\rho)}\right)^{\frac{\delta/2}{1+\delta}} \leq C,\end{aligned}$$

since $\frac{1+\delta/2}{\delta/2}\varepsilon\delta$ is less than half of the α-invariant. Combining the above with the fact that F is bounded on E_2, we have

$$\int_M e^{(1+\delta/2)F} \leq C.$$

□

The following proof of Theorem 1.2 is slightly different from that of Chen and Cheng [2].

Proof of Theorem 1.2. As in (2) and (3), we define functions ψ and ρ as the solutions of the following:

$$(\omega + \sqrt{-1}\ \partial\bar{\partial}\psi)^n = A_F^{-n}e^F\Phi(F)\omega^n = A_F^{-n}\Phi(F)\omega_\varphi^n, \quad \sup_M \psi = 0; \tag{6}$$

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\rho)^n = A_{R,p'}^{-n} e^F \Phi(R)^{p'} \omega^n = A_{R,p'}^{-n} \Phi(R)^{p'} \omega_\varphi^n, \quad \sup_M \rho = 0, \tag{7}$$

where $p' = (p+n)/2$.

We shall use the result of Kołodziej [7] to prove that the functions φ, ψ, ρ are uniformly bounded.

That φ is bounded directly follows from Theorem 2.1 and Kołodziej's theorem.

Since $x^{1+\delta}e^{-x} \le C$ for any real number $x > 0$, for $\delta < \delta_0/2$, we have

$$\int_M \Phi(F)^{1+\delta} e^{(1+\delta)F} \le C(n, p, \delta_0, \omega, \eta, A_F, ||R||_p).$$

Hence, Kołodziej's theorem implies that ψ is uniformly bounded.

Finally, we prove that ρ is uniformly bounded. Let $0 < \sigma < \delta < \delta_0/2$ and $a = 1 + \sigma$. We have

$$\begin{aligned}\int_M \Phi(R)^{ap'} e^{aF} &= \int_M \Phi(R)^{ap'} e^{\sigma F} \omega_\varphi^n \\ &\le \left(\int_M \Phi(R)^{ap' \frac{\delta}{\delta-\sigma}} \omega_\varphi^n\right)^{\frac{\delta-\sigma}{\delta}} \left(\int_M e^{\delta F} \omega_\varphi^n\right)^{\frac{\sigma}{\delta}} \\ &\le C\left(\int_M \Phi(R)^{ap' \frac{\delta}{\delta-\sigma}} \omega_\varphi^n\right)^{\frac{\delta-\sigma}{\delta}}.\end{aligned} \tag{8}$$

The first inequality follows from the Hölder inequality and the last inequality follows from Theorem 2.1. Now choose σ sufficiently small such that $ap' \frac{\delta}{\delta - \sigma} < p$. Therefore, the Hölder inequality implies that

$$\int_M \Phi(R)^{ap'} e^{aF} \le C \int_M \Phi(R)^p \omega_\varphi^n.$$

This, together with Kołodziej's theorem implies that $||\rho||_\infty \le C = C(n, \omega, \eta A_F, A_{R,n+1})$.

Let $u = F + \psi + \rho - \lambda\varphi$, where $\lambda = |\eta|_\omega + 2$. Then, we have

$$\Delta_\varphi u \ge -R + nA_F^{-1}\Phi(F)^{\frac{1}{n}} + nA_R^{-1}\Phi(R)^{\frac{p+n}{2n}} - C \ge nA_F^{-1}\Phi(F)^{\frac{1}{n}} - C,$$

since $(p+n)/2n > 1$. Let x_0 be a maximum point of u. Then, by the above,

$$F(x_0) \le C.$$

As a result, for any $x \in M$, we have

$$u(x) \le u(x_0) = F(x_0) + \psi(x_0) + \rho(x_0) - \lambda\varphi(x_0) \le C.$$

This implies that $F(x) \le C$.

