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Research Article

Takumi Yamada*

On line bundles arising from the LCK structure over locally conformal Kähler solvmanifolds

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Abstract: We can construct a real line bundle arising from the locally conformal Kähler (LCK) structure over an LCK manifold. We study the properties of this line bundle over an LCK solvmanifold whose complex structure is left-invariant. Mainly, we prove that this line bundle L over an LCK solvmanifold $\Gamma\backslash G$ with left-invariant complex structure is flat and G has a global closed 2-form, which induces an Hermitian structure on the holomorphic tangent bundle twisted by the line bundle $L^{\mathbb{C}} = L \otimes \mathbb{C}$ if the Lee form is cohomologous to a left-invariant 1-form on G.

Keywords: locally conformal Kähler manifold, solvmanifold

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1 Introduction

An Hermitian manifold (M, J, g) is called a locally conformal Kähler (LCK) manifold if g can be made Kähler locally by a conformal change. Some well-known complex manifolds admit no Kählerian metric, but admit an LCK metric. The main non-Kähler examples of LCK manifolds are complex Hopf manifolds, Inoue surfaces, and the Kodaira-Thurston manifold [3]. Inoue surfaces and the Kodaira-Thurston manifold have a structure of solvmanifold $\Gamma\backslash G$ with a complex structure, which is induced from a left-invariant complex structure on a solvable Lie group G, where Γ is a lattice in G [4].

In this study, we consider the line bundle arising from an LCK structure over a solvmanifold $\Gamma\backslash G$, where a solvmanifold means a quotient space of a simply connected solvable Lie group G by a lattice Γ in G. A complex structure J on $\Gamma\backslash G$ is called *left-invariant* if it is induced by a left-invariant complex structure on G. We will prove the following theorem (for the definition of Lee form, see Section 1).

Theorem 1.1. *Let $(\Gamma\backslash G, g, J)$ be an LCK solvmanifold such that J is left-invariant and α the Lee form. Assume that there exists a left-invariant closed 1-form α_0 such that $[\alpha] = [\alpha_0]$ in $H^1_{dR}(\Gamma\backslash G)$. Let f be a function on G that satisfies $\alpha_0 = df$ and $f(e) = 0$. Then, the real line bundle L arising from the LCK structure over $\Gamma\backslash G$ satisfies the following properties*:

(1) *L is a flat line bundle, which is trivial as a C^∞-line bundle.*

(2) *The LCK structure induces a global closed 2-form $\tilde{\Omega}$ on G, which satisfies the following conditions*:

   (*a*) *$L_g^*\tilde{\Omega} = e^{-f(g)}\tilde{\Omega}$, where L_g is the left translation by $g \in G$.*

   (*b*) *$\tilde{\Omega}$ induces an Hermitian structure on the complex vector bundle $T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}}$, where $T^+(\Gamma\backslash G)$ is the holomorphic tangent bundle of $\Gamma\backslash G$.*

*** Corresponding author: Takumi Yamada,** Department of Mathematics, Shimane University, Nishikawatsu-cho 1060, Matsue, 690-8504, Japan, e-mail: t_yamada@riko.shimane-u.ac.jp

(3) *The global 1-form* $\omega = -\partial f$ *is the connection form of the canonical connection on* $L^{\mathbb{C}}$ *with Hermitian structure* $h = e^{-f}$ *with respect to a frame* $s = e^{-\frac{1}{2}f}$. *In particular,* ω *satisfies* $\omega + \bar{\omega} = -2\alpha$.

We also prove a converse of this construction of a flat line bundle from an LCK solvmanifold in Section 3.

2 Preliminaries

Let E be a C^∞ complex vector bundle over a complex manifold M. An *Hermitian structure* on E is a C^∞ field of Hermitian inner products on the fibers of E.

