Research Article

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On maximal totally real embeddings

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Abstract: We consider complex structures with totally real zero section of the tangent bundle. We assume that the complex structure tensor is real-analytic along the fibers of the tangent bundle. This assumption is quite natural in view of a well-known result by Bruhat and Whitney. We provide explicit integrability equations for such complex structures in terms of the fiberwise Taylor expansion. In a particular geometric case considered in the literature, we explicit much further the fiberwise Taylor expansion of the complex structure as well as the integrability equations.

Keywords: totally real embeddings, integrability equations, linear and non-linear connections over vector bundles

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1 Introduction and statement of the main result

Let (E, π_E, M) be a smooth vector bundle over a manifold M. Let E_p be the fiber of E over a point $p \in M$, and let $\eta \in E_p$. We consider the transition map $\tau_\eta(v) \coloneqq \eta + v$ acting over E_p, and we consider its differential

$$d_0\tau_\eta : T_{E_p,0} \longrightarrow T_{E_p,\eta},$$

at the point 0. Composing $d_0\tau_\eta$ with the canonical isomorphism $E_p \simeq T_{E_p,0}$, we obtain an isomorphism map

$$T_\eta : E_p \longrightarrow T_{E_p,\eta}. \tag{1.1}$$

We denote by 0_M the zero section of E. Differentiating the identity $\mathrm{id}_M = \pi_E \circ 0_M$, we obtain $\mathbb{I}_{T_{M,p}} = d_{0_p}\pi_E \circ d_p 0_M$. This implies the decomposition

$$T_{E,0_p} = d_p 0_M(T_{M,p}) \oplus \operatorname{Ker} d_{0_p}\pi_E.$$

We notice also the obvious equalities $\operatorname{Ker} d_\eta \pi_E = d_0\tau_\eta(T_{E_p,0}) = T_\eta(E_p) \simeq E_p$, for any $\eta \in E_p$. Now applying this to $\eta = 0_p$, using the previous decomposition and the canonical isomorphism $d_p 0_M(T_{M,p}) \simeq T_{M,p}$, we infer the existence of the canonical isomorphism $T_{E,0_p} \simeq T_{M,p} \oplus E_p$, which we rewrite as follows:

$$T_{E|M} \simeq T_M \oplus E. \tag{1.2}$$

Definition 1. A real sub-manifold M of an almost complex manifold (X, J) is called totally real if $T_{M,p} \cap J(T_{M,p}) = 0_p$ for all $p \in M$. A totally real sub-manifold M of an almost complex manifold (X, J) is called maximally totally real if $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$.

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1.1 M-totally real almost complex structures over T_M

We consider M included inside T_M via the zero section. We know by the isomorphism (1.2) with $E = T_M$, which this embedding induces the canonical isomorphism $T_{T_M|M} \simeq T_M \oplus T_M$. The vector bundle $T_{T_M|M}$ is a complex one with the canonical complex structure $J^{\text{can}} : (u, v) \longmapsto (-v, u)$ acting on the fibers.

Any almost complex structure that is a continuous extension of J^{can} in a neighborhood of M inside T_M makes M a maximally totally real sub-manifold of T_M.

Over an arbitrary small neighborhood of M inside T_M, the complex distribution $T^{0,1}_{T_M}$ is horizontal with respect to the natural projection $\pi : T_M \longrightarrow M$.

We remind that the data of a smooth complex horizontal distribution over T_M coincide with the one of section

$$A \in C^\infty(T_M, \pi^*\mathbb{C}T^*_M \otimes_{\mathbb{C}} \mathbb{C}T_{T_M}),$$

such that $d\pi \cdot A = \mathbb{I}_{\pi^*\mathbb{C}T_M}$.

For any complex vector field $\xi \in C^\infty(M, \mathbb{C}T_M)$, we will denote by abuse of notation $A(\xi) \equiv A \cdot (\xi \circ \pi)$. Section A evaluated at the point $\eta \in T_M$ will be denoted by A_η.

We notice that we can write $A = \alpha + i\beta$, with

$$\alpha, \beta \in C^\infty(T_M, \pi^*T^*_M \otimes_{\mathbb{R}} T_{T_M}),$$

such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^*T_M}$ and $\beta_\eta = T_\eta B_\eta$, with $B \in C^\infty(T_M, \pi^* \operatorname{End}(T_M))$. Section A determines an almost complex structure J_A over T_M such that

$$T^{0,1}_{T_M, J_A, \eta} = A_\eta(\mathbb{C}T_{M,\pi(\eta)}) \subset \mathbb{C}T_{T_M,\eta},$$

if and only if

$$A_\eta(\mathbb{C}T_{M,\pi(\eta)}) \cap \overline{A_\eta(\mathbb{C}T_{M,\pi(\eta)})} = 0. \tag{1.3}$$

This condition is equivalent to the property:

$$\overline{A_\eta(\bar{\xi}_1)} = A_\eta(\xi_2), \tag{1.4}$$

implies $\xi_1 = \xi_2 = 0$. Taking $d_\eta\pi$ in the equality (1.4), we infer $\xi_1 = \xi_2$. Thus, equality (1.4) is equivalent to $(\overline{A} - A)(\xi_1) = 0$, and the previous property is equivalent to $\operatorname{Ker}(\overline{A} - A) = 0$, i.e.,

$$B \in C^\infty(T_M, \pi^* \operatorname{GL}(T_M)).$$

We notice that with respect to the canonical complex structure of $T_{T_M|M}$ we have the equality $(u, v)^{0,1} = (\xi, i\xi)$, with $\xi := (u - iv)/2$. Then J_A is an extension of this complex structure over an open neighborhood $U \subseteq T_M$ of M if and only if for any point $p \in M$, we have $\alpha_{0_p} = d_p 0_M$ and $B_{0_p} = \mathbb{I}_{T_{M,p}}$. We denote by

$$T \in C^\infty(T_M, \pi^*T^*_M \otimes_{\mathbb{R}} T_{T_M}),$$

the canonical section which at the point $\eta \in T_M$ takes the value T_η.

Definition 2. Let M be a smooth manifold. An M-totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the image of the zero section 0_M is a couple (α, B) with

$$\alpha \in C^\infty(U, \pi^*T^*_M \otimes_{\mathbb{R}} T_{T_M})$$

and

$$B \in C^\infty(U, \pi^* \operatorname{GL}(T_M)),$$

such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^*T_M}$ over U and such that $\alpha_{0_p} = d_p 0_M$, $B_{0_p} = \mathbb{I}_{T_{M,p}}$, for all $p \in M$. With $A := \alpha + iTB$, the almost complex structure J_A associated to (α, B) is the one which satisfies

$$T^{0,1}_{T_M, J_A, \eta} = A_\eta(\mathbb{C}T_{M,\pi(\eta)}) \subset \mathbb{C}T_{T_M,\eta},$$

for all $\eta \in U \subseteq T_M$.

Every almost complex smooth extension of the canonical complex structure $J^{\,\mathrm{can}}$ of $T_{T_M|M}$ over a neighborhood of M inside T_M can be expressed, over a sufficiently small neighborhood $U \subseteq T_M$ of M, as the almost complex structure associated with a unique M-totally real almost complex structure over U.

We provide below an explicit formula for the almost complex structure J_A. For this purpose, we notice first that for any vector $\xi \in T_{T_M,\eta}$,

$$\xi^{0,1}_{J_A} = \frac{1}{2} A_\eta [d_\eta \pi - i B_\eta^{-1} T_\eta^{-1} (\mathbb{I}_{T_{T_M}} - \alpha_\eta d_\eta \pi)]\xi,$$

$$\xi^{1,0}_{J_A} = \frac{1}{2} \overline{A}_\eta [d_\eta \pi + i B_\eta^{-1} T_\eta^{-1} (\mathbb{I}_{T_{T_M}} - \alpha_\eta d_\eta \pi)]\xi.$$

Indeed $\xi^{0,1}_{J_A} \in T^{0,1}_{T_M,J_A,\eta}$, $\xi^{1,0}_{J_A} \in T^{1,0}_{T_M,J_A,\eta}$, and $\xi = \xi^{1,0}_{J_A} + \xi^{0,1}_{J_A}$. We deduce the expression

$$J_{A,\eta} = -\alpha_\eta B_\eta^{-1} T_\eta^{-1} (\mathbb{I}_{T_{T_M}} - \alpha_\eta d_\eta \pi) + T_\eta B_\eta d_\eta \pi. \tag{1.5}$$

This shows that for any α-horizontal vector $\xi \in T_{T_M,\eta}$, i.e., $\xi = \alpha_\eta d_\eta \pi \xi$, we have

$$J_{A,\eta} \xi = T_\eta B_\eta d_\eta \pi \xi.$$

In equivalent terms,

$$J_{A,\eta} \alpha_\eta v = T_\eta B_\eta v, \tag{1.6}$$

for any $\eta \in U \subset T_M$ and any $v \in T_{M,\pi(\eta)}$. Moreover, (1.5) implies

$$J_{A,\eta|\,\mathrm{Ker}\, d_\eta \pi} = -\alpha_\eta B_\eta^{-1} T_\eta^{-1}. \tag{1.7}$$

A well-known theorem by Bruhat and Whitney [6] states that for any real-analytic manifold M there exist a complex manifold (X, J) and a real-analytic embedding of M in X such that as a sub-manifold of X, M is maximally totally real. In addition, one can arrange that X is an open neighborhood $U \subseteq T_M$ of the zero section and $J_{|M} = J^{\,\mathrm{can}}$.

Moreover, Bruhat and Whitney [6] show that if X is a real-analytic manifold equipped with two different real-analytic complex structures J_1 and J_2, which contains a real analytic sub-manifold M, which is maximally totally real with respect to both J_1 and J_2, then there exist neighborhoods U_1 and U_2 of M inside X and a real-analytic diffeomorphism $\kappa : U_1 \longrightarrow U_2$, which is the identity on M and is a holomorphic mapping of (U_1, J_1) onto (U_2, J_2).

In other words, the structure J constructed by Bruhat and Whitney [6] is unique up to complex isomorphisms.

We state below our results on the integrability conditions for J.

1.2 The integrability equations for M-totally real almost complex structures

Let (E, π_E, M) be a vector bundle over a manifold M. For an arbitrary section $B \in C^\infty(E, \pi_E^*(T_M^* \otimes E))$, we define the derivative along the fiber

$$DB \in C^\infty(E, \pi_E^*(E^* \otimes T_M^* \otimes E)),$$

by the formula

$$D_\eta B(v) \;\coloneqq\; \frac{d}{dt}_{|_{t=0}} B_{\eta + tv} \in T^*_{M,p} \otimes E_p,$$

for any $\eta, v \in E_p$. We denote by Alt_2 the alternating operator (without normalizing coefficient!), which acts on the first two entries of a tensor. For any morphism $A : T_M \longrightarrow E$ and any bilinear form $\beta : E \times T_M \longrightarrow E$, we define the contraction operation:

$$A \neg \beta \ := \ \mathrm{Alt}_2(\beta \circ A),$$

where the composition operator $\circ$ act on the first entry of β. For a given covariant derivative operator ∇ acting on the smooth sections of T_M, we denote by H^∇ the linear projection to the associated horizontal distribution. (See Lemmas 14 and 16 and Definition 5 in subsection 9.1 of the Appendix for precise definitions and properties of H^∇.)

Theorem 1. *Let M be a smooth manifold and let J_A with $A = \alpha + iTB$ be an M-totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the image of the zero section. Let also ∇ be a covariant derivative operator acting on the smooth sections of T_M. Then J_A is integrable over U if and only if the complex section $S \ := \ T^{-1}(H^\nabla - \overline{A})$ satisfies the equation*

$$H^\nabla_\eta \neg (\nabla^{\mathrm{End}(T_M),\pi} S)_\eta - S_\eta \neg D_\eta S + S_\eta \tau^\nabla + R^\nabla \cdot \eta = 0, \tag{1.8}$$

for any point $\eta \in U$, where $\nabla^{\mathrm{End}(T_M),\pi}$ is the covariant derivative operator acting on the smooth sections of $\pi^* \mathrm{End}(T_M)$ induced by ∇ and where τ^∇ and R^∇ are, respectively, the torsion and curvature forms of ∇.

We notice that $S_{|M} = i\mathbb{I}_{T_M}$ by the conditions $\alpha_{0_p} = H^\nabla_{0_p} = d_p 0_M$ and $B_{0_p} = \mathbb{I}_{T_{M,p}}$.

Notation for the statement of the main theorem.

For any $A \in T_M^{*,\otimes p} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}T_M)$ and for any $\theta \in T_M^{*,\otimes q} \otimes \mathbb{C}T_M$, the product operations of tensors $A \cdot \theta, A \neg \theta \in T_M^{*,\otimes(p+q)} \otimes \mathbb{C}T_M$ are defined by

$$(A \cdot \theta)(u_1, \dots, u_p, v_1, \dots, v_q) := A(u_1, \dots, u_p) \cdot \theta(v_1, \dots, v_q),$$

$$(A \neg \theta)(u_1, \dots, u_p, v_1, \dots, v_q) := \sum_{j=1}^{q} \theta(v_1, \dots, A(u_1, \dots, u_p) \cdot v_j, \dots, v_q).$$

We will denote for notation simplicity $R^\nabla . \, \theta \ := \ R^\nabla \cdot \theta - R^\nabla \neg \theta$. We will denote by Circ the circular operator

$$(\mathrm{Circ}\, \theta)(v_1, v_2, v_3, \bullet) \ := \ \theta(v_1, v_2, v_3, \bullet) + \theta(v_2, v_3, v_1, \bullet) + \theta(v_3, v_1, v_2, \bullet),$$

acting on the first three entries of any q-tensor θ, with $q \geqslant 3$. We define also the permutation operation $\theta_2(v_1, v_2, \bullet) \ := \ \theta(v_2, v_1, \bullet)$.

For any covariant derivative ∇ acting on the smooth sections of $\mathbb{C}T_M$, we define the operator

$$d_1^\nabla : C^\infty(M, T_M^{*,\otimes k} \otimes_{\mathbb{R}} \mathbb{C}T_M) \longrightarrow C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^{*,\otimes(k-1)} \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

with $k \geqslant 1$ as follows:

$$d_1^\nabla A(\xi_1, \xi_2, \mu) \ := \ \nabla_{\xi_1} A(\xi_2, \mu) - \nabla_{\xi_2} A(\xi_1, \mu),$$

with $\xi_1, \xi_2 \in T_M$ and with $\mu \in T_M^{\oplus(k-1)}$. Moreover, for any

$$A \in C^\infty(M, T_M^{*,\otimes(k+1)} \otimes_{\mathbb{R}} \mathbb{C}T_M),$$
$$B \in C^\infty(M, T_M^{*,\otimes(l+1)} \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

we define the exterior product

$$A \wedge_1 B \in C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^{*,\otimes(k+l-1)} \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

as

$$(A \wedge_1 B)(\xi_1, \xi_2, \eta, \mu) \ := \ A(\xi_1, B(\xi_2, \eta), \mu) - A(\xi_2, B(\xi_1, \eta), \mu),$$

with $\xi_1, \xi_2 \in T_M, \eta \in T_M^{\oplus l}$ and $\mu \in T_M^{\oplus(k-1)}$. We denote by $\mathrm{Sym}_{r_1, \dots, r_s}$ the symmetrizing operator (without normalizing coefficient!) acting on the entries $r_1, \dots, r_s$ of a multi-linear form. We use in this article the common convention that a sum and a product running over an empty set is equal to 0 and 1, respectively.

With these notations, we can state our main theorem.

Theorem 2. (Integrability in the fiberwise real analytic case) *Let M be smooth manifold equipped with a torsion free covariant derivative operator ∇ acting on the smooth sections of the tangent bundle T_M, let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers, let J_A be an M-totally real almost complex structure over U, real-analytic along the fibers of U, and consider the fiberwise Taylor expansion at the origin*

$$T_\eta^{-1}(H^\nabla - \overline{A})_\eta \cdot \xi = i\xi + \sum_{k\geqslant 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a neighborhood of the image of the zero section, with $\xi \in T_{M,\pi(\eta)}$ arbitrary, with

$$S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

with $\eta^k := \eta^{\times k} \in T_{M,\pi(\eta)}^{\oplus k}$ and let ∇^{S_1} be the complex covariant derivative operator acting on the smooth sections of $\mathbb{C}T_M$ defined by

$$\nabla_\xi^{S_1}\eta := \nabla_\xi \eta + S_1(\xi, \eta).$$

Then J_A is integrable over U if and only if $S_1 \in C^\infty(M, S^2T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$, (i.e. ∇^{S_1} is torsion free), and for all $k \geqslant 2$,

$$\begin{aligned}
S_k &= \frac{i}{k}\nabla^{S_1}\sigma_{k-1} + \frac{i}{(k+1)!}\operatorname{Sym}_{2,\dots,k+1}\beta_{k-1}(\sigma_{k-2}) + \sigma_k,\\
\sigma_k &\in C^\infty(M, S^{k+1}T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),\\
&\operatorname{Circ}\beta_{k+1}(\sigma_k) = 0,
\end{aligned}$$

where $\sigma_1 := 0$, $\beta_1(\sigma_0) := R^{\nabla^{S_1}}$, $\beta_2(\sigma_1) := -\frac{i}{3}(\nabla^{S_1}R^{\nabla^{S_1}})_2$, and for all $k \geqslant 3$,

$$\begin{aligned}
\beta_k(\sigma_{k-1}) &:= \frac{i}{k}R^{\nabla^{S_1}} . \, \sigma_{k-1} + \frac{1}{(k+1)!k!}\operatorname{Sym}_{3,\dots,k+2}\theta_k(\sigma_{k-1}),\\
\theta_k(\sigma_{k-1}) &:= i\sum_{r=3}^{k-1}\frac{(r+1)!}{r}(id_1^{\nabla^{S_1}})^{k-r}(R^{\nabla^{S_1}} . \, \sigma_{r-1}) - 2i(id_1^{\nabla^{S_1}})^{k-2}(\nabla^{S_1}R^{\nabla^{S_1}})_2\\
&\quad + \sum_{r=4}^{k+1} r! \sum_{p=2}^{r-2}(id_1^{\nabla^{S_1}})^{k+1-r}(pS_p \wedge_1 S_{r-p}).
\end{aligned}$$

In more explicit terms,

$$S_2 = S_2^0 + \sigma_2, \tag{1.9}$$

$$S_2^0(\xi_1, \xi_2, \xi_3) := \frac{i}{6}[R^{\nabla^{S_1}}(\xi_1, \xi_2)\xi_3 + R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_2], \tag{1.10}$$

$$\sigma_2 \in C^\infty(M, S^3T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M), \tag{1.11}$$

$$\operatorname{Circ}\beta_3(\sigma_2) = 0, \tag{1.12}$$

$$\beta_3(\sigma_2) := \frac{i}{3}R^{\nabla^{S_1}} . \, \sigma_2 + \frac{1}{4!3!}\operatorname{Sym}_{3,4,5}\theta_3(\sigma_2), \tag{1.13}$$

$$\theta_3(\sigma_2) := 2d_1^{\nabla^{S_1}}(\nabla^{S_1}R^{\nabla^{S_1}})_2 + 4!2S_2 \wedge_1 S_2. \tag{1.14}$$

The assumption that the complex structure tensor is real-analytic along the fibers of the tangent bundle is quite natural. Indeed if M is real analytic, then the M-totally real complex structure constructed by Bruhat and Whitney [6] is also real analytic with respect to the real analytic structure of the tangent bundle induced by M.

In this article, we request from the readers very good knowledge of the geometric theory of linear connections. Basics of such theory can be found in the Appendix.

1.3 Application of the main integrability result

Over a Riemannian manifold (M, g), we denote by

$$\mathcal{V}^g \ni (\eta, t) \longmapsto \Phi_t^g(\eta) \in T_M,$$

the geodesic flow, where $\mathcal{V}^g \subset T_M \times \mathbb{R}$ is an open neighborhood of $T_M \times \{0\}$. Let ∇^g be the Levi-Civita connection of the metric g. We denote by H^g the liner projection to the associated horizontal distribution. We state the following corollary of the main Theorem 2.

Corollary 1. *Let (M, g) be a smooth Riemannian manifold, let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers, let $J \equiv J_A$ be an M-totally real almost complex structure over U, real analytic along the fibers of U, and consider the fiberwise Taylor expansion at the origin*

$$T_\eta^{-1}(H^g - \overline{A})_\eta \cdot \xi = i\xi + \sum_{k \geqslant 1} S_k(\xi, \eta^k),$$

with $\eta \in T_M$ in a neighborhood of the image of the zero section, with $\xi \in T_{M,\pi(\eta)}$ arbitrary, with

$$S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

and with $\eta^k := \eta^{\times k} \in T_{M,\pi(\eta)}^{\oplus k}$. Then statements (a) and (b) are equivalent.

(*a*) *The almost complex structure J is integrable over U and for any $\eta \in U$ the smooth map $\psi_\eta : t + is \longmapsto s\Phi_t^g(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$ is J-holomorphic.*

(*b*) *The components S_k satisfy $S_1 = 0$,*

$$S_k = \frac{i}{(k+1)!k!} \operatorname{Sym}_{2,\dots,k+1} \Theta_k(g),$$

for all $k \geqslant 2$, with $\Theta_2(g) := 2R^g$ and with

$$\Theta_k(g) := -2i(id_1^{\nabla^g})^{k-3}(\nabla^g R^g)_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2} (id_1^{\nabla^g})^{k-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 3$, and the equations $\operatorname{CircSym}_{3,\dots,k+1}\Theta_k(g) = 0$ are satisfied for all $k \geqslant 4$.