Now, let $u' = -F + \psi + \rho - \lambda\varphi$. Then, by a similarly computation, we have

$$\Delta_\varphi u' \ge \varepsilon n A_F^{-1} \Phi(F)^{\frac{1}{n}} - C.$$

The same argument would imply that $F \ge -C$. This completes the proof of the theorem. □

3 $W^{2,p}$ estimate

In this section, we prove that for any $p > 0$, $n + \Delta\varphi$, where φ is the solution of (1), is in $L^p(M)$.

This section's constants C and C_i depend on $n = \dim M$, $p > 0$, ω, and η, which may differ line by line.

Theorem 3.1. *Let $\gamma = \frac{n}{n-1}$ and $p > 0$ be a positive number. Then*

$$\int_M (n + \Delta\varphi)^p \le C.$$

where C depends on $n, p, \omega, \eta, \|\varphi\|_\infty, \|F\|_\infty$, and $\|R\|_{\frac{(n-1)p}{\gamma}}$.

To prove Theorem 3.1, we first prove the following gradient estimate.

Proposition 3.1. *For any $p \ge 1$, there exist constants c_1 and c_2 depending on $n, p, \omega, \eta, \|\varphi\|_\infty, \|F\|_\infty$, and $\|R\|_{(n-1)p}$ such that*

$$\|\nabla\varphi\|_{2p} \le c_1 + c_2\|R\|_{(n-1)p}^{(n-1)/2}.$$

Proof. Let

$$u = e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}(|\nabla\varphi|^2 + K),$$

where K is an absolute constant (e.g., we can take $K = 10$). Then, we have

$$\Delta_\varphi u \ge Cu^{\frac{n}{n-1}} - (c + |R|)u$$

by [1, p. 918, equation (2.31)], where C, c are positive constants depending on $n, p, \omega, \eta, \|\varphi\|_\infty, \|F\|_\infty$. Let $p > 0$ and let γ be defined in Theorem 3.1. Then, we have

$$\frac{1}{p+1}\Delta_\varphi u^{p+1} = u^p\Delta_\varphi u + pu^{p-1}|\nabla_\varphi u|^2 \ge u^p\Delta_\varphi u \ge Cu^{p+\gamma} - (c + |R|)u^{p+1}.$$

Using Young's inequality $|R|u^{p+1} \le |R|^{(p+\gamma)(n-1)} + u^{p+\gamma}$, we have

$$\frac{1}{p+1}\Delta_\varphi u^{p+1} \ge Cu^{p+\gamma} - C_1 - C_2|R|^{(n-1)(p+\gamma)}.$$

Integrating the above inequality to the volume form ω_φ^n, we have

$$C\int_M u^{p+\gamma}\omega_\varphi^n \le C_1 + C_2\int_M |R|^{(n-1)(p+\gamma)}.$$

Since F is bounded, ω_φ^n and ω^n are equivalent. Thus, we have

$$\|u\|_{L^{p+\gamma}} \le c_1 + c_2\|R\|_{(n-1)(p+\gamma)}^{n-1}.$$

Thus, the proposition is valid for $p > \gamma$. But, then from the Hölder inequality, it is valid for any $p > 0$. □

Proof of Theorem 3.1. Let $\alpha > 2$ be a constant depending on p only, and to be determined later. Let λ be a constant depending on M. Let

$$u = e^{-\alpha(F+\lambda\varphi)}(n + \Delta\varphi).$$

By Yau's estimate, we have

$$\begin{aligned}\Delta_\varphi u \ge{}& e^{-(\alpha+\frac{1}{n-1})F-\alpha\lambda\varphi}\left(\frac{\lambda\alpha}{2} - C\right)(n+\Delta\varphi)^{1+\frac{1}{n-1}} \\ &- \lambda\alpha n e^{-\alpha(F+\lambda\varphi)}(n+\Delta\varphi) + \alpha e^{-\alpha(F+\lambda\varphi)}R(n+\Delta\varphi) + e^{-\alpha(F+\lambda\varphi)}(\Delta F - R_\omega),\end{aligned}$$

where R_ω is the scalar curvature of the metric ω. By choosing λ big enough such that $\frac{\lambda\alpha}{2} - C \ge \frac{\lambda\alpha}{4}$, we have

$$\Delta_\varphi u \ge C_1u^\gamma - C_2|R|u + e^{-\alpha(F+\lambda\varphi)}\Delta F - C_3.$$