Let (M, J, g) be a compact Hermitian manifold. Let Ω be the fundamental 2-form on (M, J, g), i.e., the 2-form defined by $\Omega(X, Y) = g(JX, Y)$. Then, (M, J, g) is an LCK manifold if there exists an open cover $\{U_i\}$ and a family $\{f_i\}$ of C^∞ functions $f_i : U_i \longrightarrow \mathbb{R}$ so that each metric

$$g_i = \exp(-f_i)g\,|_{U_i}$$

is Kählerian [3]. Then, we have the fundamental closed 2-form Ω_i on U_i. From $\{(U_i, f_i)\}$, we can define a mapping $\rho_{ij} : U_i \cap U_j \longrightarrow \mathbb{R}^+$ by $\rho_{ij} = e^{\frac{1}{2}(f_i - f_j)}$. Thus, we can consider the line bundle L whose family of transition functions is $\{\rho_{ij}\}$. Let $L^{\mathbb{C}}$ be the complexification of L. Because $e^{-\frac{1}{2}f_i}\rho_{ij} = e^{-\frac{1}{2}f_j}$, which means that $s = e^{-\frac{1}{2}f_i}$ on U_i is a global C^∞-section, this line bundle is trivial as a C^∞-line bundle. It is well-known that an Hermitian manifold (M, J, g) is LCK if and only if there exists a globally defined closed 1-form α such that $d\Omega = \alpha \wedge \Omega$, the closed 1-form α is called the Lee form. Indeed, if (M, J, g) is an LCK manifold, then $\alpha = df_i$ on U_i is a global 1-form.

Because $e^{-f_i}\,|\rho_{ij}|^2 = e^{-f_j}$, we can define an Hermitian structure h on $L^{\mathbb{C}}$ by

$$h_x(\zeta_i, \zeta_i) = e^{-f_i(x)}\,|\zeta_i|^2,$$

where $x \in U_i \subset M$, $\zeta_i \in \mathbb{C} \cong T_x L^{\mathbb{C}}$. Assume that $L^{\mathbb{C}}$ is a holomorphic line bundle. Then, we can consider the canonical connection on $(L^{\mathbb{C}}, h)$. Thus, the connection form ω of the canonical connection on $(L^{\mathbb{C}}, h)$ with respect to this frame $e^{-\frac{1}{2}f_i}$ is

$$\omega = -h^{-1}\partial h = -\partial f_i, \quad \text{on} \quad U_i.$$

The curvature form R is $R = \bar{\partial}\omega = -\bar{\partial}\partial f_i$. Note that $\{g_i \otimes h\}_i$ is an Hermitian structure on $T^+M \otimes L^{\mathbb{C}}$. In the next section, we will show that, roughly speaking, if an LCK solvmanifold $(\Gamma\backslash G, g, J)$ satisfies some conditions, then $\{g_i \otimes h\}_i$ induces a closed 2-form on G.

Next, let M be a real manifold. Given a representation

$$\rho : \pi_1(M) \longrightarrow GL(r, \mathbb{R}),$$

we can construct a vector bundle E by setting

$$E = \tilde{M} \times_\rho \mathbb{R}^r,$$

where $\tilde{M}$ is the universal covering of M and $\tilde{M} \times_\rho \mathbb{R}^r$ denotes the quotient of $\tilde{M} \times \mathbb{R}^r$ induced by the action of $\pi_1(M)$ given by

$$\gamma : (x, v) \in \tilde{M} \times \mathbb{R}^r \longmapsto (\gamma(x), \rho(\gamma)v) \in \tilde{M} \times \mathbb{R}^r, \quad \gamma \in \pi_1(M)$$

(We are considering $\pi_1(M)$ as the covering transformation group acting on $\tilde{M}$). The vector bundle defined by above is said to be *flat* ([6, page 4]).

3 Construction of the flat line bundle

In this section, we prove that the complex line bundle arising from the LCK structure over an LCK solvmanifold is flat.

Let $(\Gamma\backslash G, g, J)$ be an LCK solvmanifold such that J is left-invariant, and α the Lee form. We now assume that there exists a left-invariant closed 1-form α_0 such that $[\alpha] = [\alpha_0]$ in $H^1_{dR}(\Gamma\backslash G)$. We use the same notation α_0 for the 1-form on $\Gamma\backslash G$ induced by this left G-invariant 1-form α_0 on G. This assumption holds for completely solvable solvmanifolds according to Hattori's theorem [5].