It has been 64 years since the existence of complex structures on Grauert Tubes was proven for the first time by Bruhat–Whitney [6]. Still, up to now, the explicit form of the Taylor expansion has remained mysterious. This is finally clarified in the main Theorem 1.5 in [18], which is based on the statement of Corollary 1. Indeed in the study by Pali and Salvy [18], Theorem 1.5, we obtain a rather simple and explicit global expression for the complex structure on Grauert tubes.

The expression in Theorem 1.5 in the study by Pali and Salvy [18] (see also Theorem 1.6 there for a more general statement) is important for applications to analytic micro local analysis over manifolds. It allows indeed an explicit global construction of the complex extension of a given global Fourier integral operator defined on a real analytic manifold.

The expression in Theorem 1.5 in the study by Pali and Salvy [18] allows also to perform useful explicit global intrinsic operator computations in the sense of Pali [17]. In more explicit terms, given a global intrinsic section over the Grauert tube, an explicit formula for the complex structure such as the one in Theorem 1.5 in [18], allows to determine if the section is holomorphic or not.

The proof of Corollary 1 will be given in the Section 6.2. In the case (M, g) is a compact real analytic Riemannian manifold, the complex structure in the statement of Corollary 1 exists thanks to the works of Guillemin and Stenzel [10], Lempert [13], Lempert and Szöke [14,15], Szöke [20,21], as well as Bielawski [5]. Thus, in this case, the integrability conditions

$$I_k := \operatorname{CircSym}_{3,\dots,k+1}\Theta_k(g) = 0$$

in the statement of Corollary 1 are satisfied for all $k \geqslant 4$. We notice in particular that in the case $k = 4$, the equation $I_4 \coloneqq \mathrm{CircSym}_{3,4,5}\Theta_4(g) = 0$, expands out to

$$\mathrm{CircSym}_{3,4,5}[3d_1^{\nabla^g}(\nabla^g R^g)_2 - 2\tilde{R}^g \wedge_1 \tilde{R}^g] = 0, \tag{1.15}$$

with $\tilde{R}^g \coloneqq \mathrm{Sym}_{2,3}R^g$. We will show in a quite general set-up that the previous equation is an identity. We have indeed the following result, which shows the vanishing of I_4.

Proposition 1. *Let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$ with curvature operator $R^\nabla(\cdot,\cdot)\cdot \equiv R^\nabla(\cdot,\cdot,\cdot)$. Let $\tilde{R}^\nabla \coloneqq \mathrm{Sym}_{2,3}R^\nabla$. Then*

$$\mathrm{CircSym}_{3,4,5}[3d_1^\nabla(\nabla R^\nabla)_2 - 2\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla] = 0. \tag{1.16}$$

The proof will be provided in Section 7. In subsection 5.1 in the study by Pali and Salvy [18], we provide a shorter proof of the vanishing of I_4 in Proposition 1 by using some more advanced combinatorial techniques. In Section 5.2 in the study by Pali and Salvy [18], we show also the vanishing of I_5. Using computer algebra (see Sections 2 and 5 in the study by Pali and Salvy [19]), we can show the vanishing of I_k for $k = 4,\ldots,7$. In Section 5 in the study by Pali and Salvy [19], we use the explicit expression in Theorems 1.5 and 1.6 in the study by Pali and Salvy [18] and we observe that in the case $k = 7$, the computer perform the computation in approximately 1 s, but we expect that the case $k = 8$ would take a computation of approximately 2 weeks. We feel confident at this point to formulate the following conjecture.

Conjecture 1. *Let M be a smooth manifold and let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$. Then the sequence of tensors $S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$, $k \geqslant 2$, defined by the inductive rule*

$$S_k \coloneqq \frac{i}{(k+1)!k!}\mathrm{Sym}_{2,\ldots,k+1}\Theta_k^\nabla,$$

with $\Theta_2^\nabla \coloneqq 2R^\nabla$ and with

$$\Theta_k^\nabla \coloneqq -2i(id_1^\nabla)^{k-3}(\nabla R^\nabla)_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2}(id_1^\nabla)^{k-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 3$, satisfies the identities

$$I_k \coloneqq \mathrm{CircSym}_{3,\ldots,k+1}\Theta_k^\nabla = 0,$$

for all $k \geqslant 4$.

A general mathematical proof for the vanishing of all the integrability conditions I_k is part of a long and difficult work in progress. A corollary of the solution of the aforementioned conjecture and of the main Theorem 2 will be the following striking result that allows canonical construction of maximal totally real embeddings.

Corollary 2. (Canonical maximal totally real embeddings). *Let M be a real analytic manifold and let ∇ be a torsion free complex covariant derivative operator acting on the real analytic sections of the complexified tangent bundle $\mathbb{C}T_M$. Then there exists an open neighborhood $U \subseteq T_M$ of the image of the zero section with connected fibers and a fiberwise real-analytic section S of $\pi^*\mathrm{End}(T_M)$ over U with fiberwise Taylor expansion at the origin*

$$S_\eta \cdot \xi = \sum_{k\geqslant 2} S_k(\xi, \eta^k),$$

for any $\eta \in U$ and any $\xi \in T_{M,\pi(\eta)}$, with $S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ for all $k \geqslant 2$, (we denote by $\eta^k \coloneqq \eta^{\times k} \in T_{M,\pi(\eta)}^{\oplus k}$) given by the recursive formula

$$S_k \;\coloneqq\; \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{\,2,\ldots,k+1}\Theta_k^{\nabla},$$

with $\Theta_2^{\nabla} \;\coloneqq\; 2R^{\nabla}$ *and*

$$\Theta_k^{\nabla} \;\coloneqq\; -2i(id_1^{\nabla})^{k-3}(\nabla R^{\nabla})_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2} (id_1^{\nabla})^{k-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 3$, *such that* J_A *with*

$$\overline{A} \;\coloneqq\; -iT\mathbb{I}_{T_M} + H^{\nabla} - TS,$$

is an M-totally real complex structure over U, which is real-analytic over U.

Indeed in the statement of the main Theorem 2, we set $\sigma_k = 0$ for all $k \geqslant 2$ and we identify the torsion free complex covariant derivative ∇^{S_1} with the arbitrary torsion free complex covariant derivative ∇ in the statements of Corollary 2 and Conjecture 1. Then the integrability equations in the statement of the main Theorem 2 reduce to the identities $I_k = 0$, for all $k \geqslant 4$ in the statement of the Conjecture 1.

We notice that the notation H^{∇} in the aforementioned definition of the section A is slightly abusive. We mean there by H^{∇} the restriction to T_M of the horizontal map over $\mathbb{C}T_M$ associated with the complex covariant derivative operator ∇. We must observe here the obvious inclusion $T_{\mathbb{C}T_M|T_M} \subset \mathbb{C}T_{T_M}$.

The expression of S_k above can an should be replaced with the explicit global expression in the Theorems 1.5 and 1.6 in [18]. That expression shows that in the case (M, ∇) with smooth regularity we can assume weaker conditions on the growth of the covariant derivatives of the curvature and still obtain convergence along the fibers.

We obtain in this more general setting a canonical M-totally real complex structure over U which is real-analytic along the fibers of U. This is sufficient for the applications to micro local analytic analysis over manifolds.

We wish to point out that in the general setting of a torsion free complex covariant derivative operator ∇ acting on the sections of the complexified tangent bundle $\mathbb{C}T_M$ there are no geodesics associated with ∇. (Cauchy's existence theorem does not apply).

Therefore there exist no geodesic flow associated with ∇ and the Jacobi field techniques of the authors [5,10,13–15,20,21] do not apply.

We wish also to point out that in mathematics and in theoretical physics there are many important natural complex differential operators that are defined via complex connections as above.

The set up of Corollary 1 is inspired by the articles [10,13–15,20,21]. The genesis of their approach will be reminded in Sub-section 6.1 and is needed for the proof of Corollary 1.

The long series of articles due to Guillemin and Stenzel [10], Lempert [13], Lempert and Szöke [14,15]

Szöke [20,21], Burns [2,3], Burns et al. [4] as well as Aslam et al. [1] are inspired by the fundamental work of Grauert [9].

Their existence results are needed in a crucial way in analytic micro-local analysis, in pluri-potential theory (see the work by Zelditch [22]) as well as in Hamiltonian dynamics and in geometric quantization (see the work by Morao and Nunes [16], Hall and Kirwin [11]).

2 General connections over vector bundles

2.1 Basic definitions

Definition 3. Let (E, π_E, M) be a smooth vector bundle over a manifold M. A connection form over E is a section $\gamma \in C^{\infty}(E, T_E^* \otimes T_E)$ such that $d\pi_E \cdot \gamma = 0$ and $\gamma_{|\,\mathrm{Ker}\, d\pi_E} = \mathbb{I}_{\,\mathrm{Ker}\, d\pi_E}$.

We will denote by γ_η the connection form γ evaluated at the point $\eta \in E$.

Lemma 1. *For any connection $\gamma \in C^\infty(E, T_E^* \otimes T_E)$ the map*

$$d_\eta \pi_{E|\,\mathrm{Ker}\gamma_\eta} : \mathrm{Ker}\gamma_\eta \longrightarrow T_{M,\pi_E(\eta)}, \tag{2.1}$$

is an isomorphism for all $\eta \in E$.

Proof. The assumption $\gamma_{|\,\mathrm{Ker}d\pi_E} = \mathbb{I}_{\mathrm{Ker}d\pi_E}$ implies $\gamma \cdot (\mathbb{I}_{T_E} - \gamma) = 0$. Thus $\mathrm{Im}(\mathbb{I}_{T_E} - \gamma) \subseteq \mathrm{Ker}\gamma$. Then $\mathrm{Im}(\mathbb{I}_{T_E} - \gamma) = \mathrm{Ker}\gamma$. Indeed if $\gamma(u) = 0$ then $u = (\mathbb{I}_{T_E} - \gamma)u$. On the other hand we notice that the condition $d\pi_E \cdot \gamma = 0$ implies $d\pi_E \cdot (\mathbb{I}_{T_E} - \gamma) = d\pi_E$ and thus

$$d_\eta \pi_{E|\,\mathrm{Ker}\gamma_\eta} \cdot (\mathbb{I}_{T_E} - \gamma) = d\pi_E. \tag{2.2}$$

This equality shows that the map (2.1) is surjective. The injectivity follows from the fact that if $u, v \in \mathrm{Ker}\gamma_\eta$ and $d_\eta\pi_E(u - v) = 0$ then $u - v = \gamma(u - v) = 0$ by the assumption $\gamma_{|\,\mathrm{Ker}d\pi_E} = \mathbb{I}_{\mathrm{Ker}d\pi_E}$. □

We denote by $H_\eta^\gamma := (d_\eta \pi_{E|\,\mathrm{Ker}\gamma_\eta})^{-1}$ the horizontal map. We deduce the existence of a section

$$H^\gamma = C^\infty(E, \pi_E^* T_M^* \otimes T_E),$$

such that $d\pi_E \cdot H^\gamma = \mathbb{I}_{\pi_E^* T_M}$. (We notice that $d\pi_E \in C^\infty(E, T_E^* \otimes \pi_E^* T_M)$). Composing both sides of (2.2) with H_η^γ we infer

$$\gamma = \mathbb{I}_{T_E} - H^\gamma \cdot d\pi_E,$$

and the smooth vector bundle decomposition $T_E = \mathrm{Ker}d\pi_E \oplus \mathrm{Ker}\gamma$.

The data of a connection form γ is equivalent with the data of a horizontal form H^γ. The connection form is called linear if the horizontal form H^γ satisfies

$$d_{(\eta_1,\eta_2)}(sm_E) \cdot (H_{\eta_1}^\gamma \oplus H_{\eta_2}^\gamma) = H_{\eta_1+\eta_2}^\gamma, \quad H_{\lambda\eta}^\gamma = d_\eta(\lambda \mathbb{I}_E) \cdot H_\eta^\gamma,$$

where $sm_E : E \oplus E \longrightarrow E$ is the sum bundle map where $\eta_1, \eta_2, \eta \in E$ with $\pi_E(\eta_1) = \pi_E(\eta_2)$, and λ is a scalar.

Definition 4. The **curvature form** $\theta^\gamma \in C^\infty(E, \Lambda^2 T_E^* \otimes T_E)$ of a connection form γ is defined as

$$\theta^\gamma(\xi_1, \xi_2) := -\gamma[(\mathbb{I}_{T_E} - \gamma)\xi_1, (\mathbb{I}_{T_E} - \gamma)\xi_2],$$

for all $\xi_1, \xi_2 \in C^\infty(E, T_E)$.

The definition is tensorial. Indeed if $f \in C^\infty(E, \mathbb{R})$ then

$$[(\mathbb{I}_{T_E} - \gamma)f\xi_1, (\mathbb{I}_{T_E} - \gamma)\xi_2] = f[(\mathbb{I}_{T_E} - \gamma)\xi_1, (\mathbb{I}_{T_E} - \gamma)\xi_2] - [(\mathbb{I}_{T_E} - \gamma)\xi_2. f](\mathbb{I}_{T_E} - \gamma)\xi_1.$$

The conclusion follows from the fact that $\gamma \cdot (\mathbb{I}_{T_E} - \gamma) = 0$. We notice that

$$\theta^\gamma \in C^\infty(E, \Lambda^2(\mathrm{Ker}\gamma)^* \otimes \mathrm{Ker}\, d\pi),$$

and such element is uniquely determined by the **curvature field** Θ^γ defined as

$$\Theta^\gamma(\xi_1, \xi_2)(\eta) := T_\eta^{-1}\theta_\eta^\gamma(H_\eta^\gamma \xi_1, H_\eta^\gamma \xi_2),$$

for all $\xi_1, \xi_2 \in T_{M,\pi_E(\eta)}$. In the case γ is linear then

$$\Theta^\gamma \in C^\infty(M, \Lambda^2 T_M^* \otimes \mathrm{End}(E)),$$

is called the **curvature operator**. The terminology is consistent with the fact that if we denote by ∇^γ the covariant derivative associated with γ then the identity $R^{\nabla^\gamma} = \Theta^\gamma$ holds, thanks to Lemma A6 in the Appendix.

Parallel transport. Given any horizontal form $\alpha \in C^{\infty}(E, \pi^* T^*_M \otimes_{\mathbb{R}} T_E)$ over a vector bundle E, the parallel transport with respect to α is defined as follows. We consider a smooth curve $c : (-\varepsilon, \varepsilon) \longrightarrow M$ and the section $\sigma \in C^1((-\varepsilon, \varepsilon), c^* E)$ which satisfies the equation

$$\dot{\sigma} = (\alpha \circ \sigma) \cdot \dot{c},$$

over $(-\varepsilon, \varepsilon)$ with $\sigma(0) = \eta \in E_{c(0)}$. We define the parallel transport map $\tau^{\alpha}_{c,t} : E_{c(0)} \longrightarrow E_{c(t)}$, $t \in (-\varepsilon, \varepsilon)$ along c with respect to α as $\tau^{\alpha}_{c,t}(\eta) = \sigma(t)$.

We consider now a C^1-vector field ξ over M and let $\varphi_{\xi,t}$ be the associated 1-parameter subgroup of transformations of M. Let $\Phi^{\alpha}_{\xi,t} : E \longrightarrow E$ be the parallel transport map along the flow lines of $\varphi_{\xi,t}$. In equivalent terms the map $\Phi^{\alpha}_{\xi,t}$ is determined by the ODE

$$\dot{\Phi}^{\alpha}_{\xi,t} = (\alpha \circ \Phi^{\alpha}_{\xi,t}) \cdot (\xi \circ \varphi_{\xi,t} \circ \pi_E),$$

with initial condition $\Phi^{\alpha}_{\xi,0} \equiv \mathbb{I}_E$. We observe that by definition of parallel transport, the map $\Phi^{\alpha}_{\xi,t}$ satisfies $\pi_E \circ \Phi^{\alpha}_{\xi,t} = \varphi_{\xi,t} \circ \pi_E$. This follows also from the equalities

$$(d\pi_E \circ \Phi^{\alpha}_{\xi,t}) \cdot \dot{\Phi}^{\alpha}_{\xi,t} = \xi \circ \varphi_{\xi,t} \circ \pi_E = \dot{\varphi}_{\xi,t} \circ \pi_E.$$

Moreover the vector field $\Xi^{\alpha} := \alpha \cdot (\xi \circ \pi_E)$ over E satisfies $\dot{\Phi}^{\alpha}_{\xi,t} = \Xi^{\alpha} \circ \Phi^{\alpha}_{\xi,t}$. Indeed

$$\Xi^{\alpha} \circ \Phi^{\alpha}_{\xi,t} = (\alpha \circ \Phi^{\alpha}_{\xi,t}) \cdot (\xi \circ \pi_E \circ \Phi^{\alpha}_{\xi,t}) = (\alpha \circ \Phi^{\alpha}_{\xi,t}) \cdot (\xi \circ \varphi_{\xi,t} \circ \pi_E).$$

We deduce that $t \longmapsto \Phi^{\alpha}_{\xi,t}$ is also a 1-parameter subgroup of transformations of E.

2.2 The geometric meaning of the curvature field

The following result provides a clear geometric meaning of the curvature field.

Lemma 2. *Let (E, π_E, M) be a smooth vector bundle over a manifold M and consider a horizontal form $\alpha \in C^{\infty}(E, \pi^* T^*_M \otimes_{\mathbb{R}} T_E)$ over bundle E. Then the curvature field Θ^{α} associated with α satisfies*

$$\Theta^{\alpha}(\xi_1, \xi_2)(\eta) = T^{-1}_{\eta} \frac{\partial^2}{\partial t \partial s}_{|t=s=0} (\Phi^{\alpha}_{\xi_1,-s} \circ \Phi^{\alpha}_{\xi_2,-t} \circ \Phi^{\alpha}_{\xi_1,s} \circ \Phi^{\alpha}_{\xi_2,t}(\eta)).$$

for any $\xi_1, \xi_2 \in C^{\infty}(M, T_M)$ such that $[\xi_1, \xi_2] \equiv 0$ and for any $\eta \in E$.

Proof. We observe first that if we have a family of transformations $(\Psi_s)_s$ over a manifold with $\Psi_0 = \mathrm{id}$ and a curve c then

$$\frac{d}{ds}_{|s=0} \Psi_s(c_s) = \dot{\Psi}_0(c_0) + d\Psi_0(\dot{c}_0) = \dot{\Psi}_0(c_0) + \dot{c}_0.$$

By applying the last equality to $\Psi_s = \varphi_{\xi_2,-s}$ and $c_s := \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}$, we infer

$$\frac{d}{ds}_{|s=0} (\varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}) = -\xi_2 + \frac{d}{ds}_{|s=0} (\varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}),$$

and thus,

$$\begin{aligned}[\xi_1, \xi_2] &= \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (\varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}) \\ &= \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (\varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}).\end{aligned}$$

In a similar way,

$$[\Xi_2^\alpha, \Xi_1^\alpha] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (\Phi^\alpha_{\xi_1,-s} \circ \Phi^\alpha_{\xi_2,-t} \circ \Phi^\alpha_{\xi_1,s} \circ \Phi^\alpha_{\xi_2,t}),$$

with $\Xi_j^\alpha := \alpha \cdot (\xi_j \circ \pi_E)$, $j = 1, 2$. Let $\eta \in E_p$ and observe that

$$\Phi^\alpha_{\xi_1,-s} \circ \Phi^\alpha_{\xi_2,-t} \circ \Phi^\alpha_{\xi_1,s} \circ \Phi^\alpha_{\xi_2,t}(\eta) \in E_p,$$

for all parameters t, s, since $\varphi_{\xi_1,-s} \circ \varphi_{\xi_2,-t} \circ \varphi_{\xi_1,s} \circ \varphi_{\xi_2,t}(p) = p$ thanks to the assumption $[\xi_1, \xi_2] \equiv 0$. We conclude the required geometric identity. □

2.3 Comparison of the curvature fields of two connections

We consider now two connection forms γ_j, $j = 1, 2$ over E, and let $\alpha_j := H^{\gamma_j}$ be the corresponding horizontal forms. The fact that $d\pi_E(\alpha_1 - \alpha_2) = 0$ implies that there exist a section

$$B := T^{-1}(\alpha_1 - \alpha_2) \in C^\infty(E, \pi_E^*(T_M^* \otimes E)),$$

which satisfies

$$\gamma_1 = \gamma_2 - TB \cdot d\pi_E.$$

We want to compare the curvature fields $\Theta_j := \Theta^{\gamma_j}$. We will denote by abuse of notation $\alpha_j \xi \equiv \alpha_j \cdot (\xi \circ \pi_E)$ and $B\xi \equiv B \cdot (\xi \circ \pi_E)$ for any $\xi \in C^\infty(M, T_M)$.

Lemma 3. *In the aforementioned set up, the identity*

$$\Theta_1(\xi_1, \xi_2) = (\Theta_2 - B \neg DB)(\xi_1, \xi_2) - T^{-1}([\alpha_2\xi_1, TB\xi_2] - [\alpha_2\xi_2, TB\xi_1]) + B[\xi_1, \xi_2], \tag{2.3}$$

holds for any $\xi_1, \xi_2 \in C^\infty(M, T_M)$.