We then have

$$
\begin{aligned}
\frac{1}{p+1}\Delta_\varphi u^{p+1} &= u^p\Delta_\varphi u + pu^{p-1}\,|\nabla_\varphi u|^2 \\
&\geq pu^{p-1}\,|\nabla_\varphi u|^2 + C_1 u^{p+\gamma} - C_2\,|R|^{(p+\gamma)(n-1)} + e^{-\alpha(F+\lambda\varphi)}\Delta F\ u^p - C_3,
\end{aligned}
\tag{9}
$$

where we used Young's inequality (with possibly different $C_1 > 0$, C_2, and C_3). Integrating the above to the volume form ω_φ^n and using the fact that F is bounded and R is in $L^{(p+\gamma)(n-1)}$, we have

$$
C_1\int\limits_M u^{p+\gamma} + p\int\limits_M u^{p-1}\,|\nabla_\varphi u|^2 \leq C_3 - \int\limits_M e^{-\alpha(F+\lambda\varphi)}\Delta F\ u^p\omega_\varphi^n. \tag{10}
$$

Let $\hat{F} = (1-\alpha)F - \alpha\lambda\varphi$. Using integration by parts, we have

$$
\begin{aligned}
-\int\limits_M e^{-\alpha(F+\lambda\varphi)}\Delta F u^p\omega_\varphi^n &= -\int\limits_M \Delta F u^p e^{\hat{F}} \\
&= -(\alpha-1)\int u^p\,|\nabla F|^2 e^{\hat{F}} - \lambda\alpha\int u^p\nabla F\nabla\varphi\ e^{\hat{F}} + p\int u^{p-1}\nabla F\nabla u\ e^{\hat{F}}.
\end{aligned}
\tag{11}
$$

By the AM-GM inequality, we have

$$
-\frac{\alpha-1}{2}\int u^p\,|\nabla F|^2 e^{\hat{F}} - \lambda\alpha\int u^p\nabla F\nabla\varphi\ e^{\hat{F}} \leq \frac{\lambda^2\alpha^2}{2(\alpha-1)}\int u^p\,|\nabla\varphi|^2 e^{\hat{F}}. \tag{12}
$$

Since $\hat{F}$ is bounded, we have

$$
\frac{\lambda^2\alpha^2}{2(\alpha-1)}\int u^p\,|\nabla\varphi|^2 e^{\hat{F}} \leq C_4\int u^p\,|\nabla\varphi|^2,
$$

where C_4 is a constant that depends on λ, α, and $||\hat{F}||_{L^\infty}$. Using Young's Inequality, we obtain

$$
\int u^p\,|\nabla\varphi|^2 \leq \frac{1}{2}\int u^{p+\gamma} + C_5\int |\nabla\varphi|^{2(p+\gamma)/\gamma}
$$

for a constant depending only on n. By Proposition 3.1, we have

$$
\int\limits_M |\nabla\varphi|^{2(p+\gamma)/\gamma} \leq (c_1 + c_2||R||^{(n-1)/2}_{(n-1)^2(p+\gamma)/n}).
$$

Since $(n-1)^2(p+\gamma)/n \leq (p+\gamma)(n-1)$, from (12), we conclude that

$$
-\frac{\alpha-1}{2}\int u^p\,|\nabla F|^2 e^{\hat{F}} - \lambda\alpha\int u^p\nabla F\nabla\varphi\ e^{\hat{F}} \leq C_6, \tag{13}
$$

where C_6 depends on λ, α, and $||\hat{F}||_{L^\infty}$.