From the assumption $[\alpha] = [\alpha_0]$ in $H^1_{dR}(\Gamma\backslash G)$, we have

$$\alpha - \alpha_0 = d\phi,$$

where $\phi \in C^\infty(\Gamma\backslash G)$. Let π be the natural map from G to $\Gamma\backslash G$. Then,

$$\pi^*\alpha - \alpha_0 = d\pi^*\phi.$$

It is known that a simply connected solvable Lie group is homeomorphic to some Euclidean space ([2, page 668]). Therefore, for $\pi^*\alpha$, α_0, there exist $F, f \in C^\infty(G)$ such that

$$\pi^*\alpha = dF, \alpha_0 = df.$$

We may assume that $f(e) = 0$ without the loss of generality. Therefore, we have a relation

$$F - f = \pi^*\phi + c,$$

where c is the constant. Because α_0 is left-invariant, we have

$$L_g^*\alpha_0 - \alpha_0 = d(L_g^*f - f) = 0.$$

Thus, we have

$$L_g^*f - f = \varphi(g),$$

which means that $f(gh) - f(h) = \varphi(g)$ for $g, h \in G$. Let $h = e$. Then, we have $\varphi(g) = f(g)$. Hence, we have

$$f(gh) = f(g) + f(h),$$

for $g, h \in G$. Put

$$\rho(g) = \exp\left(\frac{1}{2}f(g)\right).$$

Then, $\rho : G \longrightarrow \mathbb{R}^+$ is a homomorphism. Thus, we define an action of Γ on $G \times \mathbb{R}$ by

$$\gamma \cdot (g, v) = (\gamma \cdot g, \rho(\gamma)v).$$

Hence, we have a real flat line bundle

$$L = \Gamma\backslash(G \times \mathbb{R}) \longrightarrow \Gamma\backslash G.$$

Note that $e^{-F}\pi^*g$ is a Kähler metric on G. Indeed,

$$d(e^{-F}\pi^*\Omega) = -e^{-F}dF \wedge \pi^*\Omega + e^{-F}\pi^*\alpha \wedge \pi^*\Omega = 0.$$

Because $\pi : G \longrightarrow \Gamma\backslash G$ is the universal covering, we can consider that $e^{-F}\pi^*g$ is a Kähler metric on $\pi(U) \cong U$, where U is an open set of G, which is mapped homeomorphically onto $\pi(U)$. Therefore, the line bundle arising from this LCK structure is $L_{CK} = \sim\backslash G \times \mathbb{R}$, where

$$(g, v) \sim (h, w) \Leftrightarrow h = \gamma \cdot g, w = \exp\frac{1}{2}(F(\gamma \cdot g) - F(g))v, \quad \gamma \in \Gamma.$$

Indeed, assume that U and V are the open sets of G that are homeomorphic to $\pi(U)$ and $\pi(V)$, respectively. Assume that $\pi(g) \in \pi(U) \cap \pi(V)$ and $g \in U$. Then, there exists a unique $\gamma \in \Gamma$ such that $\gamma \cdot g \in V$. Moreover, $e^{-F}g$ and $e^{-L_\gamma^*F}g$ induce Kähler metrics on $\pi(U)$ and $\pi(V) \subset \Gamma\backslash G$, respectively. Because

$$\begin{aligned}\exp\frac{1}{2}(F(\gamma\cdot g)-F(g)) &= \exp\frac{1}{2}((f(\gamma\cdot g)+\pi^*\phi(\gamma\cdot g)+c)-(f(g)+\pi^*\phi(g)+c))\\ &= \exp\frac{1}{2}(f(\gamma\cdot g)-f(g)) = \exp\left(\frac{1}{2}f(\gamma)\right),\end{aligned}$$

this line bundle L coincides with L_{CK}. Note that L is trivial as a C^∞-line bundle, because $\rho|_\Gamma$ can be extended to $\rho : G \longrightarrow \mathbb{R}^+$. Because L is a real flat line bundle, $L^\mathbb{C}$ is a holomorphic flat line bundle.