Proof. We notice first the equalities

$$\begin{aligned} T\Theta_1(\xi_1, \xi_2) &= \theta^{\gamma_1}(\alpha_1\xi_1, \alpha_1\xi_2) \\ &= -\gamma_1[\alpha_1\xi_1, \alpha_1\xi_2] \\ &= -\gamma_2[\alpha_1\xi_1, \alpha_1\xi_2] + TB \cdot d\pi_E[\alpha_1\xi_1, \alpha_1\xi_2] \\ &= T\Theta_2(\xi_1, \xi_2) - \gamma_2([\alpha_2\xi_1, TB\xi_2] + [TB\xi_1, \alpha_2\xi_2] + [TB\xi_1, TB\xi_2]) + TB[\xi_1, \xi_2]. \end{aligned}$$

In the last line, we use the well-known identity $d\pi_E[\alpha_1\xi_1, \alpha_1\xi_2] = [\xi_1, \xi_2] \circ \pi_E$, which follows from the fact that $d\pi_E \alpha_1 \xi_j = \xi_j \circ \pi_E$, $j = 1, 2$. Let now $\Phi_{TB\xi_2,t}$ be the one-parameter subgroup of transformations of E associated with the vertical vector field $TB\xi_2$. It satisfies $\pi_E \circ \Phi_{TB\xi_2,t} = \pi_E$. Using the standard expression of the Lie bracket

$$[\alpha_2\xi_1, TB\xi_2] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} (\Phi^{\alpha_2}_{\xi_1,-t} \circ \Phi_{TB\xi_2,s} \circ \Phi^{\alpha_2}_{\xi_1,t}),$$

we deduce that this vector field is vertical. In the same way, $[TB\xi_1, \alpha_2\xi_2]$ is vertical. It is obvious that the vector field $[TB\xi_1, TB\xi_2]$ is also vertical. We infer the identity

$$T\Theta_1(\xi_1, \xi_2) = T\Theta_2(\xi_1, \xi_2) - [TB\xi_1, TB\xi_2] - [\alpha_2\xi_1, TB\xi_2] - [TB\xi_1, \alpha_2\xi_2] + TB[\xi_1, \xi_2].$$

The required formula (2.3) follows from the identity

$$[TB\xi_1, TB\xi_2] = T(B \neg DB)(\xi_1, \xi_2), \tag{2.4}$$

which we show now. We first remind the reader that for any vector space V, the canonical translation operator $T : C^\infty(V, V) \longrightarrow C^\infty(V, T_V)$ defined as $(T\xi)(v) := T_v\xi_v$ is a Lie algebra isomorphism, where the Lie algebra structure over $C^\infty(V, V)$ is defined by $[\xi, \eta]_v := D_v\eta \cdot \xi_v - D_v\xi \cdot \eta_v$. Indeed if we define the action of $C^\infty(V, V)$ over $C^\infty(V, \mathbb{R})$ as follows:

$$(\xi. f)(v) \coloneqq D_v f \cdot \xi_v = \frac{d}{dt}_{|t=0} f(v + t\xi_v) = [(T\xi). f](v),$$

then

$$(\xi \cdot \eta \cdot f)(v) = \frac{d}{dt}_{|t=0}(\eta \cdot f)(v + t\xi_v) = \frac{d}{dt}_{|t=0}(D_{v+t\xi_v} f \cdot \eta_{v+t\xi_v}) = D^2_v f(\xi_v, \eta_v) + D_v f \cdot D_v \eta \cdot \xi_v.$$

The fact that the bilinear form $D^2_v f$ is symmetric implies

$$\xi \cdot \eta \cdot f - \eta \cdot \xi \cdot f = [\xi, \eta] \cdot f.$$

On the other hand, by definition,

$$\begin{aligned} T\xi \cdot T\eta \cdot f - T\eta \cdot T\xi \cdot f &= \xi \cdot \eta \cdot f - \eta \cdot \xi \cdot f, \\ [\xi, \eta] \cdot f &= T[\xi, \eta] \cdot f. \end{aligned}$$

We conclude the required identity $[T\xi, T\eta] = T[\xi, \eta]$. We apply this remark to our setup. For any point $p \in M$, we denote by $B\xi(p) \in C^\infty(E_p, E_p)$ the map $\eta \in E_p \longmapsto B_\eta \xi(p) \in E_p$ and we denote by $TB\xi(p) \in C^\infty(E_p, T_{E_p})$ the section $\eta \in E_p \longmapsto T_\eta B_\eta \xi(p) \in T_{E_p,\eta}$. Then for any $\eta \in E_p$,

$$\begin{aligned} [TB\xi_1, TB\xi_2]_\eta &= [TB\xi_1(p), TB\xi_2(p)]_\eta \\ &= T_\eta[B\xi_1(p), B\xi_2(p)]_\eta \\ &= T_\eta[D_\eta B(B_\eta \xi_1(p))\xi_2(p) - D_\eta B(B_\eta \xi_2(p))\xi_1(p)], \end{aligned}$$

which shows (2.4). □

We notice now that for any covariant derivative ∇ over E, the identity (A8) can be expressed as follows:

$$[H^\nabla \xi, T\pi_E^* s] = T\pi_E^*(\nabla_\xi s), \tag{2.5}$$

for any vector field $\xi \in C^\infty(M, T_M)$ and any section $s \in C^\infty(M, E)$. We need to show the following more general formula.

Lemma 4. *Let (E, π_E, M) be a smooth vector bundle over a manifold M, and let ∇ be a covariant derivative operator acting on the smooth sections of E. Then the equality holds*

$$[H^\nabla \xi, T\sigma] = T\nabla^{\pi_E}_{H^\nabla \xi}\sigma, \tag{2.6}$$

for any vector field $\xi \in C^\infty(M, T_M)$ and for any section $\sigma \in C^\infty(E, \pi_E^* E)$.

We observe that (2.6) implies (2.5), since $\nabla^{\pi_E}_{H^\nabla \xi}\sigma = \pi_E^*(\nabla_\xi s)$, thanks to the functorial property (A6).

Proof. To show the identity (2.6) we notice first that the assumption $\sigma \in C^\infty(E, \pi_E^* E)$ means that σ is a map $\sigma : E \longrightarrow E$ such that $\pi_E \circ \sigma = \pi_E$. Then the one-parameter subgroup of transformations of E associated with the vector field $T\sigma$ satisfies $\Phi_{T\sigma,t}(\eta) = \eta + t\sigma(\eta)$. Moreover, with the notation in the proof of identity (A8),

$$[H^\nabla \xi, T\sigma] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0}(\Phi_{\xi,-t} \circ \Phi_{T\sigma,s} \circ \Phi_{\xi,t}).$$

The fact that $\Phi_{\xi,-t}$ is linear on the fibers of E implies

$$\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} = \Phi_{\xi,-t}[\Phi_{\xi,t} + s\sigma \circ \Phi_{\xi,t}] = \mathbb{I}_E + s\Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}.$$

We infer

$$\frac{d}{ds}_{|s=0}(\Phi_{\xi,-t} \circ \Phi_{T\sigma,s} \circ \Phi_{\xi,t})(\eta) = T_\eta \Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}(\eta),$$

for any $\eta \in E_p$. We observe that $\sigma \circ \Phi_{\xi,t}(\eta) \in E_{\varphi_{\xi,t}(p)}$. Indeed, using the property $\pi_E \circ \sigma = \pi_E$, we deduce

$$\pi_E \circ \sigma \circ \Phi_{\xi,t}(\eta) = \pi_E \circ \Phi_{\xi,t}(\eta) = \varphi_{\xi,t}(p).$$

We remind now that if $t \longmapsto \eta_t \in E$ is a smooth curve such that $c_t := \pi_E(\eta_t)$, then

$$T_{\eta_0}^{-1}\gamma_{\eta_0}^{\nabla}\dot{\eta}_0 = \frac{d}{dt}_{|_{t=0}}(\tau_{c,t}^{-1}\eta_t),$$

thanks to formula (A7). We apply the previous identity to the curve $\eta_t := \sigma \circ \Phi_{\xi,t}(\eta) \in E_{\varphi_{\xi,t}(p)}$. We obtain

$$T_{\sigma(\eta)}^{-1}\gamma_{\sigma(\eta)}^{\nabla}\frac{d}{dt}_{|_{t=0}}[\sigma \circ \Phi_{\xi,t}(\eta)] = \frac{d}{dt}_{|_{t=0}}[\Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}(\eta)] = T_\eta^{-1}[H^\nabla\xi, T\sigma](\eta).$$

Moreover,

$$\frac{d}{dt}_{|_{t=0}}[\sigma \circ \Phi_{\xi,t}(\eta)] = d_\eta\sigma \cdot \dot{\Phi}_{\xi,0}(\eta) = d_\eta\sigma \cdot H_\eta^\nabla\xi(p).$$

We conclude the equality

$$T_{\sigma(\eta)}^{-1}\gamma_{\sigma(\eta)}d_\eta\sigma \cdot H_\eta^\nabla\xi(p) = T_\eta^{-1}[H^\nabla\xi, T\sigma](\eta),$$

which represents the required formula (2.6). □

We can show now the following result.

Lemma 5. *Let (E, π_E, M) be a smooth vector bundle over a manifold M, and let ∇ and ∇^{T_M} be covariant derivative operators acting, respectively, on the smooth sections of the bundles E and T_M.*

Then for any section $B \in C^\infty(E, \pi_E^*(T_M^* \otimes E))$ the curvature field Θ^α of the horizontal form $\alpha := H^\nabla + TB$ satisfies

$$\Theta^\alpha = -H^\nabla \neg \nabla^{T_M^*\otimes E,\pi_E}B - B \neg DB - B\tau^{\nabla^{T_M}} + R^\nabla, \tag{2.7}$$

where $\nabla^{T_M^*\otimes E,\pi_E}$ is the covariant derivative acting on the smooth sections of the bundle $\pi_E^*(T_M^* \otimes E)$, induced by ∇ and ∇^{T_M}, where $\tau^{\nabla^{T_M}}$ is the torsion form of ∇^{T_M}.

Proof. In the case $\alpha_2 = H^\nabla$ in the identity (2.3), we can apply the formula (2.6) to the sections $B\xi_j \in C^\infty(E, \pi_E^*E)$. We obtain

$$\Theta_1(\xi_1, \xi_2) = (R^\nabla - B \neg DB)(\xi_1, \xi_2) - \nabla^{\pi_E}_{H^\nabla\xi_1}(B\xi_2) + \nabla^{\pi_E}_{H^\nabla\xi_2}(B\xi_1) + B[\xi_1, \xi_2].$$

By using functorial properties of the pull-back, we have (with no abuse of notation)

$$\nabla^{\pi_E}_{H^\nabla\xi_1}(B \cdot \pi_E^*\xi_2) = \nabla^{T_M^*\otimes E,\pi_E}_{H^\nabla\xi_1}B \cdot \pi_E^*\xi_2 + B \cdot \nabla^{T_M,\pi_E}_{H^\nabla\xi_1}(\pi_E^*\xi_2) = \nabla^{T_M^*\otimes E,\pi_E}_{H^\nabla\xi_1}B \cdot \pi_E^*\xi_2 + B \cdot \pi_E^*(\nabla^{T_M}_{\xi_1}\xi_2).$$

We conclude by (2.3) that if $\alpha_1 = \alpha = H^\nabla + TB$, then the curvature field Θ^α of α satisfies the identity

$$\Theta^\alpha(\xi_1, \xi_2) = (R^\nabla - B \neg DB)(\xi_1, \xi_2) - \nabla^{T_M^*\otimes E,\pi_E}_{H^\nabla\xi_1}B\xi_2 + \nabla^{T_M^*\otimes E,\pi_E}_{H^\nabla\xi_2}B\xi_1 - B\tau^{\nabla^{T_M}}(\xi_1, \xi_2).$$

We infer the required formula (2.7). □

3 First reduction of the integrability equations

Proof of Theorem 1

Proof. Let γ^A be the connection form associated with the horizontal form A. Then the integrability of J_A is equivalent to the condition

$$\gamma^A[A\xi_1, A\xi_2] = 0, \tag{3.1}$$

for all smooth complex vector fields ξ_1, ξ_2 over M. (We remind here the use of the abusive notation $A\xi \equiv A(\xi \circ \pi)$.) We denote, respectively, by Θ^A and Θ^α the curvature fields of the horizontal distributions A and α. The integrability condition (3.1) is equivalent to the condition $\Theta^A \equiv 0$. Then by applying the identity (2.3) with $\alpha_1 = A, \alpha_2 = \alpha$ and separating real and imaginary parts, we deduce that the integrability of J_A is equivalent to the system

$$\begin{cases}\Theta^\alpha + B \neg DB = 0,\\ TB[\xi_1, \xi_2] = [\alpha\xi_1, TB\xi_2] - [\alpha\xi_2, TB\xi_1].\end{cases} \tag{3.2}$$

Let $\Gamma \in C^\infty(U, \pi^*\operatorname{End}(T_M))$ such that $\alpha = H^\nabla - T\Gamma$. By using the formula (2.7) in the case $E = T_M$ and $\nabla = \nabla^{T_M}$ we can write the previous equation of the system (3.2) as follows:

$$H^\nabla \neg \nabla^{\operatorname{End}(T_M),\pi}\Gamma - \Gamma \neg D\Gamma + \Gamma\tau^\nabla + B \neg DB + R^\nabla = 0.$$

We express the second equation of the system (3.2) as follows:

$$TB[\xi_1, \xi_2] = [H^\nabla\xi_1, TB\xi_2] - [T\Gamma\xi_1, TB\xi_2] - [H^\nabla\xi_2, TB\xi_1] + [T\Gamma\xi_2, TB\xi_1].$$

By using formula (2.6), we infer

$$\begin{aligned}B[\xi_1, \xi_2] = {}& \nabla^{\operatorname{End}(T_M),\pi}_{H^\nabla\xi_1}B\xi_2 - \nabla^{\operatorname{End}(T_M),\pi}_{H^\nabla\xi_2}B\xi_1 + B(\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1)\\ & - DB(\Gamma\xi_1)\xi_2 + D\Gamma(B\xi_2)\xi_1 + DB(\Gamma\xi_2)\xi_1 - D\Gamma(B\xi_1)\xi_2,\end{aligned}$$

which can be expressed as follows:

$$H^\nabla \neg \nabla^{\operatorname{End}(T_M),\pi}B - \Gamma \neg DB - B \neg D\Gamma + B\tau^\nabla = 0.$$

We conclude that the system (3.2) is equivalent to the system

$$\begin{cases}H^\nabla \neg \nabla^{\operatorname{End}(T_M),\pi}\Gamma - \Gamma \neg D\Gamma + \Gamma\tau^\nabla + B \neg DB + R^\nabla &= 0,\\ H^\nabla \neg \nabla^{\operatorname{End}(T_M),\pi}B - \Gamma \neg DB - B \neg D\Gamma + B\tau^\nabla &= 0.\end{cases} \tag{3.3}$$

It follows that, using the identification $S = \Gamma + iB$, the system (3.3) is equivalent to the complex equation (1.8). □

Remark 1. We notice that in the case $(\alpha, B) = (H^\nabla, \mathbb{I}_{\pi^*T_M})$, i.e., in the case $J_A = J_{H^\nabla}$, the system (3.3) reduces to

$$\begin{cases}R^\nabla &= 0,\\ \tau^\nabla &= 0.\end{cases}$$

In this way, we re-obtain the statement of Lemma 13.

Lemma 6. *Under the assumptions of the Theorem 2, the M-totally real almost complex structure J_A is integrable over U if and only if*

$$S_1 \in C^\infty(M, S^2T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

(*i.e.*, ∇^{S_1} *is torsion free*),

$$R^{\nabla^{S_1}} = -2i \operatorname{Alt}_2 S_2, \tag{3.4}$$

$$\left[d_1^{\nabla^{S_1}}S_k + \sum_{p=2}^{k-1} pS_p \wedge_1 S_{k-p+1} + i(k+1)\operatorname{Alt}_2 S_{k+1}\right](\xi_1, \xi_2, \eta^k) = 0. \tag{3.5}$$

for all $k \geqslant 2$ and for all $\xi_1, \xi_2, \eta \in T_{M,\pi(\eta)}$.

Proof. Let $S := T^{-1}(H^\nabla - \overline{A})$. In the case the connection ∇ is torsion free, equation (1.8) reduces to

$$H^\nabla \neg \nabla^{\operatorname{End}(T_M),\pi}S - S \neg DS + R^\nabla = 0. \tag{3.6}$$

The identification $\mathcal{S}_{k,\eta}\cdot\xi\equiv S_k(\xi,\eta^k)$ shows that $\mathcal{S}_{k,\eta}\in T^*_{M,\pi(\eta)}\otimes\mathbb{C}T_{M,\pi(\eta)}$, i.e.,

$$\mathcal{S}_k\in C^\infty(T_M,\pi^*(T^*_M\otimes_{\mathbb{R}}\mathbb{C}T_M))$$

and

$$S=\sum_{k\geqslant 0}\mathcal{S}_k. \tag{3.7}$$

We remind the reader of the formula

$$\nabla^\pi_{H^\nabla\xi_1}(\mathcal{S}_k\cdot\xi_2)=\nabla^{\mathrm{End}(T_M),\pi}_{H^\nabla\xi_1}\mathcal{S}_k\cdot\xi_2+\mathcal{S}_k\cdot\nabla_{\xi_1}\xi_2,$$

for any vector field ξ_1,ξ_2 over M. On the other hand, by definition,

$$\begin{aligned}\nabla^\pi_{H^\nabla\xi_1}(\mathcal{S}_k\cdot\xi_2)_{|\eta}&=T^{-1}_{S_k(\xi_2,\eta^k)}y^\nabla_{S_k(\xi_2,\eta^k)}d_\eta(\mathcal{S}_k\cdot\xi_2)(H^\nabla\xi_1)\\&=T^{-1}_{S_k(\xi_2,\eta^k)}y^\nabla_{S_k(\xi_2,\eta^k)}\frac{d}{dt}_{|t=0}[S_k(\xi_2\circ\varphi_{\xi_1,t}\circ\pi(\eta),\Phi_{\xi_1,t}(\eta)^k)].\end{aligned}$$

Let now $\boldsymbol{\eta}$ be the vector field over $\mathrm{Im}(\varphi_{\xi_1,\bullet}\circ\pi(\eta))$ defined by

$$\boldsymbol{\eta}(\varphi_{\xi_1,t}\circ\pi(\eta))=\Phi_{\xi_1,t}(\eta).$$

Then

$$\nabla^\pi_{H^\nabla\xi_1}(\mathcal{S}_k\cdot\xi_2)_{|\eta}=\nabla_{\xi_1}[S_k(\xi_2,\boldsymbol{\eta}^k)]_{|\pi(\eta)}=\nabla_{\xi_1}S_k(\xi_2\circ\pi(\eta),\eta^k)+S_k(\nabla_{\xi_1}\xi_{2|\pi(\eta)},\eta^k),$$

since $\nabla_{\xi_1}\boldsymbol{\eta}=0$. We conclude the identity

$$(\nabla^{\mathrm{End}(T_M),\pi}_{H^\nabla\xi_1}\mathcal{S}_k)_{|\eta}\cdot\xi_2=\nabla_{\xi_1}S_k(\xi_2,\eta^k),$$

$\xi_1,\xi_2\in T_{M,\pi(\eta)}$. We infer the formula

$$H^\nabla\neg\nabla^{\mathrm{End}(T_M),\pi}\mathcal{S}_k=d_1^\nabla S_k. \tag{3.8}$$

We notice now the equalities

$$D_\eta\mathcal{S}_k(v)\cdot\xi=\frac{d}{dt}_{|t=0}[S_k(\xi,(\eta+tv)^k)]=\sum_{j=1}^k S_k(\xi,\eta^{j-1},v,\eta^{k-j})=kS_k(\xi,v,\eta^{k-1}),$$

and

$$(\mathcal{S}_l\neg D\mathcal{S}_k)_{|\eta}(\xi_1,\xi_2)=kS_k(\xi_2,\mathcal{S}_{l,\eta}\cdot\xi_1,\eta^{k-1})-kS_k(\xi_1,\mathcal{S}_{l,\eta}\cdot\xi_2,\eta^{k-1}).$$

We infer the equality

$$(\mathcal{S}_l\neg D\mathcal{S}_k)_{|\eta}(\xi_1,\xi_2)=-k(S_k\wedge_1 S_l)(\xi_1,\xi_2,\eta^{k+l-1}). \tag{3.9}$$

Let $W\subset U$ be any set containing the zero section of T_M such that $W\cap T_{M,p}$ is a neighborhood of 0_p for any $p\in M$ and such that the fiberwise expansion (3.7) converges over $W\cap T_{M,p}$. The fact that by assumption $U\cap T_{M,p}$ is connected implies by the fiberwise real analyticity of S that S is a solution of (3.6) over U if and only if it satisfies (3.6) over W.