On the other hand, we have

$$
-\frac{\alpha-1}{2}\int u^p\,|\nabla F|^2 e^{\hat{F}} + p\int u^{p-1}|\nabla F|\ |\nabla u|\ e^{\hat{F}} \leq \frac{p^2}{2(\alpha-1)}\int u^{p-2}\,|\nabla u|^2 e^{\hat{F}}. \tag{14}
$$

By the Cauchy-Schwarz inequality, we have

$$
|\nabla u|^2 = \left(\sum_i \sqrt{1+\varphi_{i\bar{i}}}\cdot\frac{|u_i|}{\sqrt{1+\varphi_{i\bar{i}}}}\right)^2 \leq (n+\Delta\varphi)\cdot|\nabla_\varphi u|^2.
$$

Thus

$$
|\nabla u|^2 \leq e^{\alpha(F+\lambda\varphi)}\ u\ \ |\nabla_\varphi u|^2 \leq C_7 u\ \ |\nabla_\varphi u|^2\ e^{-\hat{F}}
$$

for $C_7 = ||e^F||_{L^\infty}$. Hence, we have

$$
\frac{p^2}{2(\alpha-1)}\int u^{p-2}\,|\nabla u|^2 e^{\hat{F}} \leq \frac{C_7 p^2}{2(\alpha-1)}\int u^{p-1}\,|\nabla_\varphi u|^2.
$$

We choose α large enough so that

$$\frac{C_7 p^2}{2(\alpha-1)} \le \frac{p}{2}.$$

Then, we have

$$\frac{p^2}{2(\alpha-1)}\int u^{p-2}\,|\nabla u|^2 e^{\hat{F}} \le \frac{p}{2}\int u^{p-1}\,|\nabla_\varphi u|^2. \tag{15}$$

Thus, from (11), using (13) and (15), we have

$$-\int_M e^{-\alpha(F+\lambda\varphi)}\Delta F u^p \omega_\varphi^n \le C_6 + \frac{p}{2}\int u^{p-1}\,|\nabla_\varphi u|^2.$$

Combining with (10), we have

$$C_1\int_M u^{p+\gamma} + p\int_M u^{p-1}\,|\nabla_\varphi u|^2 \le C_3 + C_6 + \frac{p}{2}\int u^{p-1}\,|\nabla_\varphi u|^2.$$

Thus,

$$\int_M (n+\Delta\varphi)^p \le C$$

is valid for any $p > \gamma$. By the monotonicity of the L^p-norm, the above inequality is valid for any $p > 0$. □

4 C^2-estimate

In this section, we shall give the C^2 and high-order estimates. This section's constants C and C_i depend on n, ω, and η, which may differ line by line. But contrary to the previous section, these constants are independent of $p > 0$.

Theorem 4.1. *For each n, there exist positive numbers p_n, q_n (depending only on n) and C such that $\|n+\Delta\varphi\|_\infty \le C$. Here, C depends on n, ω, η, $\|\varphi\|_\infty$, $\|F\|_\infty$, $\|R\|_{p_n}$, and $\|n+\Delta\varphi\|_{q_n}$.*

We start with a Sobolev-type of inequality proved in [2].

Lemma 4.1. *Let n be the complex dimension of M. Then, for any $\varepsilon \in (0, \frac{1}{n+1})$, there exists a constant C that depends on ω and ε such that*

$$\|u\|_\beta^2 \le C\left(\|n+\Delta\varphi\|_{\frac{1-\varepsilon}{\varepsilon}}^2 \int_M |\nabla_\varphi u|_\varphi^2 + \|u\|_1^2\right),$$

where $\beta = 2\left(1 + \frac{1-(n+1)\varepsilon}{n-1+\varepsilon}\right) = \frac{2n(1-\varepsilon)}{n-1+\varepsilon}$.