4 Properties of the flat line bundle

In this section, we consider the properties of the line bundle arising from the LCK structure over an LCK solvmanifold.

Let $(\Gamma\backslash G, g, J)$ be an LCK solvmanifold such that J is left-invariant, and α the Lee form such that there exists a left-invariant closed 1-form α_0 such that $[\alpha] = [\alpha_0]$ in $H^1_{dR}(\Gamma\backslash G)$. Because α and α_0 are cohomologous, there exists a function ϕ on $\Gamma\backslash G$ such that $\alpha - \alpha_0 = d\phi$. Hence, we define an inner product $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ of G by

$$\langle X, Y\rangle = \int (e^{-\phi}g)(X, Y)d\mu,$$

for $X, Y \in \mathfrak{g}$, where $d\mu$ is the volume element induced by a bi-invariant volume element on G. Let Ω be the fundamental 2-form of $(\Gamma\backslash G, J, \langle , \rangle)$. Belgun [1, Proof of Theorem 7] proved that $(\Gamma\backslash G, J, \langle\ ,\ \rangle)$ is also an LCK solvmanifold. Therefore, from now on, we assume that the LCK structure Ω is left-invariant.

Let $\rho(g) = \exp\left(\frac{1}{2}f(g)\right)$, where $\alpha_0 = \pi^*\alpha = df$ as in Section 2. Put

$$\tilde{\Omega}_g = \rho(g)^{-2}\Omega_g = e^{-f(g)}\Omega_g.$$

It is obvious that this 2-form $\tilde{\Omega}$ is a closed 2-form on G, which induces an Hermitian structure on the vector bundle $T^+(\Gamma\backslash G) \otimes L^\mathbb{C}$ by the definition of LCK structure. Indeed, let $\mathfrak{g}^+ \subset \mathfrak{g}^\mathbb{C}$ be the $\sqrt{-1}$ eigenspace of the complex structure. Let us consider the following C^∞-section $s_{Z^+} \in \Gamma_\infty(G, T^+(G) \otimes L^\mathbb{C})$ over G defined by

$$s_{Z^+} : g \longmapsto \rho(g)\cdot(Z^+)_g,$$

for $Z^+ \in \mathfrak{g}^+$. Note that Z^+_g and $Z^+_{\gamma\cdot g}$ are identified on $T^+_{\pi(g)}(\Gamma\backslash G)$. Because

$$s_{Z^+} : \gamma\cdot g \longmapsto \rho(\gamma\cdot g)\cdot(Z^+)_{\gamma\cdot g} = \rho(\gamma)\cdot\rho(g)\cdot(Z^+)_{\gamma\cdot g},$$

each s_{Z^+} induces a C^∞-section of $T^+(\Gamma\backslash G) \otimes L^\mathbb{C}$ over $\Gamma\backslash G$. Then,

$$\begin{aligned}\tilde{\Omega}_{\gamma\cdot g}(\rho(\gamma\cdot g)Z^+_{\gamma\cdot g}, J\rho(\gamma\cdot g)\overline{W}^+_{\gamma\cdot g}) &= \tilde{\Omega}_{\gamma\cdot g}\left(e^{\frac{1}{2}f(\gamma\cdot g)}Z^+_{\gamma\cdot g}, Je^{\frac{1}{2}f(\gamma\cdot g)}\overline{W}^+_{\gamma\cdot g}\right)\\ &= \Omega_g(Z^+_g, J\overline{W}^+_g)\\ &= \tilde{\Omega}_g(e^{\frac{1}{2}f(g)}Z^+_g, Je^{\frac{1}{2}f(g)}\overline{W}^+_g)\\ &= \tilde{\Omega}_g(\rho(g)Z^+_g, J\rho(g)\overline{W}^+_g),\end{aligned}$$

for $Z^+, W^+ \in \mathfrak{g}^+$. Thus, $\tilde{\Omega}$ induces an Hermitian structure on the complex vector bundle $T^+(\Gamma\backslash G) \otimes L^\mathbb{C}$. Meanwhile,

$$d\tilde{\Omega} = d(e^{-f}\Omega) = de^{-f} \wedge \Omega + e^{-f}d\Omega = 0,$$

on G. This proves Theorem 1.1.