By using (3.8), we can write the equation (3.6) under the form

$$\sum_{k\geqslant 1}d_1^\nabla S_k-\sum_{l,p\geqslant 0}(\mathcal{S}_l\neg D\mathcal{S}_p)+R^\nabla=0, \tag{3.10}$$

over W. We decompose the sum

$$\begin{aligned}
\sum_{l,p\geqslant 0}(\mathcal{S}_l \neg D\mathcal{S}_p) &= \sum_{l\geqslant 0,p\geqslant 1}(\mathcal{S}_l \neg D\mathcal{S}_p)\\
&= \sum_{l,p\geqslant 1}(\mathcal{S}_l \neg D\mathcal{S}_p) + i\sum_{k\geqslant 0}(\mathbb{I}_{T_M} \neg D\mathcal{S}_{k+1})\\
&= \sum_{k\geqslant 1}\sum_{p=1}^{k}(\mathcal{S}_{k-p+1} \neg D\mathcal{S}_p) + i\sum_{k\geqslant 0}(\mathbb{I}_{T_M} \neg D\mathcal{S}_{k+1})\\
&= -\sum_{k\geqslant 1}\sum_{p=1}^{k}p(S_p \wedge_1 S_{k-p+1}) - i\sum_{k\geqslant 0}(k+1)(S_{k+1}\wedge_1 \mathbb{I}_{T_M}),
\end{aligned}$$

thanks to the equality (3.9). If we denote by $\deg_\eta$ the degree with respect to the fiber variable $\eta \in E_{\pi(\eta)}$, we have

$$\begin{aligned}
\deg_\eta d_1^\nabla S_k = \deg_\eta(S_p \wedge_1 S_{k-p+1}) &= k,\\
\deg_\eta(S_{k+1}\wedge_1 \mathbb{I}_{T_M}) &= k,\\
\deg_\eta R^\nabla &= 1.
\end{aligned}$$

Thus, by homogeneity, equation (3.10) is equivalent to the countable system

$$\begin{cases}
S_1 \wedge_1 \mathbb{I}_{T_M} = 0,\\
d_1^\nabla S_1 + S_1\wedge_1 S_1 + 2iS_2\wedge_1 \mathbb{I}_{T_M} + R^\nabla = 0,\\
\left[d_1^\nabla S_k + \sum_{p=1}^{k}p(S_p\wedge_1 S_{k-p+1}) + i(k+1)S_{k+1}\wedge_1 \mathbb{I}_{T_M}\right](\xi_1,\xi_2,\eta^k) = 0,\\
\forall k\geqslant 2,\ \forall \xi_1,\xi_2,\eta\in T_M.
\end{cases} \tag{3.11}$$

The first equation in the system means $S_1 \in C^\infty(M, S^2T_M^*\otimes_{\mathbb{R}} \mathbb{C}T_M)$, i.e., the complex connection ∇^{S_1} is torsion free. The second equation in system (3.11) rewrites as (3.4). We show now that the equation for $k \geqslant 2$ in system (3.11) rewrites as (3.5). Indeed using the formula

$$\nabla_\xi^\Gamma\theta(v_1,\dots,v_p) = \nabla_\xi\theta(v_1,\dots,v_p) + \Gamma(\xi,\theta(v_1,\dots,v_p)) - \sum_{j=1}^{p}\theta(v_1,\dots,v_{j-1},\Gamma(\xi,v_j),v_{j+1},\dots,v_p),$$

where $\Gamma \in C^\infty(M, T_M^{*,\otimes 2}\otimes_{\mathbb{R}} \mathbb{C}T_M)$, $\theta \in C^\infty(M, T_M^{*,\otimes p}\otimes_{\mathbb{R}} \mathbb{C}T_M)$ and $\xi, v_k \in T_M$, we infer

$$\begin{aligned}
d_1^{\nabla^{S_1}}S_k(\xi_1,\xi_2,\eta^k) &= \nabla_{\xi_1}S_k(\xi_2,\eta^k) - \nabla_{\xi_2}S_k(\xi_1,\eta^k) + S_1(\xi_1,S_k(\xi_2,\eta^k)) - S_k(S_1(\xi_1,\xi_2),\eta^k)\\
&\quad - kS_k(\xi_2,S_1(\xi_1,\eta),\eta^{k-1}) - S_1(\xi_2,S_k(\xi_1,\eta^k)) + S_k(S_1(\xi_2,\xi_1),\eta^k) + kS_k(\xi_1,S_1(\xi_2,\eta),\eta^{k-1})\\
&= [d_1^\nabla S_k + S_1\wedge_1 S_k + kS_k\wedge_1 S_1](\xi_1,\xi_2,\eta^k),
\end{aligned}$$

since S_1 is symmetric and S_k is symmetric in the last k variables. We conclude (3.5). □

Remark 2. In the case $S_k = 0$, for all $k \geqslant 2$, the previous system reduces to the equation

$$d_1^\nabla S_1 + S_1\wedge_1 S_1 + R^\nabla = 0. \tag{3.12}$$

The equation (3.12) means that the complex connection ∇^{S_1} acting on sections of $\mathbb{C}T_M$ is flat. In the case $B = \mathbb{I}_{\pi^*T_M}$, the second equation in system (3.11) implies

$$d_1^\nabla\Gamma_1 + \Gamma_1\wedge_1\Gamma_1 + R^\nabla = 0,$$

with $\Gamma_1 := S_1$. This means that the real connection ∇^{Γ_1} is flat.

4 Second reduction of the integrability equations

In this section, we will prove the following result.

Proposition 2. *Under the assumptions of the Theorem 2, the M-totally real almost complex structure J_A is integrable over U if and only if*

$$
\begin{aligned}
&S_1 \in C^\infty(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M), \ \textit{i.e.}, \ \nabla^{S_1} \textit{ is torsion free},\\
&S_2 = S_2^0 + \sigma_2,\\
S_2^0(\xi_1, \xi_2, \xi_3) &\coloneqq \frac{i}{6}[R^{\nabla^{S_1}}(\xi_1, \xi_2)\xi_3 + R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_2],\\
&\sigma_2 \in C^\infty(M, S^3 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),\\
&S_3 = \frac{i}{3}\nabla^{S_1}\sigma_2 + \frac{1}{4!3}\operatorname{Sym}_{2,3,4}(\nabla^{S_1}R^{\nabla^{S_1}})_2 + \sigma_3,\\
&\sigma_3 \in C^\infty(M, S^4 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),
\end{aligned}
$$

$(\nabla^{S_1}R^{\nabla^{S_1}})_2(\xi_1, \xi_2, \xi_3, \xi_4) \coloneqq \nabla^{S_1}_{\xi_2}R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_4$, *for all* $\xi_1, \xi_2, \xi_3, \xi_4 \in T_{M,\pi(\xi_1)}$ *and for all* $k \geqslant 3$,

$$\left[d_1^{\nabla^{S_1}} S_k + \sum_{p=2}^{k-1} p S_p \wedge_1 S_{k-p+1} + i(k+1)\operatorname{Alt}_2 S_{k+1}\right](\xi_1, \xi_2, \eta^k) = 0,$$

for all $\xi_1, \xi_2, \eta \in T_{M,\pi(\eta)}$.

We first remind the reader that for any complex connection ∇ acting over the sections of $\mathbb{C}T_M$ its torsion τ^∇ satisfies the identity

$$\tau^\nabla = d^\nabla \mathbb{I}_{T_M},$$

where d^∇ is the covariant exterior differentiation and $\mathbb{I}_{T_M} \in C^\infty(M, T_M^* \otimes T_M)$. Then

$$d^\nabla \tau^\nabla = R^\nabla \wedge \mathbb{I}_{T_M},$$

and

$$(R^\nabla \wedge \mathbb{I}_{T_M})(\xi_1, \xi_2, \xi_3) = R^\nabla(\xi_1, \xi_2)\xi_3 + R^\nabla(\xi_2, \xi_3)\xi_1 + R^\nabla(\xi_3, \xi_1)\xi_2.$$

We conclude that if a connection is torsion free then then its curvature operator satisfies the algebraic Bianchi identity.

We denote by Alt_p the alternating operator (without normalizing coefficients!) acting on the first $p \geqslant 2$ entries of a tensor, counted from the left to the right. We notice the following very elementary fact.

Lemma 7. *Let V be a vector space over a field $\mathbb{K}$ of characteristic zero. Then for any integer $p \geqslant 2$, the sequence*

$$0 \longrightarrow S^{p+1}V^* \longrightarrow V^* \otimes S^p V^* \xrightarrow{\operatorname{Alt}_2} \Lambda^2 V^* \otimes S^{p-1}V^* \xrightarrow{\operatorname{Alt}_3} \Lambda^3 V^* \otimes S^{p-2}V^*,$$

is exact.

Proof. The equality

$$S^{p+1}V^* = \operatorname{Ker}(V^* \otimes S^p V^* \xrightarrow{\operatorname{Alt}_2} \Lambda^2 V^* \otimes S^{p-1}V^*),$$

is obvious. We show now the equality

$$\operatorname{Im}(V^* \otimes S^p V^* \xrightarrow{\operatorname{Alt}_2} \Lambda^2 V^* \otimes S^{p-1}V^*) = \operatorname{Ker}(\Lambda^2 V^* \otimes S^{p-1}V^* \xrightarrow{\operatorname{Alt}_3} \Lambda^3 V^* \otimes S^{p-2}V^*). \tag{4.1}$$

We show first the inclusion $\subseteq$ in (4.1). We notice the equality

$$(\Lambda^2 V^* \otimes S^{p-1}V^* \xrightarrow{\operatorname{Alt}_3} \Lambda^3 V^* \otimes S^{p-2}V^*) = (\Lambda^2 V^* \otimes S^{p-1}V^* \xrightarrow{2\operatorname{Circ}} \Lambda^3 V^* \otimes S^{p-2}V^*).$$

Let now $\beta \coloneqq \operatorname{Alt}_2\alpha$, with $\alpha \in V^* \otimes S^p V^*$. Then summing up the two equalities

$$\beta(v_1, v_2;\ v_3, v_4, \dots, v_{p+1}) = \alpha(v_1;\ v_2, v_3, v_4, \dots, v_{p+1}) - \alpha(v_2;\ v_1, v_3, v_4, \dots, v_{p+1}),$$
$$-\beta(v_1, v_3;\ v_2, v_4, \dots, v_{p+1}) = -\alpha(v_1;\ v_3, v_2, v_4, \dots, v_{p+1}) + \alpha(v_3; \mathring{v}_1, v_2, v_4, \dots, v_{p+1}),$$

we obtain

$$\begin{aligned} & \beta(v_1, v_2;\ v_3, v_4, \dots, v_{p+1}) - \beta(v_1, v_3;\ v_2, v_4, \dots, v_{p+1}) \\ = & \ -\alpha(v_2;\ v_3, v_1, v_4, \dots, v_{p+1}) + \alpha(v_3;\ v_2, v_1, v_4, \dots, v_{p+1}) \\ = & \ -\beta(v_2, v_3;\ v_1, v_4, \dots, v_{p+1}), \end{aligned}$$

which rewrites as follows:

$$\beta(v_1, v_2;\ v_3, v_4, \dots, v_{p+1}) + \beta(v_2, v_3;\ v_1, v_4, \dots, v_{p+1}) + \beta(v_3, v_1;\ v_2, v_4, \dots, v_{p+1}) \ = \ 0,$$

i.e., $\operatorname{Circ}\beta = 0$, which shows the inclusion $\subseteq$ in (4.1). To show the reverse inclusion in (4.1), we consider $\beta \in \Lambda^2 V^* \otimes S^{p-1} V^*$ with $\operatorname{Circ}\beta = 0$ and we will prove that $\beta = C_p \operatorname{Alt}_2 \alpha$, with

$$\alpha \ := \ \operatorname{Sym}_{2,\dots,p+1} \beta \in V^* \otimes S^p V^*,$$

and with $C_p \ := \ p/(p+1)!$. Indeed

$$\frac{1}{(p-1)!} \alpha(v_1;\ v_2, \dots, v_{p+1}) = \sum_{j=2}^{p+1} \beta(v_1, v_j;\ v_2, \dots, \hat{v}_j, \dots, v_{p+1})$$

and

$$\begin{aligned} & \frac{1}{(p-1)!} (\operatorname{Alt}_2 \alpha)(v_1, v_2;\ \dots, v_{p+1}) \\ = & \ \frac{1}{(p-1)!} \alpha(v_1;\ v_2, \dots, v_{p+1}) - \frac{1}{(p-1)!} \alpha(v_2;\ v_1, \hat{v}_2, \dots, v_{p+1}) \\ = & \ \sum_{j=2}^{p+1} \beta(v_1, v_j;\ v_2, \dots, \hat{v}_j, \dots, v_{p+1}) - \sum_{\substack{j=1 \\ j \neq 2}}^{p+1} \beta(v_2, v_j;\ v_1, \hat{v}_2, \dots, \hat{v}_j, \dots, v_{p+1}) \\ = & \ \beta(v_1, v_2;\ v_3, \dots, v_{p+1}) + \sum_{j=3}^{p+1} \beta(v_1, v_j;\ v_2, v_3, \dots, \hat{v}_j, \dots, v_{p+1}) \\ & + \beta(v_1, v_2;\ v_3, \dots, v_{p+1}) + \sum_{j=3}^{p+1} \beta(v_j, v_2;\ v_1, \hat{v}_2, v_3, \dots, \hat{v}_j, \dots, v_{p+1}). \end{aligned}$$

By using the circular identity $\operatorname{Circ}\beta = 0$, we obtain

$$\frac{1}{(p-1)!} (\operatorname{Alt}_2 \alpha)(v_1, v_2;\ \dots, v_{p+1}) = 2\beta(v_1, v_2;\ v_3, \dots, v_{p+1}) - \sum_{j=3}^{p+1} \beta(v_2, v_1;\ v_j, v_3, \dots, \hat{v}_j, \dots, v_{p+1}).$$

This combined with the fact that $\beta \in \Lambda^2 V^* \otimes S^{p-1} V^*$ implies

$$\begin{aligned} \frac{1}{(p-1)!} (\operatorname{Alt}_2 \alpha)(v_1, v_2;\ \dots, v_{p+1}) &= 2\beta(v_1, v_2;\ v_3, \dots, v_{p+1}) + (p-1)\beta(v_1, v_2;\ v_3, \dots, v_{p+1}) \\ &= (p+1)\beta(v_1, v_2;\ \dots, v_{p+1}), \end{aligned}$$

which shows the required identity. □

A direct consequence of the proof of Lemma 7 is the following fact.

Corollary 3. *Let* $R \in C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ *satisfying the algebraic Bianchi identity. Then a tensor* $S \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ *satisfies* $3R = \operatorname{Alt}_2 S$ *if and only if* $S = \operatorname{Sym}_{2,3} R + \sigma$, *with* $\sigma \in C^\infty(M, S^3 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$.

We infer by Corollary 3 that equation (3.4) is satisfied by $S_2 = S_2^0 + \sigma_2$, with

$$S_2^0(\xi_1, \xi_2, \xi_3) = \frac{i}{6}[R^{\nabla^{S_1}}(\xi_1, \xi_2)\xi_3 + R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_2], \tag{4.2}$$

and with $\sigma_2 \in C^\infty(M, S^3T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$. We consider now equation (3.5) for $k = 2$, which writes as follows:

$$[d_1^{\nabla^{S_1}} S_2 + 3i \operatorname{Alt}_2 S_3](\xi_1, \xi_2, \eta^2) = 0. \tag{4.3}$$

The fact that the tensor

$$d_1^{\nabla^{S_1}} S_2 + 3i \operatorname{Alt}_2 S_3$$

is symmetric in the last two variables implies that equation (4.3) is equivalent to the equation

$$d_1^{\nabla^{S_1}} S_2 + 3i \operatorname{Alt}_2 S_3 = 0,$$

which we can rewrite under the form

$$d_1^{\nabla^{S_1}} S_2^0 + 3i \operatorname{Alt}_2 \hat{S}_3 = 0, \tag{4.4}$$

with

$$\hat{S}_3 := S_3 - \frac{i}{3}\nabla^{S_1}\sigma_2.$$

Then by using expression (4.2), we can rewrite equation (4.4) in the explicit form

$$\begin{aligned} &\nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_3)\xi_4 + \nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_4)\xi_3 - \nabla^{S_1}_{\xi_2} R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_4 - \nabla^{S_1}_{\xi_2} R^{\nabla^{S_1}}(\xi_1, \xi_4)\xi_3 \\ &= -18[\hat{S}_3(\xi_1, \xi_2, \xi_3, \xi_4) - \hat{S}_3(\xi_2, \xi_1, \xi_3, \xi_4)]. \end{aligned} \tag{4.5}$$

We notice that the fact that the complex connection ∇^{S_1} is torsion free implies that the tensor ρ given by $\rho(\xi_1, \xi_2, \xi_3, \xi_4) := \nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_3)\xi_4$ satisfies the circular identity with respect to the first and last three entries. Moreover, ρ is obviously skew-symmetric with respect to the variables ξ_2, ξ_3.

Lemma 8. *Let ρ be a four-linear form, which satisfies the circular identity with respect to the first and last three entries and which is skew-symmetric with respect to the second and third variables. Then a four-linear form S, which is symmetric with respect to the last three entries satisfies the equation*

$$\operatorname{Alt}_2[8 \operatorname{Sym}_{3,4}\rho - S] = 0, \tag{4.6}$$

if and only if

$$S = -2\operatorname{Sym}_{2,3,4}\rho_2 + \sigma = 2\operatorname{Sym}_{2,3,4}\rho_3 + \sigma,$$

with $\rho_2(\xi_1, \xi_2, \xi_3, \xi_4) := \rho(\xi_2, \xi_1, \xi_3, \xi_4)$, with $\rho_3(\xi_1, \xi_2, \xi_3, \xi_4) := \rho(\xi_2, \xi_3, \xi_1, \xi_4)$, for all $\xi_1, \xi_2, \xi_3, \xi_4 \in T_{M,\pi(\xi_1)}$ and with σ a four-linear form which is symmetric with respect to all its entries.

Proof. We observe first that the assumptions on ρ imply $\operatorname{Circ}\operatorname{Alt}_2 \operatorname{Sym}_{3,4}\rho = 0$. Indeed

$$(\operatorname{Alt}_2 \operatorname{Sym}_{3,4}\rho)(\xi_1, \xi_2, \xi_3, \xi_4) = \rho(\xi_1, \xi_2, \xi_3, \xi_4) + \rho(\xi_1, \xi_2, \xi_4, \xi_3) - \rho(\xi_2, \xi_1, \xi_3, \xi_4) - \rho(\xi_2, \xi_1, \xi_4, \xi_3),$$

and

$$\begin{aligned} &(\operatorname{Circ}\operatorname{Alt}_2 \operatorname{Sym}_{3,4}\rho)(\xi_1, \xi_2, \xi_3, \xi_4) \\ &= \rho(\xi_1, \xi_2, \xi_3, \xi_4)_1 + \rho(\xi_1, \xi_2, \xi_4, \xi_3)_4 - \rho(\xi_2, \xi_1, \xi_3, \xi_4)_2 - \rho(\xi_2, \xi_1, \xi_4, \xi_3)_5 \\ &\quad + \rho(\xi_2, \xi_3, \xi_1, \xi_4)_2 + \rho(\xi_2, \xi_3, \xi_4, \xi_1)_5 - \rho(\xi_3, \xi_2, \xi_1, \xi_4)_3 - \rho(\xi_3, \xi_2, \xi_4, \xi_1)_6 \\ &\quad + \rho(\xi_3, \xi_1, \xi_2, \xi_4)_3 + \rho(\xi_3, \xi_1, \xi_4, \xi_2)_6 - \rho(\xi_1, \xi_3, \xi_2, \xi_4)_1 - \rho(\xi_1, \xi_3, \xi_4, \xi_2)_4, \end{aligned}$$

where we denote by $\rho(\cdot,\cdot,\cdot,\cdot)_j$ the terms we group together. By using the assumption that ρ is skew-symmetric with respect to the second and third variables, we infer

$$
\begin{aligned}
&(\mathrm{CircAlt}_2\mathrm{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)\\
&\quad= 2\rho(\xi_1,\xi_2,\xi_3,\xi_4)_1+\rho(\xi_1,\xi_2,\xi_4,\xi_3)_4+\rho(\xi_2,\xi_4,\xi_1,\xi_3)_5+2\rho(\xi_2,\xi_3,\xi_1,\xi_4)_1+\rho(\xi_2,\xi_3,\xi_4,\xi_1)_5\\
&\qquad+\rho(\xi_3,\xi_4,\xi_2,\xi_1)_6+2\rho(\xi_3,\xi_1,\xi_2,\xi_4)_1+\rho(\xi_3,\xi_1,\xi_4,\xi_2)_6+\rho(\xi_1,\xi_4,\xi_3,\xi_2)_4.
\end{aligned}
$$

By using the circular assumptions on ρ, we infer

$$(\mathrm{CircAlt}_2\mathrm{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)=-\rho(\xi_1,\xi_3,\xi_2,\xi_4)-\rho(\xi_2,\xi_1,\xi_3,\xi_4)-\rho(\xi_3,\xi_2,\xi_1,\xi_4)=0.$$

Then by the proof of Lemma 7 in the case $p=3$, we infer that a four-linear form S, which is symmetric with respect to the last three entries satisfies equation (4.6) if and only if

$$S=\mathrm{Sym}_{2,3,4}\mathrm{Alt}_2\mathrm{Sym}_{3,4}\rho+\sigma,$$

with σ any four-linear form, which is symmetric with respect to all its entries, satisfies (4.6). We write now

$$
\begin{aligned}
&(\mathrm{Sym}_{2,3,4}\mathrm{Alt}_2\mathrm{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)\\
&\quad= \rho(\xi_1,\xi_2,\xi_3,\xi_4)_1+\rho(\xi_1,\xi_2,\xi_4,\xi_3)_2-\rho(\xi_2,\xi_1,\xi_3,\xi_4)_3-\rho(\xi_2,\xi_1,\xi_4,\xi_3)_4\\
&\qquad+\rho(\xi_1,\xi_2,\xi_4,\xi_3)_1+\rho(\xi_1,\xi_2,\xi_3,\xi_4)_2-\rho(\xi_2,\xi_1,\xi_4,\xi_3)_4-\rho(\xi_2,\xi_1,\xi_3,\xi_4)_3\\
&\qquad+\rho(\xi_1,\xi_3,\xi_2,\xi_4)_1+\rho(\xi_1,\xi_3,\xi_4,\xi_2)_2-\rho(\xi_3,\xi_1,\xi_2,\xi_4)_5-\rho(\xi_3,\xi_1,\xi_4,\xi_2)_6\\
&\qquad+\rho(\xi_1,\xi_3,\xi_4,\xi_2)_1+\rho(\xi_1,\xi_3,\xi_2,\xi_4)_2-\rho(\xi_3,\xi_1,\xi_4,\xi_2)_6-\rho(\xi_3,\xi_1,\xi_2,\xi_4)_5\\
&\qquad+\rho(\xi_1,\xi_4,\xi_2,\xi_3)_1+\rho(\xi_1,\xi_4,\xi_3,\xi_2)_2-\rho(\xi_4,\xi_1,\xi_2,\xi_3)_7-\rho(\xi_4,\xi_1,\xi_3,\xi_2)_8\\
&\qquad+\rho(\xi_1,\xi_4,\xi_3,\xi_2)_1+\rho(\xi_1,\xi_4,\xi_2,\xi_3)_2-\rho(\xi_4,\xi_1,\xi_3,\xi_2)_8-\rho(\xi_4,\xi_1,\xi_2,\xi_3)_7.
\end{aligned}
$$

The fact that ρ is skew-symmetric with respect to the second and third variables implies that $\mathrm{Sym}_{2,3,4}\rho=0$. We infer

$$
\begin{aligned}
(\mathrm{Sym}_{2,3,4}\mathrm{Alt}_2\mathrm{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)&=-2\rho(\xi_2,\xi_1,\xi_3,\xi_4)-2\rho(\xi_2,\xi_1,\xi_4,\xi_3)-2\rho(\xi_3,\xi_1,\xi_2,\xi_4)-2\rho(\xi_3,\xi_1,\xi_4,\xi_2)\\
&\quad-2\rho(\xi_4,\xi_1,\xi_2,\xi_3)-2\rho(\xi_4,\xi_1,\xi_3,\xi_2)\\
&=2\rho(\xi_2,\xi_3,\xi_1,\xi_4)+2\rho(\xi_2,\xi_4,\xi_1,\xi_3)+2\rho(\xi_3,\xi_2,\xi_1,\xi_4)+2\rho(\xi_3,\xi_4,\xi_1,\xi_2)\\
&\quad+2\rho(\xi_4,\xi_2,\xi_1,\xi_3)+2\rho(\xi_4,\xi_3,\xi_1,\xi_2),
\end{aligned}
$$

which shows the required expressions for S. □

By equation (4.5), we can apply Lemma 8 to the tensor $\rho := \nabla^{S_1}R^{\nabla^{S_1}}$. We infer the equation

$$S_3=\frac{1}{4!3}\mathrm{Sym}_{2,3,4}(\nabla^{S_1}R^{\nabla^{S_1}})_2+\frac{i}{3}\nabla^{S_1}\sigma_2+\sigma_3. \tag{4.7}$$

We deduce that equation (4.4) is equivalent to equation (4.7). This concludes the proof of the Proposition 2 thanks to Lemma 6.