Proof. The proof is given in [2]. For the reader's convenience, we include the argument here. We have the following Sobolev inequality:

$$\int_M |u|^{2n/(2n-1)} \le C\left(\int_M |\nabla u| + \int_M |u|\right)^{\frac{2n}{2n-1}}.$$

Replacing u by $u^{\frac{2n-1}{2n}\beta}$ in the above inequality, and by interpolation, we obtain

$$\int_M |u|^\beta \leq C\left(\int_M |\nabla u|^{2\alpha} + \left(\int_M |u|\right)^{2\alpha}\right)^{\frac{\beta}{2\alpha}}, \tag{16}$$

where $\alpha = 1 - \varepsilon$.

By the Cauchy-Schwarz inequality, we have

$$|\nabla u|^2 = \left(\sum_i \sqrt{1+\varphi_{i\bar{i}}} \cdot \frac{|u_i|}{\sqrt{1+\varphi_{i\bar{i}}}}\right)^2 \leq (n + \Delta\varphi)\cdot|\nabla_\varphi u|^2.$$

Thus, using (16), we have

$$\begin{aligned}\left(\int_M |u|^\beta\right)^{\frac{2\alpha}{\beta}} &\leq C\left(\int_M |\nabla u|^{2\alpha} + \left(\int_M u\right)^{2\alpha}\right)\\ &\leq C\left(\int_M |\nabla_\varphi u|^{2\alpha}(n+\Delta\varphi)^\alpha + \left(\int_M |u|\right)^{2\alpha}\right)\\ &\leq C\left(\int_M |\nabla_\varphi u|^2\right)^\alpha\left(\int_M (n+\Delta\varphi)^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} + C\left(\int_M |u|\right)^{2\alpha}.\end{aligned}$$

□

Proof of Theorem 4.1. We let

$$u = e^{F/2}\,|\nabla_\varphi F|_\varphi^2 + (n+\Delta\varphi) + 1.$$

Then, by [1, equation (4.13)], we have

$$\Delta_\varphi u \geq -C(n+\Delta\varphi)^{n-1}u + 2e^{F/2}\langle\nabla_\varphi F, \nabla_\varphi\Delta_\varphi F\rangle - C|R|u - C. \tag{17}$$

Multiplying (17) by u^{2p} and integrating by parts and using the fact that F is bounded, we have

$$2p\int_M u^{2p-1}\,|\nabla_\varphi u|^2\omega_\varphi^n \leq C\int_M (n+\Delta\varphi)^{n-1}u^{2p+1} + C\int_M |R|u^{2p+1} + C\int_M u^{2p} - 2\int_M e^{F/2}\langle\nabla_\varphi F, \nabla_\varphi\Delta_\varphi F\rangle u^{2p}\omega_\varphi^n. \tag{18}$$

In the above last term, we use the same idea as in the proof of Theorem 3.1 to obtain

$$\begin{aligned}&-\int_M e^{F/2}\langle\nabla_\varphi F, \nabla_\varphi\Delta_\varphi F\rangle u^{2p}\omega_\varphi^n\\ &\quad= \int_M e^{F/2}(\Delta_\varphi F)^2u^{2p}\omega_\varphi^n + \frac{1}{2}\int e^{F/2}(\Delta_\varphi F)|\nabla_\varphi F|^2u^{2p}\omega_\varphi^n + 2p\int_M e^{F/2}(\Delta_\varphi F)\langle\nabla_\varphi F, \nabla_\varphi u\rangle u^{2p-1}\omega_\varphi^n.\end{aligned}$$