From the aforementioned argument, we consider the following concept.

Definition 4.1. Let (G, J) be a simply connected solvable Lie group such that J is left-invariant and Γ a lattice in G. Let $\rho : \Gamma \longrightarrow \mathbb{R}^+$ be a homomorphism. Let $L = \Gamma\backslash(G \times \mathbb{R}) \longrightarrow \Gamma\backslash G$ be the line bundle defined by the action $\gamma \cdot (g, v) = (\gamma \cdot g, \rho(\gamma)v)$. We say that $T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}}$ has a closed twisted Hermitian form, which is induced by a closed $(1,1)$-form $\tilde{\Omega}$ on G if a closed $(1,1)$-form $\tilde{\Omega}$ on G induces an Hermitian structure h on $T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}}$ by defining

$$h_{\pi(g)}(v, w) = \tilde{\Omega}_g(Z_g^+, J\overline{W}_g^+) \quad \text{for } v, w \in (T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}})_{\pi(g)},$$

where $Z^+, W^+ \in \mathfrak{g}^+$ satisfy $(\pi_*)_g Z^+ = v$, $(\pi_*)_g W^+ = w$.

Conversely, we have the following proposition.

Proposition 4.2. *Let (G, J) be a simply connected solvable Lie group such that J is left-invariant and Γ a lattice in G. Let $\rho : \Gamma \longrightarrow \mathbb{R}^+$ be a homomorphism. Let $L = \Gamma\backslash(G \times \mathbb{R}) \longrightarrow \Gamma\backslash G$ be the line bundle defined by the action $\gamma \cdot (g, v) = (\gamma \cdot g, \rho(\gamma)v)$. If*
(1) *ρ can be extended to a homomorphism $\tilde{\rho} : G \longrightarrow \mathbb{R}^+$*
(2) *$T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}}$ has a closed twisted Hermitian form, which is induced by a closed $(1,1)$-form $\tilde{\Omega}$ on G,*

then $(\Gamma\backslash G, J)$ has an LCK structure.

Proof. Let $\Omega = \tilde{\rho}^2\tilde{\Omega}$. Using the equations $\tilde{\rho}(\gamma \cdot g) = \rho(\gamma) \cdot \tilde{\rho}(g)$ and

$$\tilde{\Omega}_{\gamma \cdot g}(\tilde{\rho}(\gamma \cdot g)Z^+_{\gamma \cdot g}, J\tilde{\rho}(\gamma \cdot g)\overline{W}^+_{\gamma \cdot g}) = \tilde{\Omega}_g(\tilde{\rho}(g)Z_g^+, J\tilde{\rho}(g)\overline{W}_g^+),$$

for $Z^+, W^+ \in \mathfrak{g}^+$, we have

$$\Omega_{\gamma \cdot g}(Z^+_{\gamma \cdot g}, J\overline{W}^+_{\gamma \cdot g}) = \Omega_g(Z_g^+, J\overline{W}_g^+).$$

Because $d\tilde{\Omega} = 0$, $d\Omega = 2\tilde{\rho}^{-1}d\tilde{\rho} \wedge \Omega$. □

Example 4.3. (Kodaira-Thurston manifold) Let us consider the following nilpotent Lie group and its lattice:

$$G = \left\{ \begin{pmatrix} 1 & \bar{z} & w \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} ; z, w \in \mathbb{C} \right\}, \Gamma = \left\{ \begin{pmatrix} 1 & \bar{\mu} & v \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} ; \mu, v \in \mathbb{Z}[\sqrt{-1}] \right\}.$$

The manifold $\Gamma\backslash G$ is called the Kodaira-Thurston manifold. We consider the global complex coordinate system on G defined by

$$\begin{pmatrix} 1 & \bar{z} & w \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (z, w).$$