5 Third reduction of the integrability equations and proof of the main theorem

In this section, we will prove the following result.

Lemma 9. *Under the assumptions of the Theorem* 2, *the M-totally real almost complex structure J_A is integrable over U if and only if*

$$\begin{aligned}
S_1 &\in C^\infty(M, S^2T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M), \quad \textit{i.e.}, \ \nabla^{S_1} \textit{ is torsion free},\\
S_2 &= S_2^0 + \sigma_2,\\
S_2^0(\xi_1, \xi_2, \xi_3) &\coloneqq \frac{i}{6}[R^{\nabla^{S_1}}(\xi_1, \xi_2)\xi_3 + R^{\nabla^{S_1}}(\xi_1, \xi_3)\xi_2],\\
\sigma_2 &\in C^\infty(M, S^3T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),
\end{aligned}$$

and for all $k \geqslant 3$,

$$\begin{aligned}
S_k &= \frac{i}{k}\nabla^{S_1}\sigma_{k-1} + \frac{i}{(k+1)!}\operatorname{Sym}_{2,\dots,k+1}\beta_{k-1} + \sigma_k,\\
\sigma_k &\in C^\infty(M, S^{k+1}T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),\\
\beta_k &\coloneqq \frac{i}{k}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{k-1} + \frac{i}{(k+1)!}d_1^{\nabla^{S_1}}\operatorname{Sym}_{2,\dots,k+1}\beta_{k-1}\\
&\quad + \frac{1}{k!}\operatorname{Sym}_{3,\dots,k+2}\left(\sum_{p=2}^{k-1} pS_p \wedge_1 S_{k-p+1}\right),\\
\beta_2 &\coloneqq -\frac{i}{3}(\nabla^{S_1}R^{\nabla^{S_1}})_2,\\
\operatorname{Circ}\beta_k &= 0.
\end{aligned}$$

Proof. We show that the statement of proposition 2 is equivalent to the statement of Lemma 9. We show indeed by induction on $k \geqslant 3$ the following statement.

Statement 1. The tensors S_h, $h = 3, \dots, k+1$, satisfy the equations

$$\left[d_1^{\nabla^{S_1}}S_h + \sum_{p=2}^{h-1} pS_p \wedge_1 S_{h-p+1} + i(h+1)\operatorname{Alt}_2 S_{h+1}\right](\xi_1, \xi_2, \eta^h) = 0, \tag{5.1}$$

for all $h = 3, \dots, k$, for all $\xi_1, \xi_2, \eta \in T_{M,\pi(\eta)}$ and

$$S_3 = \frac{i}{3}\nabla^{S_1}\sigma_2 + \frac{1}{4!3}\operatorname{Sym}_{2,3,4}(\nabla^{S_1}R^{\nabla^{S_1}})_2 + \sigma_3,$$

with $\sigma_3 \in C^\infty(M, S^4T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$, if and only if the tensors S_h satisfy for all $h = 3, \dots, k+1$, the identities

$$S_h = \frac{i}{h}\nabla^{S_1}\sigma_{h-1} + \frac{i}{(h+1)!}\operatorname{Sym}_{2,\dots,h+1}\beta_{h-1} + \sigma_h, \tag{5.2}$$

with $\sigma_h \in C^\infty(M, S^{h+1}T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M)$ and where for all $r = 3, \dots, k$,

$$\beta_r \coloneqq \frac{i}{r}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{r-1} + \frac{i}{(r+1)!}d_1^{\nabla^{S_1}}\operatorname{Sym}_{2,\dots,r+1}\beta_{r-1} + \frac{1}{r!}\operatorname{Sym}_{3,\dots,r+2}\left(\sum_{p=2}^{r-1} pS_p \wedge_1 S_{r-p+1}\right),$$

with $\beta_2 \coloneqq -\frac{i}{3}(\nabla^{S_1}R^{\nabla^{S_1}})_2$ satisfies equation $\operatorname{Circ}\beta_r = 0$.

Statement 1 follows directly from the following fact.

Fact 1. Let S_h, for some $h = 3, \dots, k$, be the tensor given by (5.2). Then the tensor S_{h+1} satisfies equation (5.1) if and only if S_{h+1} satisfies identity (5.2), with h replaced by $h+1$ and β_h satisfies equation $\operatorname{Circ}\beta_h = 0$.

To show the fact 1 we observe first that (5.1) rewrites as follows:

$$d_1^{\nabla^{S_1}}S_h + \frac{1}{h!}\operatorname{Sym}_{3,\dots,h+2}\left(\sum_{p=2}^{h-1} pS_p \wedge_1 S_{h-p+1}\right) + i(h+1)\operatorname{Alt}_2 S_{h+1} = 0.$$

By using expression (5.2) for S_h and the definition of β_h, we can rewrite the previous identity as follows:

$$\beta_h = -\operatorname{Alt}_2[\nabla^{S_1}\sigma_h + i(h+1)S_{h+1}]. \tag{5.3}$$

By the proof of Lemma 7, we deduce $\operatorname{Circ}\beta_h = 0$ and

$$-\nabla^{S_1}\sigma_h - i(h+1)S_{h+1} = C_{h+1}\operatorname{Sym}_{2,\dots,h+2}\beta_h - i(h+1)\sigma_{h+1}.$$

Therefore, identity (5.3) is equivalent to $\operatorname{Circ}\beta_h = 0$, and S_{h+1} satisfies (5.2), with h replaced by $h+1$. This concludes the proof fact 1. We infer the required conclusion of Lemma 9. □

Proof of the main theorem

Proof. We show that the recursive definition of β_k in the statement of Lemma 9 yields the formula

$$\begin{aligned}
\beta_k &= \frac{i}{k}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{k-1} + \frac{1}{(k+1)!k!}\operatorname{Sym}_{3,\dots,k+2}\theta_k,\\
\theta_k &:= \sum_{r=2}^{k-2}\frac{(r+2)!}{r+1}(id_1^{\nabla^{S_1}})^{k-r}\nabla^{S_1}\sigma_r + 3!(id_1^{\nabla^{S_1}})^{k-2}\beta_2\\
&\quad + \sum_{r=3}^{k}(r+1)!\sum_{p=2}^{r-1}(id_1^{\nabla^{S_1}})^{k-r}(pS_p \wedge_1 S_{r-p+1}),
\end{aligned} \tag{5.4}$$

for all $k \geqslant 3$. We show (5.4) by induction on k. We notice first that the recursive definition of β_k rewrites as follows:

$$\beta_k = \frac{i}{k}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{k-1} + \operatorname{Sym}_{3,\dots,k+2}\left[\frac{i}{(k+1)!}d_1^{\nabla^{S_1}}\beta_{k-1} + \frac{1}{k!}\sum_{p=2}^{k-1}pS_p \wedge_1 S_{k-p+1}\right],$$

and we write

$$\beta_{k+1} = \frac{i}{k+1}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_k + \operatorname{Sym}_{3,\dots,k+3}\left[\frac{i}{(k+2)!}d_1^{\nabla^{S_1}}\beta_k\right], + \operatorname{Sym}_{3,\dots,k+3}\left[\frac{1}{(k+1)!}\sum_{p=2}^{k}pS_p \wedge_1 S_{k-p+2}\right].$$

By using the inductive assumption (5.4), we infer the expressions

$$\begin{aligned}
\frac{i}{(k+2)!}d_1^{\nabla^{S_1}}\beta_k &= \frac{1}{(k+2)!k}(id_1^{\nabla^{S_1}})^2\nabla^{S_1}\sigma_{k-1} + \frac{1}{(k+2)!(k+1)!k!}\operatorname{Sym}_{4,\dots,k+3}id_1^{\nabla^{S_1}}\theta_k,\\
id_1^{\nabla^{S_1}}\theta_k &= \sum_{r=2}^{k-2}\frac{(r+2)!}{r+1}(id_1^{\nabla^{S_1}})^{k+1-r}\nabla^{S_1}\sigma_r + 3!(id_1^{\nabla^{S_1}})^{k-1}\beta_2 + \sum_{r=3}^{k}(r+1)!\sum_{p=2}^{r-1}(id_1^{\nabla^{S_1}})^{k+1-r}(pS_p \wedge_1 S_{r-p+1}).
\end{aligned}$$

This combined with the identity $\operatorname{Sym}_{3,\dots,k+3}\operatorname{Sym}_{4,\dots,k+3} = k!\operatorname{Sym}_{3,\dots,k+3}$, yields

$$\begin{aligned}
\beta_{k+1} &= \frac{i}{k+1}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_k + \frac{1}{(k+2)!k}\operatorname{Sym}_{3,\dots,k+3}(id_1^{\nabla^{S_1}})^2\nabla^{S_1}\sigma_{k-1}\\
&\quad + \frac{1}{(k+2)!(k+1)!}\operatorname{Sym}_{3,\dots,k+3}\sum_{r=2}^{k-2}\frac{(r+2)!}{r+1}(id_1^{\nabla^{S_1}})^{k+1-r}\nabla^{S_1}\sigma_r\\
&\quad + \frac{3!}{(k+2)!(k+1)!}\operatorname{Sym}_{3,\dots,k+3}(id_1^{\nabla^{S_1}})^{k-1}\beta_2\\
&\quad + \frac{1}{(k+2)!(k+1)!}\operatorname{Sym}_{3,\dots,k+3}\sum_{r=3}^{k}(r+1)!\sum_{p=2}^{r-1}(id_1^{\nabla^{S_1}})^{k+1-r}(pS_p \wedge_1 S_{r-p+1})\\
&\quad + \frac{1}{(k+1)!}\operatorname{Sym}_{3,\dots,k+3}\sum_{p=2}^{k}pS_p \wedge_1 S_{k-p+2}.
\end{aligned}$$

Putting the terms together, we obtain (5.4) for β_{k+1}. Then the obvious identity $d_1^{\nabla}\nabla = \operatorname{Alt}_2\nabla^2$, combined with the formula (5.5), allows to conclude the required expression of $\beta_k \equiv \beta_k(\sigma_{k-1})$ in the statement of the main theorem.

(We perform the change of indices $r' := r + 1$ in the above expression of θ_k.) This concludes the proof of the main theorem. □

We remind first the following elementary and well-known fact.

Lemma 10. *For any covariant derivative operator ∇ acting on the smooth sections of $\mathbb{C}T_M$ and for any tensor $\theta \in C^\infty(X, T_M^{*,\otimes q} \otimes \mathbb{C}T_M)$ holds the identity*

$$\mathrm{Alt}_2 \nabla^2 \theta = R^\nabla \cdot \theta. \tag{5.5}$$

6 The symplectic approach

6.1 General facts

Let M be a smooth manifold and let $\theta \in C^\infty(T_M^*, T_{T_M^*}^*)$ be the canonical 1-form on the total space of the cotangent bundle defined as $\theta_\lambda := \lambda \cdot d_\lambda \pi_{T_M^*}$, for any $\lambda \in T_M^*$. The canonical symplectic form over the total space T_M^* is defined as $\Omega := -d\theta$. Let now g be a Riemann metric over M viewed as a vector bundle map $g : T_M \longrightarrow T_M^*$. We define also the forms $\theta^g := g^*\theta$ and $\Omega^g := g^*\Omega = -d\theta^g$ over the total space of the tangent bundle. In explicit terms $\theta_\eta^g = g(\eta) \cdot d_\eta \pi_{T_M}$, for all $\eta \in T_M$, i.e.,

$$\theta_\eta^g(\xi) = g_{\pi_{T_M}(\eta)}(\eta, d_\eta \pi_{T_M} \cdot \xi),$$

for all $\xi \in T_{T_M,\eta}$. Let ∇^g be the Levi-Civita connection, defined as

$$2\nabla_\xi^g \eta := g^{-1}[\xi \neg d(g\eta) + \eta \neg d(g\xi) + d\langle \xi, \eta \rangle_g] + [\xi, \eta],$$

for any $\xi, \eta \in C^\infty(M, T_M)$. Let also $\gamma^g \in C^\infty(T_M, T_{T_M}^* \otimes T_{T_M})$ be the Levi-Civita 1-form, which is determined along any section $\eta \in C^\infty(M, T_M)$, by the identity $\gamma_\eta^g \cdot d\eta = T_\eta \nabla^g \eta$.

For any curve $\eta : t \longmapsto \eta_t \in T_M$, we define the covariant derivative

$$\frac{\nabla^g \eta}{dt} := T_{\eta_t}^{-1} \gamma_{\eta_t}^g \dot{\eta}_t \in T_{M,\pi(\eta_t)}.$$

We consider now two curves $\eta_j : t \longmapsto \eta_{j,t} \in T_M$, $j = 1, 2$, such that $\pi_{T_M}(\eta_{1,t}) = \pi_{T_M}(\eta_{2,t}) = x_t$. Then

$$\frac{d}{dt} g_{|x_t}(\eta_{1,t}, \eta_{2,t}) = g_{|x_t}\left(\frac{\nabla^g \eta_1}{dt}, \eta_{2,t}\right) + g_{|x_t}\left(\eta_{1,t}, \frac{\nabla^g \eta_2}{dt}\right).$$

With the previous notation hold the following well-known lemma (see also Klingenberg's book [12] for a proof using local coordinates).

Lemma 11. *The formula*

$$\Omega_\eta^g(\xi_1, \xi_2) = g_p(d_\eta \pi_{T_M} \xi_1, T_\eta^{-1} \gamma_\eta^g \xi_2) - g_p(d_\eta \pi_{T_M} \xi_2, T_\eta^{-1} \gamma_\eta^g \xi_1),$$

hold for any $\eta \in T_M$, $p = \pi_{T_M}(\eta)$ and for any $\xi_1, \xi_2 \in T_{T_M,\eta}$.

Proof. With respect to a local coordinate trivialization of the tangent bundle, we can extend in a linear way the vectors ξ_1 and ξ_2 in to vector fields Ξ_1, Ξ_2 in a neighborhood of $T_{M,p}$ inside T_M. In this way, $[\Xi_1, \Xi_2] = 0$, and thus, $\Omega^g(\Xi_1, \Xi_2) = \Xi_2 . \, \theta^g(\Xi_1) - \Xi_1 . \, \theta^g(\Xi_2)$. We denote by $\eta_{j,t}$, $j = 1, 2$ the corresponding flow lines starting from η. Then

$$\Omega_\eta^g(\xi_1, \xi_2) = \frac{d}{dt}_{|t=0} [g_{\pi_{T_M}(\eta_{2,t})}(\eta_{2,t}, d_{\eta_{2,t}} \pi_{T_M} \cdot \Xi_1(\eta_{2,t}))] - \frac{d}{dt}_{|t=0} [g_{\pi_{T_M}(\eta_{1,t})}(\eta_{1,t}, d_{\eta_{1,t}} \pi_{T_M} \cdot \Xi_2(\eta_{1,t}))].$$

We distinguish two cases.

- In the case when $d_\eta \pi_{T_M} \xi_j = 0$ for some j, say $j = 1$, then $d_{\eta_{2,t}} \pi_{T_M} \Xi_1(\eta_{2,t}) = 0$ and

$$\frac{d}{dt} d_{\eta_{1,t}} \pi_{T_M} \Xi_2(\eta_{1,t}) = 0,$$

by the linear nature of the local extension. Then

$$\Omega^g_\eta(\xi_1, \xi_2) = -g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2).$$

The case $j = 2$ is quite similar.

- In the case when $d_\eta \pi_{T_M} \xi_j$, do not vanish for $j = 1, 2$, then the vector fields $\zeta_j := d\pi_{T_M} \Xi_j$ are well defined and $[\zeta_1, \zeta_2] = 0$. Then

$$\begin{aligned}\Omega^g_\eta(\xi_1, \xi_2) &= g_p(T^{-1}_\eta \gamma^g_\eta \xi_2, d_\eta \pi_{T_M} \xi_1) + g_p(\eta, \nabla^g_{\zeta_2(p)} \zeta_1 - \nabla^g_{\zeta_1(p)} \zeta_2) - g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2)\\ &= g_p(T^{-1}_\eta \gamma^g_\eta \xi_2, d_\eta \pi_{T_M} \xi_1) + g_p(\eta, [\zeta_1, \zeta_2](p)) - g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2),\end{aligned}$$

which implies the required conclusion. □

We need to remind in detail also the following very well-known lemma [12].

Lemma 12. *Let $2\zeta^g := \Omega^{g,-1} d|\cdot|^2_g$ and let Φ^g_t be the corresponding one-parameter subgroup of transformations of T_M. Then for any $\eta \in T_M$, the curve $c_t := \pi_{T_M} \circ \Phi^g_t(\eta)$ is the geodesic with initial speed $\dot{c}_0 = \eta$ and $\dot{c}_t = \Phi^g_t(\eta)$.*

Proof. For any $\eta \in T_M$ and for any $\xi \in T_{T_M,\eta}$, let $t \longmapsto \eta_t \in T_M$ be the curve such that $\dot{\eta}_0 = \xi$. Then

$$\xi.\ |\cdot|^2_g = \frac{d}{dt}_{|t=0} [g_{\pi_{T_M}(\eta_t)}(\eta_t, \eta_t)] = 2g_p(\eta, T^{-1}_\eta \gamma^g_\eta \xi),$$

and thus,

$$\Omega^g_\eta(\zeta^g_\eta, \xi) = g_p(\eta, T^{-1}_\eta \gamma^g_\eta \xi),$$

by the definition of the vector field ζ^g_η. By using Lemma 11, we infer

$$g_p(d_\eta \pi_{T_M} \zeta^g_\eta, T^{-1}_\eta \gamma^g_\eta \xi) - g_p(d_\eta \pi_{T_M} \xi, T^{-1}_\eta \gamma^g_\eta \zeta^g_\eta) = g_p(\eta, T^{-1}_\eta \gamma^g_\eta \xi). \tag{6.1}$$

In the case $d_\eta \pi_{T_M} \xi = 0$, the identity (6.1) yields

$$g_p(d_\eta \pi_{T_M} \zeta^g_\eta, T^{-1}_\eta \xi) = g_p(\eta, T^{-1}_\eta \xi),$$

and thus, $d_\eta \pi_{T_M} \zeta^g_\eta = \eta$. In the case $\gamma^g_\eta \xi = 0$, the identity (6.1) yields

$$g_p(d_\eta \pi_{T_M} \xi, T^{-1}_\eta \gamma^g_\eta \zeta^g_\eta) = 0,$$

and thus, $\gamma^g_\eta \zeta^g_\eta = 0$. We deduce the formula

$$\zeta^g_\eta = H^g_\eta \cdot \eta. \tag{6.2}$$

Thus, the flow line $\eta_t := \Phi^g_t(\eta)$ satisfies the identity

$$\dot{\eta}_t = H^g_{\eta_t} \cdot \eta_t. \tag{6.3}$$

We deduce

$$\dot{c}_t = d_{\eta_t} \pi_{T_M} \cdot \dot{\eta}_t = d_{\eta_t} \pi_{T_M} \cdot H^g_{\eta_t} \cdot \eta_t = \eta_t,$$

and $\ddot{c}_t = H^g_{\dot{c}_t} \cdot \dot{c}_t$, which is the geodesic equation. □

We provide now a proof of the following well-known result due to Lempert and Szöke [14]. See also, Guillemin and Stenzel [10], Burns [2,3], and Burns et al. [4].