Using the Cauchy-Schwarz inequality, for any $\varepsilon_0 > 0$, we have

$$\begin{aligned}&\int_M e^{F/2}(\Delta_\varphi F)\langle\nabla_\varphi F, \nabla_\varphi u\rangle u^{2p-1}\omega_\varphi^n\\ &\quad\leq C\varepsilon_0^{-1}\int_M (\Delta_\varphi F)^2u^{2p}\omega_\varphi^n + \varepsilon_0\int_M |\langle\nabla_\varphi F, \nabla_\varphi u\rangle|^2u^{2p-2}\omega_\varphi^n\\ &\quad\leq C\varepsilon_0^{-1}\int_M (\Delta_\varphi F)^2u^{2p} + C\varepsilon_0\int_M |\nabla_\varphi u|^2u^{2p-1}.\end{aligned} \tag{19}$$

As a result, we have

$$-\int_M e^{F/2}\langle\nabla_\varphi F, \nabla_\varphi\Delta_\varphi F\rangle u^{2p}\omega_\varphi^n \leq C\varepsilon_0\int_M |\nabla_\varphi u|^2u^{2p-1} + C(\varepsilon_0^{-1}+1)\int_M (\Delta_\varphi F)^2u^{2p} + C_2p\int_M |\Delta_\varphi F|u^{2p+1}.$$

By choosing ε_0 small enough, from (18), we have

$$p\int_M |\nabla_\varphi u|^2 u^{2p-1} \le C_1\int_M (n+\Delta\varphi)^{n-1}u^{2p+1} + C_2\int_M |R|u^{2p+1} + C_3\int_M (\Delta_\varphi F)^2 u^{2p} + C_4 p\int_M |\Delta_\varphi F|u^{2p+1}. \tag{20}$$

Using equation (1), we have

$$|\Delta_\varphi F| \le |R| + |\mathrm{Tr}_{\omega_\varphi}\eta| \le |R| + C(n+\Delta\varphi)^{n-1}.$$

Therefore, from (20), we obtain

$$\int_M |\nabla_\varphi u|^2 u^{2p-1} \le C_1\int_M (n+\Delta\varphi)^{2n-2}u^{2p+1} + C_2\int_M (1+|R|^2)u^{2p+1}.$$

Hence,

$$p^{-2}\int_M \left|\nabla_\varphi u^{p+\frac{1}{2}}\right|^2 \le C\int_M u^{2p-1}\,|\nabla_\varphi u|^2 \le C\int_M ((n+\Delta\varphi)^{2n-2} + 1 + |R|^2)u^{2p+1}.$$

Now, we fix an $\varepsilon \in (0, \frac{1}{n+1})$. Let $\beta = 2(1+\delta)$, where

$$\delta = \frac{1-(n+1)\varepsilon}{n-1+\varepsilon}$$

as in Lemma 4.1. Then, we have

$$\begin{aligned}\left\|u^{p+\frac{1}{2}}\right\|_\beta^2 &\le C\|n+\Delta\varphi\|_{\frac{1-\varepsilon}{\varepsilon}}\int_M \left|\nabla_\varphi u^{p+\frac{1}{2}}\right|_\varphi^2 + C\left\|u^{p+\frac{1}{2}}\right\|_1^2\\ &\le Cp^2\|n+\Delta\varphi\|_{\frac{1-\varepsilon}{\varepsilon}}\left(\int_M ((n+\Delta\varphi)^{2n-2} + 1 + |R|^2)u^{2p+1}\right).\end{aligned}$$

On the other hand, let $2 < \theta < \beta$ and let $\theta^* = (1-2\theta^{-1})^{-1}$. Then, for any function H, by the Hölder inequality, we have

$$\int_M Hu^{2p+1} \le \|H\|_{\theta^*}\cdot\left(\int_M u^{(2p+1)\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

In particular, we have

$$\int_M |R|^2u^{2p+1} \le \|R\|_{2\theta^*}^2\cdot\left(\int_M u^{(2p+1)\frac{\theta}{2}}\right)^{\frac{2}{\theta}}$$

and

$$\int_M (n+\Delta\varphi)^{2n-2}u^{2p+1} \le \|n+\Delta\varphi\|_{(2n-2)\theta^*}^{2n-2}\cdot\left(\int_M u^{(2p+1)\frac{\theta}{2}}\right)^{\frac{2}{\theta}}.$$