Thus, $G \cong \mathbb{R}^4$ as a C^∞ manifold. Then,

$$Z = \frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial w}, \quad \text{and} \quad W = \frac{\partial}{\partial w}$$

are left G-invariant vector fields on G. Put

$$\omega_z = dz, \ \ \omega_w = dw - \bar{z}dz.$$

Because ω_z and ω_w are left G-invariant $(1,0)$-forms,

$$\Omega = \sqrt{-1}(\omega_z \wedge \bar{\omega}_z + \omega_w \wedge \bar{\omega}_w)$$

induces an Hermitian metric on $\Gamma\backslash G$. Consider the function defined by

$$f(z, w) = w + \bar{w} - |z|^2.$$

Then,

$$f((a, b)\cdot(z, w)) = w + \bar{w} - |z|^2 + (b + \bar{b} - |a|^2).$$

Thus, we have $L_g^* df = df$. We also have

$$d\Omega = (\bar{\omega}_w + \omega_w) \wedge \Omega = df \wedge \Omega.$$

Hence, Ω is an LCK structure on $\Gamma\backslash G$. Put

$$\rho(z, w) = e^{\frac{1}{2}f(z,w)}.$$

Then, we can easily see that $\rho : G \longrightarrow \mathbb{R}^+$ is a homomorphism, because $f(0, 0) = 0$. Thus, we have a real flat line bundle

$$L = \Gamma\backslash(G \times \mathbb{R}) \longrightarrow \Gamma\backslash G.$$

We also see that a closed 2-form

$$\tilde{\Omega} = e^{-f(z,w)}\Omega = \sqrt{-1}\exp(|z|^2 - w - \bar{w})(\omega_z \wedge \bar{\omega}_z + \omega_w \wedge \bar{\omega}_w)$$

on G induces an Hermitian structure on $T^+(\Gamma\backslash G) \otimes L^{\mathbb{C}}$. The connection form of the canonical connection on $L^{\mathbb{C}}$ is $-\omega_w$.

Remark 4.4. The line bundle arising from the LCK structure over a Hopf surface has the same properties as LCK solvmanifolds, even though they are not homeomorphic to solvmanifolds.

Let $\Gamma = \{2^k | k \in \mathbb{Z}\}$. Consider the action of Γ on $D = \mathbb{C}^2 - \{0\}$ defined by $2^k : (z_1, z_2) \mapsto (2^k z_1, 2^k z_2)$. The quotient space $\mathbb{C}^2 - \{0\}/\Gamma$ is called a Hopf surface. A Hopf surface has an LCK structure induced by

$$\frac{\sqrt{-1}}{|z_1|^2 + |z_2|^2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

The Lee form α is

$$\alpha = \frac{-1}{|z_1|^2 + |z_2|^2}(z_1 d\bar{z}_1 + \bar{z}_1 dz_1 + z_2 d\bar{z}_2 + \bar{z}_2 dz_2) = -d\log(|z_1|^2 + |z_2|^2).$$

Thus, the line bundle $L^{\mathbb{C}}$ arising from this LCK structure is a flat line bundle such that the connection form is $\omega = \partial\log(|z_1|^2 + |z_2|^2)$. Indeed, put

$$f(z_1, z_2) = \log(|z_1|^2 + |z_2|^2).$$

Because $f(2^k(z_1, z_2)) - f(z_1, z_2) = -2\log 2^k$,

$$\exp\left(\frac{1}{2}f(2^k(z_1, z_2)) - f(z_1, z_2)\right) = 2^{-k}.$$

Therefore, we consider the action $\rho(2^k) = 2^{-k}$ of Γ on $\mathbb{C}$. Then,

$$L^{\mathbb{C}} = \Gamma\backslash D \times \mathbb{C},$$

where Γ acts $D \times \mathbb{C}$ by $2^k \cdot (z, v) = (2^k \cdot z, \rho(2^k)v)$.

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References

[1] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.

[2] C. Chevalley, *On the topological structure of solvable groups*, Ann. of Math. **42** (1941), 668–675.

[3] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progr. Math., vol. 155, Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] K. Hasegawa, *Complex and Kähler structures on compact solvmanifolds*, J. Symplectic Geom. **3** (2005), 749–767.

[5] A. Hattori, *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 289–331.

[6] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan vol. 15, Princeton University Press, New Jersey, 1987.