Corollary 4. *Let (M, g) be a smooth Riemannian manifold. A complex structure J over the total space of the tangent bundle T_M satisfies the conditions*

$$J_{|M} = J^{\,\mathrm{can}}\,, \tag{6.4}$$

$$2\theta^g = d|\cdot|_g^2 \cdot J, \tag{6.5}$$

if and only if for any $\eta \in T_M$, the smooth map $\psi_\eta : t + is \longmapsto s\Phi_t^g(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J-holomorphic.

Proof. We define the Reeb vector field $\Xi := \Omega^{g,-1}\theta^g$. This vector field is independent of the metric g. Indeed by Lemma 11, the identity

$$g_p(\eta, d_\eta\pi_{T_M}\xi) = g_p(d_\eta\pi_{T_M}\Xi_\eta, T_\eta^{-1}\gamma_\eta^g\xi) - g_p(d_\eta\pi_{T_M}\xi, T_\eta^{-1}\gamma_\eta^g\Xi_\eta), \tag{6.6}$$

holds for any $\xi \in T_{T_M,\eta}$. Thus, if $d_\eta\pi_{T_M}\xi = 0$, we deduce the equality

$$g_p(d_\eta\pi_{T_M}\Xi_\eta, T_\eta^{-1}\xi) = 0,$$

and thus, $d_\eta\pi_{T_M}\Xi_\eta = 0$. Then the identity (6.6) reduces as follows:

$$g_p(\eta, d_\eta\pi_{T_M}\xi) = -g_p(d_\eta\pi_{T_M}\xi, T_\eta^{-1}\Xi_\eta),$$

for any $\xi \in T_{T_M,\eta}$. We infer the formula

$$\Xi_\eta = -T_\eta \cdot \eta, \tag{6.7}$$

for all $\eta \in T_M$. We notice now that the identity (6.5) is equivalent to the identity

$$\Omega^g(2\Xi, \xi) = d|\cdot|_g^2 J\xi$$

and is also equivalent to the identity $\theta^g = -d_J^c|\cdot|_g^2$. Thus,

$$\Omega^g = dd_J^c|\cdot|_g^2 = i\partial_J\bar{\partial}_J|\cdot|_g^2,$$

thanks to the fact that J^g is integrable. We infer that the symplectic form Ω^g is J-invariant. Thus,

$$\Omega^g(2J\Xi, J\xi) = d|\cdot|_g^2 J\xi,$$

i.e.,

$$J\Xi = \zeta^g. \tag{6.8}$$

This combined with (6.7) and with (6.2) implies that (6.5) is equivalent to the identity

$$J_\eta H_\eta^g \cdot \eta = T_\eta \cdot \eta. \tag{6.9}$$

We show now that the later combined with (6.4) is equivalent to the J-holomorphy of the maps ψ_η. For this purpose, we observe that the differential of such maps is given by

$$d_{t_0+is_0}\psi_\eta\left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s}\right) = ad(s_0\mathbb{1}_{T_M})\dot{\Phi}_{t_0}^g(\eta) + bT_{s_0\Phi_{t_0}^g(\eta)}\Phi_{t_0}^g(\eta).$$

But

$$\dot{\Phi}_{t_0}^g(\eta) = \zeta^g \circ \Phi_{t_0}^g(\eta) = H_{\Phi_{t_0}^g(\eta)}^g \cdot \Phi_{t_0}^g(\eta),$$

thanks to (6.2). Then by using the property (A5) of the linear connection ∇^g, we infer

$$d_{t_0+is_0}\psi_\eta\left(a\frac{\partial}{\partial t}+b\frac{\partial}{\partial s}\right)=(aH^g_{s_0\Phi^g_{t_0}(\eta)}+bT_{s_0\Phi^g_{t_0}(\eta)})\cdot\Phi^g_{t_0}(\eta). \tag{6.10}$$

The smooth map ψ_η is J-holomorphic if and only if

$$d_{t_0+is_0}\psi_\eta\left(-b\frac{\partial}{\partial t}+a\frac{\partial}{\partial s}\right)=Jd_{t_0+is_0}\psi_\eta\left(a\frac{\partial}{\partial t}+b\frac{\partial}{\partial s}\right),$$

thus, if and only if

$$(-bH^g_{s_0\Phi^g_{t_0}(\eta)}+aT_{s_0\Phi^g_{t_0}(\eta)})\cdot\Phi^g_{t_0}(\eta)=J(aH^g_{s_0\Phi^g_{t_0}(\eta)}+bT_{s_0\Phi^g_{t_0}(\eta)})\cdot\Phi^g_{t_0}(\eta).$$

For $s_0\neq 0$, this is equivalent to (6.9). For $s_0=0$, this is equivalent to (6.4). We deduce the required conclusion. □

The condition (6.4) implies that J is an M-totally real complex structure. We can provide now the proof of Corollary 1.

6.2 Proof of Corollary 1

Proof. If we write $A=\alpha+iTB$ and $\alpha=H^g-T\Gamma$, then

$$S\;:=\;T^{-1}(H^\nabla-\overline{A})=\Gamma+iB.$$

We set $S_k=\Gamma_k+iB_k$. From the proof of Corollary 4, we know that in the case J is integrable over U, the curve ψ_η is J-holomorphic if and only if hold (6.9). The later rewrites as follows:

$$H^g_\eta\cdot\eta=-J_\eta T_\eta\cdot\eta.$$

By using (1.7), we infer that the previous identity is equivalent to

$$H^g_\eta\cdot\eta=\alpha_\eta B^{-1}_\eta\cdot\eta. \tag{6.11}$$

Taking $d_\eta\pi$ on both sides of (6.11), we deduce $\eta=B^{-1}_\eta\cdot\eta$. Therefore, (6.11) is equivalent to the system

$$\begin{cases}B_\eta\cdot\eta=\eta,\\ H^g_\eta\cdot\eta=\alpha_\eta\cdot\eta.\end{cases} \tag{6.12}$$

Then the system (6.12) rewrites as

$$\begin{cases}\sum\limits_{k\geqslant 1}B_k(\eta^{k+1})=0,\\ \sum\limits_{k\geqslant 1}\Gamma_k(\eta^{k+1})=0.\end{cases}$$

and thus as $S_k(\eta^{k+1})=0$ for all $k\geqslant 1$. We remind now that, according to Theorem 2, the integrability of the structure J implies the condition $S_1\in C^\infty(M,S^2T^*_M\otimes_{\mathbb{R}}\mathbb{C}T_M)$. We infer $S_1=0$. We notice that, with the notation of the statement of Theorem 2, the equation $\mathrm{Circ}\,\beta_k=0$ holds for all $k\geqslant 1$. This combined with the identity

$$[\mathrm{Circ},\mathrm{Sym}_{2,\dots,k+2}]=0,$$

implies

$$\mathrm{Circ}\mathrm{Sym}_{2,\dots,k+2}\beta_k=0, \tag{6.13}$$

for all $k\geqslant 1$. So if we apply the Circ operator to both sides of the definition of S_2 in the statement of Theorem 2, we infer $\mathrm{Circ}S_2=\mathrm{Circ}\,\sigma_2=3\sigma_2$. If we evaluate this equality to η^3, we infer $S_2(\eta^3)=\sigma_2(\eta^3)$, which implies $\sigma_2=0$. We show now by induction that $\sigma_k=0$ for all $k\geqslant 2$. Indeed by the inductive assumption,

$$S_{k+1} = \frac{i}{(k+2)!}\operatorname{Sym}_{2,\dots,k+2}\beta_k + \sigma_{k+1}.$$

By applying the Circ operator to both sides of this identity and using the equation (6.13), we infer $\operatorname{Circ} S_{k+1} = \operatorname{Circ}\sigma_{k+1} = 3\sigma_{k+1}$, which evaluated at η^{k+2} gives $S_{k+1}(\eta^{k+2}) = \sigma_{k+1}(\eta^{k+2})$. We deduce $\sigma_{k+1} = 0$. By using the identity

$$\operatorname{Sym}_{2,\dots,k+1}\operatorname{Sym}_{3,\dots,k+1} = (k-1)!\operatorname{Sym}_{2,\dots,k+1}, \tag{6.14}$$

we infer from the statement of Theorem 2 and with the notation there

$$S_k = \frac{i}{(k+1)!k!}\operatorname{Sym}_{2,\dots,k+1}\theta_{k-1},$$

for $k \geqslant 2$, with $\theta_1 := 2R^g$ and

$$\theta_k := -2i(id_1^{\nabla^g})^{k-2}(\nabla^g R^g)_2 + \sum_{r=4}^{k+1} r! \sum_{p=2}^{r-2} (id_1^{\nabla^g})^{k+1-r}(pS_p \wedge_1 S_{r-p}),$$

for all $k \geqslant 2$. Moreover, we observe that the equation $\operatorname{Circ}\beta_k = 0$, $k \geqslant 3$ rewrites as

$$\operatorname{Circ}\operatorname{Sym}_{3,\dots,k+2}\theta_k = 0.$$

If we set $\Theta_k(g) := \theta_{k-1}$, for all $k \geqslant 2$, we obtain the required expansion.

On the other hand, if the expansion in the statement of the lemma under consideration hold, then J is integrable thanks to Theorem 2 and $\operatorname{Circ} S_k = 0$ for all $k \geqslant 2$, ($S_1 = 0$). Indeed for $k = 2, 3$, this equality follows from the identities $\operatorname{Circ}\Theta_k(g) = 0$ and

$$[\operatorname{Circ}, \operatorname{Sym}_{2,\dots,k+1}] = 0. \tag{6.15}$$

For $k \geqslant 4$, we use the identities (6.15) and (6.14) and the integrability equations satisfied by the metric g. We deduce $S_k(\eta^{k+1}) = 0$, for all $k \geqslant 1$, which is equivalent to (6.11) and so to the fact that the curves ψ_η are J-holomorphic. □

7 Proof of the proposition 1

Proof. We expand first the term

$$\begin{aligned}
d_1^{\nabla}(\nabla R^{\nabla})_2(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) &= \nabla_{\xi_1}(\nabla R^{\nabla})_2(\xi_2, \xi_3, \xi_4, \xi_5) - \nabla_{\xi_2}(\nabla R^{\nabla})_2(\xi_1, \xi_3, \xi_4, \xi_5) \\
&= \nabla^2 R^{\nabla}(\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) - \nabla^2 R^{\nabla}(\xi_2, \xi_3, \xi_1, \xi_4, \xi_5) \\
&= \nabla^2 R^{\nabla}(\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + \nabla^2 R^{\nabla}(\xi_2, \xi_3, \xi_4, \xi_1, \xi_5) \\
&= \nabla^2 R^{\nabla}(\xi_3, \xi_1, \xi_2, \xi_4, \xi_5) + \nabla^2 R^{\nabla}(\xi_3, \xi_2, \xi_4, \xi_1, \xi_5) \\
&\quad + (R^{\nabla}. R^{\nabla})(\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + (R^{\nabla}. R^{\nabla})(\xi_2, \xi_3, \xi_4, \xi_1, \xi_5),
\end{aligned}$$

thanks to formula (5.5). By using the differential Bianchi identity, we infer

$$d_1^{\nabla}(\nabla R^{\nabla})_2(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = -\nabla^2 R^{\nabla}(\xi_3, \xi_4, \xi_1, \xi_2, \xi_5) + (R^{\nabla} \cdot R^{\nabla})(\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + (R^{\nabla} \cdot R^{\nabla})(\xi_2, \xi_3, \xi_4, \xi_1, \xi_5).$$

To simplify the notation in the computations that will follow we will use from now on the identification

$$\theta(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \equiv \theta(12345),$$

for any tensor θ. We expand now the term

$$\operatorname{Circ}\operatorname{Sym}_{3,4,5} d_1^{\nabla}(\nabla R^{\nabla})_2.$$

We let

$$\theta(12345) := \nabla^2 R^{\nabla}(34125),$$

and we observe the identities

$$
\begin{aligned}
(\operatorname{Sym}_{3,4,5}\theta)(12345) &= \nabla^2 R^\nabla(34125) + \nabla^2 R^\nabla(35124) + \nabla^2 R^\nabla(43125)\\
&\quad + \nabla^2 R^\nabla(45123) + \nabla^2 R^\nabla(53124) + \nabla^2 R^\nabla(54123),\\
(\operatorname{Sym}_{3,4,5}\theta)(23145) &= \nabla^2 R^\nabla(14235) + \nabla^2 R^\nabla(15234) + \nabla^2 R^\nabla(41235)\\
&\quad + \nabla^2 R^\nabla(45231) + \nabla^2 R^\nabla(51234) + \nabla^2 R^\nabla(54231),\\
(\operatorname{Sym}_{3,4,5}\theta)(31245) &= \nabla^2 R^\nabla(24315) + \nabla^2 R^\nabla(25314) + \nabla^2 R^\nabla(42315)\\
&\quad + \nabla^2 R^\nabla(45312) + \nabla^2 R^\nabla(52314) + \nabla^2 R^\nabla(54312).
\end{aligned}
$$

Summing up, we obtain

$$
\begin{aligned}
(\operatorname{CircSym}_{3,4,5}\theta)(12345) &= \nabla^2 R^\nabla(34125) + \nabla^2 R^\nabla(14235) + \nabla^2 R^\nabla(24315)\\
&\quad + \nabla^2 R^\nabla(35124) + \nabla^2 R^\nabla(15234) + \nabla^2 R^\nabla(25314)\\
&\quad + \nabla^2 R^\nabla(43125)_1 + \nabla^2 R^\nabla(41235)_1 + \nabla^2 R^\nabla(42315)_1\\
&\quad + \nabla^2 R^\nabla(45123)_2 + \nabla^2 R^\nabla(45231)_2 + \nabla^2 R^\nabla(45312)_2\\
&\quad + \nabla^2 R^\nabla(53124)_3 + \nabla^2 R^\nabla(51234)_3 + \nabla^2 R^\nabla(52314)_3\\
&\quad + \nabla^2 R^\nabla(54123)_4 + \nabla^2 R^\nabla(54231)_4 + \nabla^2 R^\nabla(54312)_4,
\end{aligned}
$$

where we denote by $\nabla^2 R^\nabla(\cdots\cdots)_j$ the terms that summed up together equal zero thanks to the differential Bianchi identity for $j = 1, 3$ and thanks to the algebraic Bianchi identity for $j = 2, 4$. By using formula (5.5), we infer

$$
\begin{aligned}
(\operatorname{CircSym}_{3,4,5}\theta)(12345) &= \nabla^2 R^\nabla(43125)_1 + \nabla^2 R^\nabla(41235)_1 + \nabla^2 R^\nabla(42315)_1\\
&\quad + \nabla^2 R^\nabla(53124)_2 + \nabla^2 R^\nabla(51234)_2 + \nabla^2 R^\nabla(52314)_2\\
&\quad + (R^\nabla . R^\nabla)(34125) + (R^\nabla . R^\nabla)(14235) + (R^\nabla . R^\nabla)(24315)\\
&\quad + (R^\nabla . R^\nabla)(35124) + (R^\nabla . R^\nabla)(15234) + (R^\nabla . R^\nabla)(25314),
\end{aligned}
$$

where as mentioned earlier we denote by $\nabla^2 R^\nabla(\cdots\cdots)_j$ the terms that summed up together equal zero thanks to the differential Bianchi identity. We deduce the expression

$$
\begin{aligned}
(\operatorname{CircSym}_{3,4,5}\theta)(12345) &= (R^\nabla . R^\nabla)(34125) + (R^\nabla . R^\nabla)(14235) + (R^\nabla . R^\nabla)(24315)\\
&\quad + (R^\nabla . R^\nabla)(35124) + (R^\nabla . R^\nabla)(15234) + (R^\nabla . R^\nabla)(25314).
\end{aligned}
\tag{7.1}
$$

We set now for notation simplicity $\rho := R^\nabla . R^\nabla$, and let

$$\Theta(12345) := \rho(13245) + \rho(23415).$$

We observe that, by definition, the tensor

$$\rho \in C^\infty(M, \Lambda^2 T_M^* \otimes_{\mathbb{R}} \Lambda^2 T_M^* \otimes_{\mathbb{R}} T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

satisfies the circular identity with respect to its last three entries. We expand now the term

$$\operatorname{CircSym}_{3,4,5}\Theta.$$

We observe the identities

$$
\begin{aligned}
(\operatorname{Sym}_{3,4,5}\Theta)(12345) &= \rho(13245) + \rho(23415) + \rho(13254) + \rho(23514) + \rho(14235) + \rho(24315)\\
&\quad + \rho(14253) + \rho(24513) + \rho(15234) + \rho(25314) + \rho(15243) + \rho(25413),\\
(\operatorname{Sym}_{3,4,5}\Theta)(23145) &= \rho(21345) + \rho(31425) + \rho(21354) + \rho(31524) + \rho(24315) + \rho(34125)\\
&\quad + \rho(24351) + \rho(34521) + \rho(25314) + \rho(35124) + \rho(25341) + \rho(35421),\\
(\operatorname{Sym}_{3,4,5}\Theta)(31245) &= \rho(32145) + \rho(12435) + \rho(32154) + \rho(12534) + \rho(34125) + \rho(14235)\\
&\quad + \rho(34152) + \rho(14532) + \rho(35124) + \rho(15234) + \rho(35142) + \rho(15432).
\end{aligned}
$$

Summing up we obtain

$$\begin{aligned}(\operatorname{CircSym}_{3,4,5}\Theta)(12345) = {}& \rho(13245)_1 + \rho(23415)_2 + \rho(21345)_3 + \rho(31425)_1 + \rho(32145)_2 + \rho(12435)_3\\ &+ \rho(13254)_4 + \rho(23514)_5 + \rho(21354)_6 + \rho(31524)_4 + \rho(32154)_5 + \rho(12534)_6\\ &+ \rho(14235)_7 + \rho(24315)_8 + \rho(24315)_8 + \rho(34125)_9 + \rho(34125)_9 + \rho(14235)_7\\ &+ \rho(14253)_7 + \rho(24513)_8 + \rho(24351)_8 + \rho(34521)_9 + \rho(34152)_9 + \rho(14532)_7\\ &+ \rho(15234)_{10} + \rho(25314)_{11} + \rho(25314)_{11} + \rho(35124)_{12} + \rho(35124)_{12} + \rho(15234)_{10}\\ &+ \rho(15243)_{10} + \rho(25413)_{11} + \rho(25341)_{11} + \rho(35421)_{12} + \rho(35142)_{12} + \rho(15432)_{10},\end{aligned}$$

where we denote by $\rho(\cdots\cdots)_j$ the terms that we sum up together using the symmetries of ρ. We obtain

$$\begin{aligned}(\operatorname{CircSym}_{3,4,5}\Theta)(12345) = {}& 2\rho(13245) + 2\rho(23415) + 2\rho(12435) + 2\rho(13254) + 2\rho(23514) + 2\rho(12534)\\ &+ 3\rho(14235) + 3\rho(24315) + 3\rho(34125) + 3\rho(15234) + 3\rho(25314) + 3\rho(35124).\end{aligned}$$

We conclude the expression

$$\begin{aligned}[\operatorname{CircSym}_{3,4,5}d_1^{\nabla}(\nabla R^{\nabla})_2](12345) = {}& 2\rho(13245) + 2\rho(23415) + 2\rho(12435) + 2\rho(13254) + 2\rho(23514) + 2\rho(12534)\\ &+ 2\rho(14235) + 2\rho(24315) + 2\rho(34125) + 2\rho(15234) + 2\rho(25314) + 2\rho(35124).\end{aligned} \tag{7.2}$$

We expand now the term

$$\operatorname{CircSym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla}).$$

From now on, we will denote for notation simplicity $(123) \equiv R^{\nabla}(123)$ and

$$[123] \coloneqq (123) + (132).$$

We observe the identities

$$\begin{aligned}[\operatorname{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](12345) = {}& [1[234]5] - [2[134]5] + [1[235]4] - [2[135]4] + [1[243]5] - [2[143]5]\\ &+ [1[245]3] - [2[145]3] + [1[253]4] - [2[153]4] + [1[254]3] - [2[154]3],\end{aligned}$$

$$\begin{aligned}[\operatorname{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](23145) = {}& [2[314]5] - [3[214]5] + [2[315]4] - [3[215]4] + [2[341]5] - [3[241]5]\\ &+ [2[345]1] - [3[245]1] + [2[351]4] - [3[251]4] + [2[354]1] - [3[254]1],\end{aligned}$$

$$\begin{aligned}[\operatorname{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](31245) = {}& [3[124]5] - [1[324]5] + [3[125]4] - [1[325]4] + [3[142]5] - [1[342]5]\\ &+ [3[145]2] - [1[345]2] + [3[152]4] - [1[352]4] + [3[154]2] - [1[354]2].\end{aligned}$$