Assuming $\|R\|_{2\theta^*} \le C$, $\|n+\Delta\varphi\|_{(2n-2)\theta^*} + \|n+\Delta\varphi\|_{\frac{1-\varepsilon}{\varepsilon}} \le C$, we have

$$\left\|u^{p+\frac{1}{2}}\right\|_\beta^2 \le Cp^2\left\|u^{p+\frac{1}{2}}\right\|_\theta^2.$$

This implies that for any $p \ge \frac{1}{2}$, we have

$$\|u\|_{(p+\frac{1}{2})\beta} \le (Cp^2)^{\frac{2}{2p+1}}\|u\|_{(p+\frac{1}{2})\theta}.$$

Applying Moser's iteration, one obtains

$$||u||_\infty \le C||u||_\theta.$$

On the other hand

$$||u||_\infty^\theta \le C||u||_\theta^\theta = \int_M |u|^\theta \le C||u||_\infty^{\theta-1}||u||_1.$$

which implies that

$$||u||_\infty \le C||u||_1 \le C\int_M (|\nabla_\varphi F|_\varphi^2 + (n+\Delta\varphi) + 1).$$

Since

$$\int_M (n+\Delta\varphi)\ \omega^n = n$$

and

$$\int_M |\nabla_\varphi F|_\varphi^2 \omega_\varphi^n = -\int_M F\Delta_\varphi F \le \int_M |F|(|R| + C(n+\Delta\varphi)^{n-1}) \le C,$$

we have

$$||u||_\infty \le C. \qquad \square$$

Remark 1. Choosing $\varepsilon = \frac{1}{2n+1}$, we obtain $q_n = 4n^2 - 4$. On the other hand, Theorem 3.1 implies that a bound on $||R||_{\frac{(n-1)^2(4n^2-4)}{n}}$ gives a bound on $||n+\Delta\varphi||_{4n^2-4}$. Therefore, we can show that C in the statement of Theorem 4.1 depends on $n, \omega, ||\varphi||_\infty, ||F||_\infty, ||R||_{p_n}$, where $p_n = \frac{4(n-1)^3(n+1)}{n}$.

One might hope to improve the estimate by lowering p_n. However, we have not been able to improve the bound yet.

Now, the proof of Theorem 1.3 is straightforward.

Proof of Theorem 1.3. Suppose that φ satisfies equation (1). Then, Theorems 1.2, 3.1, and 4.1 imply that there exists p_n such that

$$||n+\Delta\varphi||_\infty \le C = C(n, \omega, \eta, ||R||_{p_n}).$$

This implies that eigenvalues of $\omega_\varphi = \omega + \sqrt{-1}\ \partial\bar\partial\varphi$ are bounded from above by C. On the other hand, by Theorem 1.2, $||F||_\infty \le C$. Therefore, eigenvalues of $\omega_\varphi = \omega + \sqrt{-1}\ \partial\bar\partial\varphi$ are bounded below by a positive constant that only depends on $n, \omega, \eta, ||R||_{p_n}$. Hence, the equation

$$\Delta_{\omega_\varphi} F = -R + \mathrm{tr}_{\omega_\varphi}\eta$$

is uniformly elliptic. Therefore, DeGiorgi-Nash-Moser theorem implies that there exists $\alpha \in (0,1)$ such that $||F||_{C^\alpha} \le C$. This together with the C^2 bound on φ, we obtain that φ is bounded in $C^{2,\alpha}$ [11].

Hence, the Carlderon-Zygmond estimate implies that F is bounded in W^{2,p_n}. Now differentiating the Monge-Ampere equation implies that φ is bounded in W^{4,p_n}. $\square$

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