Summing up, using the symmetries of $[\cdots]$ and $(\cdots)$, we obtain

$$\begin{aligned}[\operatorname{CircSym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](12345) = {}& 6[1(234)5] + 6[2(314)5] + 6[3(124)5] + 6[1(235)4] + 6[2(315)4] + 6[3(125)4]\\ &+ 2[1[245]3]_1 - 2[2[145]3]_2 + 2[2[345]1]_3 - 2[3[245]1]_1 + 2[3[145]2]_2 - 2[1[345]2]_3.\end{aligned}$$

We combine now the terms $[\cdot[\cdots]\cdot]_j$ for each $j = 1, 2, 3$, and we explicit and simplify them by using the algebraic Bianchi identity. We obtain

$$\begin{aligned}[\operatorname{CircSym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](12345) = {}& 6[1(234)5] + 6[2(314)5] + 6[3(124)5] + 6[1(235)4] + 6[2(315)4]\\ &+ 6[3(125)4] + 6(13[245]) + 6(32[145]) + 6(21[345]).\end{aligned}$$

By expanding further, we obtain the complete expansion:

$$\begin{aligned}[\operatorname{CircSym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})](12345) = {}& 6(1(234)5) + 6(15(234)) + 6(2(314)5) + 6(25(314))\\ &+ 6(3(124)5) + 6(35(124)) + 6(1(235)4) + 6(14(235))\\ &+ 6(2(315)4) + 6(24(315)) + 6(3(125)4) + 6(34(125)) + 6(13(245)) + 6(13(254))\\ &+ 6(32(145)) + 6(32(154)) + 6(21(345)) + 6(21(354)).\end{aligned}$$

By expanding the terms ρ present in the expression (7.2), we obtain the complete expansion of the term

$$\{\mathrm{CircSym}_{3,4,5}[3d_1^{\nabla}(\nabla R^{\nabla})_2 - 2\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla}]\}(12345),$$

given by

$$\begin{aligned}
&\{\mathrm{CircSym}_{3,4,5}[3d_1^{\nabla}(\nabla R^{\nabla})_2 - 2\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla}]\}(12345)\\
&= 6(13(245))_1 - 6((132)45)_2 - 6(2(134)5)_3 - 6(24(135))_4\\
&\quad + 6(23(415))_5 - 6((234)15)_6 - 6(4(231)5)_2 - 6(41(235))_7\\
&\quad + 6(12(435))_8 - 6((124)35)_9 - 6(4(123)5)_2 - 6(43(125))_{10}\\
&\quad + 6(13(254))_{11} - 6((132)54)_{12} - 6(2(135)4)_{13} - 6(25(134))_{14}\\
&\quad + 6(23(514))_{15} - 6((235)14)_{16} - 6(5(231)4)_{12} - 6(51(234))_{17}\\
&\quad + 6(12(534))_{18} - 6((125)34)_{19} - 6(5(123)4)_{12} - 6(53(124))_{20}\\
&\quad + 6(14(235))_7 - 6((142)35)_9 - 6(2(143)5)_3 - 6(23(145))_5\\
&\quad + 6(24(315))_4 - 6((243)15)_6 - 6(3(241)5)_9 - 6(31(245))_1\\
&\quad + 6(34(125))_{10} - 6((341)25)_3 - 6(1(342)5)_6 - 6(12(345))_8\\
&\quad + 6(15(234))_{17} - 6((152)34)_{19} - 6(2(153)4)_{13} - 6(23(154))_{15}\\
&\quad + 6(25(314))_{14} - 6((253)14)_{16} - 6(3(251)4)_{19} - 6(31(254))_{11}\\
&\quad + 6(35(124))_{20} - 6((351)24)_{13} - 6(1(352)4)_{16} - 6(12(354))_{18}\\
&\quad - 12(1(234)5)_6 - 12(15(234))_{17} - 12(2(314)5)_3 - 12(25(314))_{14}\\
&\quad - 12(3(124)5)_9 - 12(35(124))_{20} - 12(1(235)4)_{16} - 12(14(235))_7\\
&\quad - 12(2(315)4)_{13} - 12(24(315))_4 - 12(3(125)4)_{19} - 12(34(125))_{10} - 12(13(245))_1 - 12(13(254))_{11}\\
&\quad - 12(32(145))_5 - 12(32(154))_{15} - 12(21(345))_8 - 12(21(354))_{18},
\end{aligned}$$

where as mentioned earlier we denote by $(\cdots\cdots)_j$ the terms that we sum up together using the symmetries of the curvature tensor R^{∇}. All the terms summed up together cancel up. This is obvious for all the sub indexes j with the exception of $j = 3, 6, 9, 13, 16, 19$ for which me must provide the detail of the computation. Indeed for $j = 3$, we have

$$\begin{aligned}
&-6((341)25) - 6(2(134)5) - 6(2(143)5) - 12(2(314)5)\\
&= 6(2(341)5) + 6(2(413)5) - 6(2(314)5)\\
&= -6(2(134)5) - 6(2(314)5)\\
&= 0.
\end{aligned}$$

For $j = 6$, we have

$$\begin{aligned}
&-12(1(234)5) - 6((234)15) - 6((243)15) - 6(1(342)5)\\
&= 6(1(243)5) + 6(1(432)5) - 6(1(234)5)\\
&= -6(1(324)5) - 6(1(234)5)\\
&= 0.
\end{aligned}$$

For $j = 9$, we have

$$\begin{aligned}
&-6((124)35) - 6((142)35) - 6(3(241)5) - 12(3(124)5)\\
&= 6(3(142)5) + 6(3(421)5) - 6(3(124)5)\\
&= -6(3(214)5) - 6(3(124)5)\\
&= 0.
\end{aligned}$$

For $j = 13$, we have

$$\begin{aligned}
&-6((351)24) - 6(2(135)4) - 6(2(153)4) - 12(2(315)4)\\
&= 6(2(351)4) + 6(2(513)4) - 6(2(315)4)\\
&= -6(2(135)4) - 6(2(315)4)\\
&= 0.
\end{aligned}$$

For $j = 16$, we have

$$\begin{aligned}
&-12(1(235)4) - 6((235)14) - 6((253)14) - 6(1(352)4)\\
&= 6(1(253)4) + 6(1(532)4) - 6(1(235)4)\\
&= -6(1(325)4) - 6(1(235)4)\\
&= 0.
\end{aligned}$$

For $j = 19$, we have

$$\begin{aligned}
&-6((125)34) - 6((152)34) - 6(3(251)4) - 12(3(125)4)\\
&= 6(3(152)4) + 6(3(521)4) - 6(3(125)4)\\
&= -6(3(215)4) - 6(3(125)4)\\
&= 0.
\end{aligned}$$

We infer the required identity (1.16) in the statement of Proposition 1. □

8 The almost complex structure associated with a connection over the tangent bundle

This section is not needed for the proof of the results in the article. We include it to clarify the integrability of an M-totally real almost complex structure over T_M associated with the horizontal distribution of a linear connection. We include it also to remind and to prove in modern terms a well-known result due to Dombrowsky [7].

It is well-known [7] that we can construct an M-totally real almost complex structure over T_M by using the horizontal distribution $\mathcal{H} \subset T_M$ associated with a linear connection ∇ acting on the sections of T_M. Indeed, in this case, we set $\alpha_\eta := H_\eta$ and $B_\eta := \mathbb{I}_{T_{M,\pi(\eta)}}$, where $\eta \longmapsto H_\eta$ is the horizontal map associated with $\mathcal{H}$. We will denote $J_{\mathcal{H}} := J_A$. If we define for any $\eta \in T_{M,p}$, the vertical projection $\operatorname{Vert}_\eta : T_{T_M,\eta} \longrightarrow T_{T_{M,p},\eta}$ as follows:

$$\operatorname{Vert}_\eta := \mathbb{I}_{T_{T_M,\eta}} - H_\eta d_\eta \pi,$$

where $\pi : T_M \longrightarrow M$ is the canonical projection, then

$$J_{\mathcal{H},\eta} := -H_\eta T_\eta^{-1} \operatorname{Vert}_\eta + T_\eta d_\eta \pi.$$

If we decompose any vector $\xi \in \mathbb{C}T_{T_M,\eta}$ in its horizontal and vertical parts $\xi = \xi^h + \xi^v$ with $\xi^v := \operatorname{Vert}_\eta(\xi)$, then we have the expressions

$$\begin{aligned}
J_{\mathcal{H},\eta}\xi &= -H_\eta T_\eta^{-1}\xi^v + T_\eta d_\eta \pi \xi^h,\\
(J_{\mathcal{H},\eta}\xi)^h &= -H_\eta T_\eta^{-1}\xi^v,\\
(J_{\mathcal{H},\eta}\xi)^v &= T_\eta d_\eta \pi \xi^h.
\end{aligned}$$

We infer

$$\begin{aligned}
\xi^{0,1}_{J_{\mathcal{H}}}(\eta) &= \frac{1}{2}[\xi^h - iH_\eta T_\eta^{-1}\xi^v + \xi^v + iT_\eta d_\eta \pi \xi^h]\\
&= \frac{1}{2}[\xi^h + H_\eta \mu + iT_\eta(d_\eta \pi \xi^h + \mu)],
\end{aligned}$$

with $\mu := -iT_\eta^{-1}\xi^v$. We notice also the identity

$$T^{0,1}_{T_M,J_{\mathcal{H}},\eta} = \frac{1}{2}(H_\eta + iT_\eta)\mathbb{C}T_{M,p}, \tag{8.1}$$

for any any $\eta \in T_{X,p}$. The distribution $T^{0,1}_{T_M,J_{\mathcal{H}}}$ is horizontal, but the associated map does not satisfies condition (A5) of linear connections thanks to the identity (A4). Therefore, this distribution does not identify a linear connection. However, its integrability implies that the vector bundle T_M is flat. Indeed, we have the following well-known lemma due to Dombrowsky [7].

Lemma 13. *The torsion form* $\tau^{J_{\mathcal{H}}}$ *of the almost complex structure* $J_{\mathcal{H}}$ *satisfies at the point* $\eta \in T_M$ *in the directions* $V_1, V_2 \in T^{0,1}_{T_M,J_{\mathcal{H}},\eta}$ *the identity*

$$8\tau^{J_{\mathcal{H}}}(V_1, V_2)(\eta) = -H_\eta[\tau^\nabla(v_1, v_2) + iR^\nabla(v_1, v_2)\eta] + T_\eta[i\tau^\nabla(v_1, v_2) - R^\nabla(v_1, v_2)\eta],$$

where $R^\nabla := \nabla^2$ *is the complex linear extension of the curvature tensor of* ∇, *where* τ^∇ *is the torsion of the complex connection* ∇ *and where* $v_j := d_\eta\pi V_j$, $j = 1, 2$. *In particular,* $J_{\mathcal{H}}$ *is a complex structure if and only if the linear connection* ∇ *is flat and torsion free.*

Proof. Let ξ_j be vector field local extensions of v_j such that $[\xi_1, \xi_2]\pi(\eta) = 0$. Then

$$\Xi_j := \frac{1}{2}(H + iT)\xi_j,$$

are local vector field extensions of V_j. We expand the bracket

$$\begin{aligned} 4[\Xi_1, \Xi_2](\eta) &= ([H\xi_1, H\xi_2] + i[H\xi_1, T\xi_2] + i[T\xi_1, H\xi_2] - [T\xi_1, T\xi_2])(\eta) \\ &= H_\eta[\xi_1, \xi_2] - T_\eta[R^\nabla(v_1, v_2)\eta] + iT_\eta[\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1]. \end{aligned}$$

The last equality follows from to the computation at the end of the proof of Lemma A6 and thanks to the identity (A8) in the Appendix. (We notice that $[T\xi_1, T\xi_2] \equiv 0$, since the vector fields $T\xi_1$ are tangent constant along the fibers.) Thanks to the assumption $[\xi_1, \xi_2]\pi(\eta) = 0$, we infer the equality

$$4[\Xi_1, \Xi_2](\eta) = T_\eta[i\tau^\nabla(v_1, v_2) - R^\nabla(v_1, v_2)\eta].$$

The required formula follows from the identity

$$\xi^{1,0}_{J_{\mathcal{H}}}(\eta) = \frac{1}{2}[\xi^h + iH_\eta T_\eta^{-1}\xi^v + \xi^v - iT_\eta d_\eta\pi\xi^h].$$

The fact that that the distribution $T^{1,0}_{T_M,J_{\mathcal{H}}}$ is horizontal implies that $\tau^{J_{\mathcal{H}}}(V_1, V_2)(\eta)$ vanishes for all V_j if and only if the quantity

$$\tau^\nabla(v_1, v_2) + iR^\nabla(v_1, v_2)\eta,$$

vanishes for all v_j. In particular, for real vectors v_j, this implies that R^∇ and τ^∇ vanish at the point $\pi(\eta)$. □

We observe that a connection over T_M is flat and torsion free if and only if there exist local parallel frames with vanishing Lie brackets.

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References

[1] V. Aslam, D. M. Burns, and D. Irvine, *Left-invariant Grauert tubes on SU(2)*, Q. J. Math. **69** (2018), no. 3, 871–885.

[2] D. M. Burns, *On the uniqueness and characterization of Grauert tubes*, Complex analysis and geometry (Trento 1993), 119–133, Lecture Notes in Pure and Appllied Mathematics, 173, Dekker, New York, 1996.

[3] D. M. Burns, *Symplectic geometry and the uniqueness of Grauert tubes*, Geom. Funct. Anal. **11** (2001), no. 1, 1–10.

[4] D. M. Burns, S. Halverscheid, and R. Hind, *The geometry of Grauert tubes andcomplexification of symmetric spaces*, Duke Math. J. **118** (2003), no. 3, 465–491.

[5] R. Bielawski, *Complexification and hypercomplexification of manifolds with a linear connection*, Int. J. Math. **14** (2003), 813–824.
[6] F. Bruhat and H. Whitney, *Quelque propriétés fondamentales des ensembles analytiques-réels*, comment. Math. Helv. **33** (1959), 132–160.
[7] P. Dombrowsky, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88.
[8] P. Gauduchon, *Connexions linéaires, classes de Chern, théorème de Riemann-Roch*, Holomorphic curves in symplectic geometry, 113–162, Progr. Math., vol. 117, Birkhäuser, Basel, 1994.
[9] H. Grauert, *On Levias problem and the imbedding of real analytic manifolds*, Ann. Math. **68** (1958), no. 2, 460–472.
[10] V. Guillemin and M. Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Differential Geom. **34** (1991), 561–570.
[11] B. C. Hall and W. D. Kirwin, *Adapted complex structures and the geodesic flow*, Math. Ann. **350** (2011), 455–474.
[12] W. Klingenberg, *Riemannian Geometry*, Walter de Gruyter, Berlin, 1982.
[13] L. Lempert, *Complex structures on the tangent bundle of Riemannian manifolds*, Complex analysis and Geometry, V. Ancona, A. Silva (eds), The university Series in Mathematics. Springer, Boston, MA, 1993, pp. 235–251.
[14] L. Lempert and R. Szöke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. **290** (1991), 689–712.
[15] L. Lempert and R. Szöke, *A new look at adapted complex structures*, Bulletin London Math. Soc. **44** (2012), 367–374.
[16] J. M. Mourao and J. P. Nunes, *On Complexified Analytic Hamiltonian Flows and Geodesics on the space of Kähler Metrics*, Int Math Res Notices **20** (2015), 10624–10656.
[17] N. Pali, *Exact Fourier inversion formula over manifolds*, J. Pseudo-Differ. Operators Appl. **8** (2017), no. 4, 623–628.
[18] N. Pali and B. Salvy, *Explicit Maximal Ttotally Real Embeddings*, 2023, p. 53, https://hal.science/hal-04266729.
[19] N. Pali and B. Salvy, *Computer verification of the claims in the article "Explicit maximal totally real embeddings"*, 2024, p. 34. (with an appendix by F. Guenard), https://hal.science/hal-04518407.
[20] R. Szöke, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), 409–928.
[21] R. Szöke, *Canonical complex structures associated with connections and complexifications of Lie groups*, Math. Ann. **328** (2004), 553–591.
[22] S. Zelditch, *Pluri-potential theory on Grauert tubes of real analytic Riemannian manifolds, I*, Spectral Geometry, Proceedings of Symposia in Pure Mathematics, vol. 84, AMS, 2012, pp. 299–339.

A Appendix

In this Appendix, we provide some well-known basic facts about the geometric theory of linear connections needed for the reading of the article [8]. We strongly recommend its reading even to experts.

A.1 The horizontal distribution associated with a linear connection

We start with the following fact.

Lemma A1. *Let ∇ be a linear connection acting on sections of a vector bundle E over a manifold M. Then the linear map*

$$T_{M,p} \ni \xi \longmapsto H_\eta(\xi) \;:=\; d_p\sigma(\xi) - T_\eta \nabla_\xi \sigma \in T_{E,\eta}$$

is independent of the sections σ such that $\sigma(p) = \eta$.

Proof. Let $e = (e_k)_{k=1}^r$ be a local frame of E over an open set $U \subset M$. We consider the local expression $\sigma = e \cdot f$ with $f \in C^1(U, \mathbb{R}^r)$. Let $A \in C^\infty(U, T_M^* \otimes \mathrm{Matrix}_{r\times r}(\mathbb{R}))$ be the connection form of ∇ with respect to the local frame e, i.e., $\nabla e = e \cdot A$. Then $\nabla\sigma = e \otimes (df + A \cdot f)$. If we denote by $\theta_e : U \times \mathbb{R}^r \longrightarrow E_{|U}$, then the differential of this map at the point $(p, f(p))$ provides an isomorphism

$$d_{p,f(p)}\theta_e : T_{U,p} \oplus \mathbb{R}^r \longrightarrow T_{E,\sigma(p)}.$$

With respect to it, the equality hold

$$d_{p,f(p)}\theta_e[\xi \oplus d_p f(\xi)] = d_p\sigma(\xi).$$

We observe now the linear identity $d\tau_{\sigma(p)} \cdot d_{p,0}\theta_{e|0\oplus\mathbb{R}^r} = d_{p,f(p)}\theta_{e|0\oplus\mathbb{R}^r}$. We infer

$$T_{\sigma(p)} \cdot \theta_{e|\{p\}\times\mathbb{R}^r} = d_{p,f(p)}\theta_{e|0\oplus\mathbb{R}^r}, \tag{A1}$$

and

$$\begin{aligned} T_{\sigma(p)}[e(p)\cdot(d_p f(\xi) + A(\xi)\cdot f(p))] &= d_{p,f(p)}\theta_e[0 \oplus \;(d_p f(\xi) + A(\xi)\cdot f(p))],\\ T_{\sigma(p)}\nabla_\xi\sigma &= d_{p,f(p)}\theta_e[0 \oplus \;(d_p f(\xi) + A(\xi)\cdot f(p))]. \end{aligned}$$

Thus,

$$H_{\sigma(p)}(\xi) = d_{p,f(p)}\theta_e[\xi \oplus (-A(\xi)\cdot f(p))],$$

i.e., if $\eta = e \cdot h$, then

$$H_\eta(\xi) = d_{p,h}\theta_e[\xi \oplus (-A(\xi)\cdot h)],$$

which shows the required conclusion. □

Let $\pi_E : E \longrightarrow M$ be the projection map and notice the equality $\operatorname{Ker} d_\eta\pi_E = T_{E_p,\eta}$, for any $\eta \in E_p$. The identity $\pi_E \circ \sigma = \mathrm{id}_M$ implies

$$d_{\sigma(p)}\pi_E \circ d_p\sigma(\xi) = \xi.$$

We deduce the identity $d_\eta\pi_E \circ H_\eta(\xi) = \xi$. We define the horizontal distribution $\mathcal{H} \subset T_E$ associated with ∇ as follows:

$$\mathcal{H}_\eta \;:=\; H_\eta(T_{M,\pi_E(\eta)}) \subset T_{E,\eta}.$$

We notice now that the tangent bundle of the vector bundle $E \oplus E$ is given by the fibers

$$T_{E\oplus E,(\eta_1,\eta_2)} = \{(v_1, v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2} | d_{\eta_1}\pi_E(v_1) = d_{\eta_2}\pi_E(v_2)\},$$

and that the differential of the sum bundle map $sm_E : E \oplus E \longrightarrow E$ satisfies

$$d_{(\eta_1,\eta_2)}(sm_E)(v_1, v_2) = T_{\eta_1+\eta_2}(T^{-1}_{\eta_1}v_1 + T^{-1}_{\eta_2}v_2),$$

for any $(v_1, v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2}$ such that $d_{\eta_1}\pi_E(v_1) = d_{\eta_2}\pi_E(v_2) = 0$. We infer that for any sections σ_j of E such that $\sigma_j(p) = \eta_j$, $j = 1, 2$, the equalities

$$\begin{aligned} H_{\eta_1+\eta_2}(\xi) &= d_p(\sigma_1 + \sigma_2)(\xi) - T_{\eta_1+\eta_2}\nabla_\xi(\sigma_1 + \sigma_2) \\ &= d_{(\eta_1,\eta_2)}(sm_E)(d_p\sigma_1(\xi), d_p\sigma_2(\xi)) - T_{\eta_1+\eta_2}\nabla_\xi\sigma_1 - T_{\eta_1+\eta_2}\nabla_\xi\sigma_2 \\ &= d_{(\eta_1,\eta_2)}(sm_E)(d_p\sigma_1(\xi) - T_{\eta_1}\nabla_\xi\sigma_1, d_p\sigma_2(\xi) - T_{\eta_2}\nabla_\xi\sigma_2) \end{aligned}$$

hold. We conclude that

$$H_{\eta_1+\eta_2}(\xi) = d_{(\eta_1,\eta_2)}(sm_E)(H_{\eta_1}(\xi), H_{\eta_2}(\xi)). \tag{A2}$$

Lemma A2. *For any section $\sigma \in C^1(M, E)$ and for any function $u \in C^1(M, \mathbb{R})$, the identity holds*

$$d_p(u\sigma) = d_pu \otimes T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E]\cdot d_p\sigma,$$

for any point $p \in M$.

Proof. With the notation in the proof of Lemma A1

$$\begin{aligned} d_p(u\sigma)(\xi) &= d_{p,uf(p)}\theta_e[\xi \oplus \ d_p(uf)(\xi)] \\ &= d_{p,uf(p)}\theta_e\{\xi \oplus \ [d_pu(\xi)f(p) + u(p)d_pf(\xi)]\} \\ &= d_{p,uf(p)}\theta_e[0 \oplus \ d_pu(\xi)f(p)] + d_{p,uf(p)}\theta_e[\xi \oplus \ u(p)d_pf(\xi)] \\ &= T_{u\sigma(p)}\theta_e(p, d_pu(\xi)f(p)) + d_p(u(p)\sigma)(\xi), \end{aligned}$$

thanks to (A1). Using the identity

$$d_p(\lambda\sigma) = d_{\sigma(p)}(\lambda\mathbb{I}_E)\cdot d_p\sigma, \tag{A3}$$

for any $\lambda \in \mathbb{R}$, we conclude

$$d_p(u\sigma)(\xi) = d_pu(\xi)T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E]\cdot d_p\sigma(\xi).\square$$

We observe also the elementary identity

$$d_\eta(\lambda\mathbb{I}_E)\cdot T_\eta = \lambda T_{\lambda\eta}, \tag{A4}$$

for all $\eta \in E$. We show now the identity

$$H_{\lambda\eta} = d_\eta(\lambda\mathbb{I}_E)\cdot H_\eta, \tag{A5}$$

for all $\eta \in E$. Indeed, let σ be a section such that $\sigma(p) = \eta$. By using (A3) and (A4), we obtain the equalities

$$\begin{aligned} H_{\lambda\eta} &= d_p(\lambda\sigma) - T_{\lambda\eta}\nabla(\lambda\sigma) \\ &= d_{\sigma(p)}(\lambda\mathbb{I}_E)\cdot d_p\sigma - \lambda T_{\lambda\eta}\nabla\sigma \\ &= d_{\sigma(p)}(\lambda\mathbb{I}_E)\cdot[d_p\sigma - T_\eta\nabla\sigma] \\ &= d_\eta(\lambda\mathbb{I}_E)\cdot H_\eta. \end{aligned}$$

The property (A5) implies in particular $H_{0_p} = d_p0_M$, where 0_M is the zero section of T_M.

Definition A1. A distribution $\mathcal{H} \subset T_E$ is called horizontal if the map

$$d_\eta\pi_{E|\mathcal{H}_\eta} : \mathcal{H}_\eta \longrightarrow T_{M,\pi_E(\eta)},$$

is an isomorphism for all $\eta \in E$.

Lemma A3. *Any horizontal distribution $\mathcal{H} \subset T_E$, which satisfies conditions* (A2) *and* (A5) *with* $H_\eta := (d_\eta \pi_{E|\mathcal{H}_\eta})^{-1}$, *determines a connection ∇ over E with associated horizontal distribution $\mathcal{H}$.*

Proof. The connection ∇ is defined by the formula

$$\nabla_\xi \sigma = T^{-1}_{\sigma(p)} \cdot [d_p\sigma - H_{\sigma(p)}](\xi),$$

for any $\xi \in T_{M,p}$. The definition is well posed because

$$[d_p\sigma - H_{\sigma(p)}](\xi) \in T_{E_p,\sigma(p)},$$

which follows from the identity

$$d_{\sigma(p)}\pi_E \cdot [d_p\sigma - H_{\sigma(p)}](\xi) = 0.$$

It is obvious that the additive property of ∇ is equivalent to the condition (A2). We observe now that with the previous definition, the covariant Leibniz property

$$\nabla_\xi(u\sigma) = d_p u(\xi)\sigma(p) + u(p)\nabla_\xi\sigma$$

is equivalent to the identity

$$d_p(u\sigma)(\xi) - H_{u\sigma(p)}(\xi) = T_{u\sigma(p)}\{d_p u(\xi)\sigma(p) + u(p)T^{-1}_{\sigma(p)} \cdot [d_p\sigma(\xi) - H_{\sigma(p)}(\xi)]\}.$$

We develop the right-hand side using (A4). We infer that the previous identity is equivalent to the following one:

$$d_p(u\sigma)(\xi) - H_{u\sigma(p)}(\xi) = d_p u(\xi) T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E] \cdot [d_p\sigma(\xi) - H_{\sigma(p)}(\xi)].$$

The later hold true thanks to Lemma A2 and Assumption (A5). □

The data of a smooth horizontal distribution over E coincides with the one of section

$$H \in C^\infty(E, \pi_E^* T_M^* \otimes T_E)$$

such that $d\pi_E \cdot H = \mathbb{I}_{\pi_E^* T_E}$. (We notice that $d\pi_E \in C^\infty(E, T_E^* \otimes \pi_E^* T_M)$.) Such type of section determines a connection if and only if it satisfies the identity (A5).

For any vector $\Xi \in T_{E,\eta}$, we denote by

$$\gamma_\eta^{\mathcal{H}}(\Xi) := \Xi - H_\eta \circ d_\eta\pi_E(\Xi),$$

its vertical component with respect to the horizontal distribution $\mathcal{H}$. In particular,

$$\gamma^{\mathcal{H}}_{\sigma(p)} \cdot d_p\sigma(\xi) = T_{\sigma(p)}[\nabla_\xi\sigma(p)].$$

A.2 The induced connection

Let $\psi : N \longrightarrow M$ be a smooth map. We define the vector bundle $\psi^* E := N \times_\psi E$ over N. In explicit terms

$$\psi^* E = \{(y, \eta) \in N \times E | \psi(y) = \pi_E(\eta)\},$$

and the projection over N is given by the restriction of the projection to the first factor. We will denote by $\Psi : \psi^* E \longrightarrow E$ the restriction of the projection to the second factor. The sections of $\psi^* E$ are identified with the maps $\sigma : N \longrightarrow E$ such that $\pi_E \circ \sigma = \psi$. In this way, if s is a section of E, then the section $\psi^* s := s \circ \psi$ is a section of $\psi^* E$. More in general if α is a section of $\Lambda^p T_M^* \otimes E$, we define the section $\psi^*\alpha \in \Lambda^p T_N^* \otimes \psi^* E$ as follows:

$$(\psi^*\alpha)(y) := (\alpha \circ \psi)(y) \cdot \Lambda^p(d_y\psi).$$

We provide a generalization of Lemma (A2).

Lemma A4. *For any section $\sigma \in C^1(N, \psi^* E)$ and for any function $u \in C^1(N, \mathbb{R})$, the identity holds*

$$d_p(u\sigma) = d_p u \otimes T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E]\cdot d_p\sigma,$$

for any point $p \in N$.

Proof. A local frame e of E induces a local frame $\psi^* e$ of $\psi^* E$ over the open set $\psi^{-1}(U)$. Then $\sigma = \psi^* e \cdot f$ with $f \in C^1(\psi^{-1}(U), \mathbb{R}^r)$. We denote by $\theta_e : U \times \mathbb{R}^r \longrightarrow E_{|U}$ the trivialization map induced by the local frame e of E. Then the differential of this map at the point $(\psi(p), f(p))$ provides an isomorphism

$$d_{\psi(p),f(p)}\theta_e : T_{U,\psi(p)} \oplus \mathbb{R}^r \longrightarrow T_{E,\sigma(p)}$$

and

$$d_p\sigma(\xi) = d_{\psi(p),uf(p)}\theta_e[d_p\psi(\xi) \oplus d_p f(\xi)].$$

For any $\xi \in T_{N,p}$, we have

$$\begin{aligned}
d_p(u\sigma)(\xi) &= d_{\psi(p),uf(p)}\theta_e[d_p\psi(\xi) \oplus \; d_p(uf)(\xi)]\\
&= d_{\psi(p),uf(p)}\theta_e\{d_p\psi(\xi) \oplus \; [d_p u(\xi) f(p) + u(p) d_p f(\xi)]\}\\
&= d_{\psi(p),uf(p)}\theta_e[0 \oplus \; d_p u(\xi) f(p)] + d_{\psi(p),uf(p)}\theta_e[d_p\psi(\xi) \oplus \; u(p) d_p f(\xi)]\\
&= T_{u\sigma(p)} \cdot \theta_e(\psi(\xi), d_p u(\xi) f(p)) + d_p(u(p)\sigma)(\xi),
\end{aligned}$$

thanks to (A1). Using the equality

$$d_p(\lambda\sigma) = d_{\sigma(p)}(\lambda\mathbb{I}_E)\cdot d_p\sigma,$$

for any $\lambda \in \mathbb{R}$, we conclude the required identity

$$d_p(u\sigma)(\xi) = d_p u(\xi) T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E]\cdot d_p\sigma(\xi). \qquad \square$$

The induced connection ∇^ψ over $\psi^* E$ is defined by the formula

$$\nabla^\psi_\xi \sigma \coloneqq T^{-1}_{\sigma(p)}\gamma^{\mathcal{H}}_{\sigma(p)} d_p\sigma(\xi) = T^{-1}_{\sigma(p)}[d_p\sigma(\xi) - H_{\sigma(p)} d_p\psi(\xi)],$$

for any $\xi \in T_{N,p}$. It is obvious that the additive property of ∇^ψ follows from the condition (A2). We show now that ∇^ψ satisfies the Leibniz property

$$\nabla^\psi_\xi(u\sigma) = d_p u(\xi)\sigma(p) + u(p)\nabla^\psi_\xi\sigma.$$

Indeed using Lemma A4 and the identity (A5), we have

$$\begin{aligned}
\nabla^\psi_\xi(u\sigma) &= T^{-1}_{u\sigma(p)}\gamma^{\mathcal{H}}_{u\sigma(p)} d_p(u\sigma)(\xi)\\
&= T^{-1}_{u\sigma(p)}\gamma^{\mathcal{H}}_{u\sigma(p)}[d_p u(\xi) T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E] d_p\sigma(\xi)]\\
&= d_p u(\xi)\sigma(p) + T^{-1}_{u\sigma(p)}[d_{\sigma(p)}[u(p)\mathbb{I}_E] d_p\sigma(\xi) - H_{u\sigma(p)} d_p\psi(\xi)]\\
&= d_p u(\xi)\sigma(p) + T^{-1}_{u\sigma(p)}[d_{\sigma(p)}[u(p)\mathbb{I}_E] d_p\sigma(\xi) - d_{\sigma(p)}[u(p)\mathbb{I}_E] H_{\sigma(p)} d_p\psi(\xi)]\\
&= d_p u(\xi)\sigma(p) + T^{-1}_{u\sigma(p)}[d_{\sigma(p)}[u(p)\mathbb{I}_E]\gamma^{\mathcal{H}}_{\sigma(p)} d_p\sigma(\xi)]\\
&= d_p u(\xi)\sigma(p) + T^{-1}_{u\sigma(p)}[u(p)\gamma^{\mathcal{H}}_{\sigma(p)} d_p\sigma(\xi)]\\
&= d_p u(\xi)\sigma(p) + u(p)\nabla^\psi_\xi\sigma.
\end{aligned}$$

We observe also that for any $s \in C^\infty(M, E)$ and $\xi \in T_{N,p}$, we have the equalities

$$\nabla^\psi_\xi(\psi^* s) = T^{-1}_{s\circ\psi(p)}\gamma^{\mathcal{H}}_{s\circ\psi(p)} d_{\psi(p)} s \cdot d_p\psi(\xi) = \nabla s(\psi(p))\cdot d_p\psi(\xi),$$

and in other terms, the functorial formula

$$\nabla^{\psi}(\psi^* s) = \psi^*(\nabla s), \tag{A6}$$

holds.

A.2.1 The induced connection (second approach)

We observe that the tangent space of $\psi^* E$ at the point (y, η) is given by the equality

$$T_{\psi^* E,(y,\eta)} = \{(\xi, \theta) \in T_{N,y} \oplus T_{E,\eta} | d_y \psi(\xi) = d_\eta \pi_E(\theta)\}.$$

Given any horizontal distribution $H \in C^\infty(E, \pi_E^* T_M^* \otimes T_E)$ over E, we define the horizontal distribution

$$H^\psi \; := \; \Psi^* H \in C^\infty(\psi^* E, \pi_{\psi^* E}^* T_N^* \otimes T_{\psi^* E}).$$

In explicit terms

$$H^\psi_{(y,\eta)} = \mathbb{I}_{T_{N,y}} \oplus H_\eta \cdot d_y \psi.$$

If H satisfies the identities (A2) and (A5), then so does H^ψ. This follows indeed from the identities

$$d_{(y,\eta_1,\eta_2)}(sm_{\psi^* E}) = \mathbb{I}_{T_{N,y}} \oplus d_{(\eta_1,\eta_2)}(sm_E),$$
$$d_{(y,\eta)}(\lambda \mathbb{I}_{\psi^* E}) = \mathbb{I}_{T_{N,y}} \oplus d_\eta(\lambda \mathbb{I}_E).$$

By definition of H^ψ, we infer that the induced connection ∇^ψ over $\psi^* E$ satisfies the formula

$$\nabla^\psi_\xi \sigma = T^{-1}_{\sigma(y)} \cdot [d_y \sigma(\xi) - H_{\sigma(y)} \cdot d_y \psi(\xi)],$$

for any $\xi \in T_{N,y}$.

The local frame e induces a local frame $\eta \; := \; e \circ \psi$ of $\psi^* E$ over $\psi^{-1}(U)$. We compute the local connection A^ψ form of ∇^ψ with respect to such frame. We notice that $\nabla^\psi \eta = \psi^*(e \cdot A) = \eta \cdot \psi^* A$ by the previous remark. We infer the equality $A^\psi = \psi^* A$.

A.2.2 Parallel transport

We consider a smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and a section $\sigma \in C^1((-\varepsilon, \varepsilon), \gamma^* E)$, which satisfies the equation

$$\nabla^\gamma_{\frac{d}{dt}} \sigma = 0,$$

over $(-\varepsilon, \varepsilon)$ with $\sigma(0) = \eta \in E_{\gamma(0)}$. If we write $\sigma(t) = e(\gamma(t)) \cdot f(t)$, then

$$\nabla^\gamma_{\frac{d}{dt}} \sigma = e(\gamma(t)) \cdot [\dot{f}(t) + A(\dot{\gamma}(t)) \cdot f(t)].$$

We infer that the parallel transport map $\tau_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$, $t \in (-\varepsilon, \varepsilon)$ given by $\tau_{\gamma,t}(\eta) = \sigma(t)$, is linear. We show the following fact.

Lemma A5. *For any smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and for any section $\sigma \in C^1((-\varepsilon, \varepsilon), \gamma^* E)$, the identity*

$$\nabla^\gamma_{\frac{d}{dt}} \sigma(0) = \frac{d}{dt}_{|t=0} [\tau^{-1}_{\gamma,t} \cdot \sigma(t)] \tag{A7}$$

holds.

Proof. We notice first that the term $\tau^{-1}_{\gamma,t} \cdot \sigma(t)$ is given by the intrinsic identities

$$\frac{du_t}{ds} + A(\dot{\gamma}(s)) \cdot u_t(s) = 0,$$
$$u_t(t) = f(t),$$
$$e(\gamma(0)) \cdot u_t(0) = \tau^{-1}_{\gamma,t} \cdot \sigma(t).$$

Integrating the first equation we infer

$$u_t(t) - u_t(0) = -\int_0^t A(\dot{\gamma}(s))\cdot u_t(s)ds.$$

Using the second equation, we obtain

$$f(t) - u_t(0) = -\int_0^t A(\dot{\gamma}(s))\cdot u_t(s)ds.$$

Deriving with respect to the variable t, we obtain

$$\frac{d}{dt}u_t(0) = \dot{f}(t) + A(\dot{\gamma}(t))\cdot u_t(t) = \dot{f}(t) + A(\dot{\gamma}(t))\cdot f(t).$$

Evaluating at $t = 0$ and multiplying both sides with $e(\gamma(0))$, we infer the required conclusion. □

We consider now a C^1-vector field ξ over M and let $\varphi_{\xi,t}$ be the associated one-parameter subgroup of transformations of M. Let $\Phi_{\xi,t} : E \longrightarrow E$ be the parallel transport map along the flow lines of $\varphi_{\xi,t}$. It is obvious by definition, that the map $\Phi_{\xi,t}$ satisfies $\pi_E \circ \Phi_{\xi,t} = \varphi_{\xi,t} \circ \pi_E$.

The vector field $\Xi := \dot{\Phi}_{\xi,0}$ over E satisfies the equality $\Xi(\eta) = H_\eta(\xi)$, for any $\eta \in E$. This is a direct consequence of the definition of the induced connection along the flow lines of ξ.

To any section $\sigma \in C^1(M, E)$ we can associate a C^1-vector field Σ over E defined as $\Sigma(\eta) := T_\eta[\sigma \circ \pi_E(\eta)]$. Let $\Phi_{\Sigma,t}$ be the associated one-parameter subgroup of transformations of E. In explicit terms, it satisfies

$$\Phi_{\Sigma,t}(\eta) = \eta + t\sigma \circ \pi_E(\eta).$$

Then

$$[\Xi, \Sigma] = \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0}(\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t}).$$

The fact that the map $\Phi_{\xi,-t}$ is linear on the fibers implies

$$\begin{aligned}\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} &= \Phi_{\xi,-t}[\Phi_{\xi,t} + s\sigma \circ \pi_E \circ \Phi_{\xi,t}]\\ &= \mathbb{I}_E + s\Phi_{\xi,-t}\cdot \sigma \circ \pi_E \circ \Phi_{\xi,t}\\ &= \mathbb{I}_E + s\Phi_{\xi,-t}\cdot \sigma \circ \varphi_{\xi,t} \circ \pi_E.\end{aligned}$$

Thus, for any $\eta \in E_p$ holds

$$\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t}(\eta) = \eta + s\Phi_{\xi,-t}\cdot \sigma \circ \varphi_{\xi,t}(p) \in E_p.$$

We conclude

$$[\Xi, \Sigma](\eta) = \frac{d}{dt}_{|t=0} T_\eta[\Phi_{\xi,-t}\cdot \sigma \circ \varphi_{\xi,t}(p)] = T_\eta[\nabla_\xi \sigma(p)],$$

i.e., for any $\eta \in E$, the equality holds

$$[\Xi, \Sigma](\eta) = T_\eta[(\nabla_\xi \sigma)\circ \pi_E(\eta)]. \tag{A8}$$

Iterating twice, we deduce the identity

$$[\Xi_1, [\Xi_2, \Sigma]](\eta) = T_\eta[(\nabla_{\xi_1}\nabla_{\xi_2}\sigma)\circ \pi_E(\eta)]. \tag{A9}$$

Moreover, the fact that by (A8) the vector fields $[\Xi_j, \Sigma]$, $j = 1, 2$ are tangent to the fibers of E and constant along them implies

$$[[\Xi_1, \Sigma], [\Xi_2, \Sigma]] = 0. \tag{A10}$$

A.3 The geometric meaning of the curvature tensor

Lemma A6. *Let $R := \nabla^2$ be the curvature tensor of the connection ∇. Then for any vector fields ξ_1, ξ_2 over M and for any $\eta \in E$, the identity holds*

$$\gamma_\eta^\nabla([\Xi_1, \Xi_2](\eta)) = T_\eta[R(\xi_2, \xi_1)\eta].$$

Proof. Let σ be a local section of E such that $\sigma(p) = \eta$. By definition of horizontal lift Ξ of a vector field ξ, we have

$$\Xi(\eta) = [d\sigma(\xi)] \circ \pi_E(\eta) - T_\eta[(\nabla_\xi \sigma) \circ \pi_E(\eta)].$$

We infer by (A8), the identity

$$[d\sigma(\xi)] \circ \pi_E = \Xi + [\Xi, \Sigma].$$

We infer $\sigma_* \xi = \Xi + [\Xi, \Sigma]$ over $\operatorname{Im} \sigma$. Thus,

$$\sigma_*[\xi_1, \xi_2] = [\sigma_* \xi_1, \sigma_* \xi_2] = [\Xi_1, \Xi_2] + [\Xi_1, [\Xi_2, \Sigma]] + [[\Xi_1, \Sigma], \Xi_2],$$

thanks to (A10). We rewrite the previous equality as

$$[\Xi_1, \Xi_2] = [\Xi_2, [\Xi_1, \Sigma]] - [\Xi_1, [\Xi_2, \Sigma]] - \sigma_*[\xi_2, \xi_1].$$

By using (A9), we deduce

$$\begin{aligned}[\Xi_1, \Xi_2](\eta) &= T_\eta[(\nabla_{\xi_2}\nabla_{\xi_1}\sigma - \nabla_{\xi_1}\nabla_{\xi_2}\sigma)(p)] - d_p\sigma([\xi_2, \xi_1]) \\ &= T_\eta[(\nabla_{\xi_2}\nabla_{\xi_1}\sigma - \nabla_{\xi_1}\nabla_{\xi_2}\sigma - \nabla_{[\xi_2, \xi_1]}\sigma)(p)] - H_\eta([\xi_2, \xi_1]) \\ &= T_\eta[R(\xi_2, \xi_1)\sigma(p)] + H_\eta([\xi_1, \xi_2]).\end{aligned}$$

We infer the required conclusion. □