Research Article

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Geodesics and magnetic curves in the 4-dim almost Kähler model space F^4

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Abstract: We study geodesics and magnetic trajectories in the model space F^4. The space F^4 is isometric to the 4-dim simply connected Riemannian 3-symmetric space due to Kowalski. We describe the solvable Lie group model of F^4 and investigate its curvature properties. We introduce the symplectic pair of two Kähler forms on F^4. Those symplectic forms induce invariant Kähler structure and invariant strictly almost Kähler structure on F^4. We explore some typical submanifolds of F^4. Next, we explore the general properties of magnetic trajectories in an almost Kähler 4-manifold and characterize Kähler magnetic curves with respect to the symplectic pair of Kähler forms. Finally, we study homogeneous geodesics and homogeneous magnetic curves in F^4.

Keywords: 4-dim Thurston space, almost Kähler 3-symmetric space F^4, geodesics, magnetic trajectories

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Dedicated to the memory of professor Mitsuhiro Itoh.

1 Introduction

From the mathematical point of view, it is natural to reinterpret static magnetic fields on Euclidean 3-space as *closed two-forms*. The equation of motion (Lorentz equation) of charged particle under the influence of magnetic field can be generalized to arbitrary Riemannian manifolds through the following procedure:

To introduce the notion of magnetic trajectories on a manifold M, we need two ingredients:
– Riemannian metric g and
– closed two-form F.

The *Lorentz force* ϕ of a pair (F, g) is a g-skew adjoint endomorphism field defined by $F = g(\phi, \cdot)$. The *Lorentz equation* is formulated as

$$\nabla_{\gamma'}\gamma' = q\phi\gamma',$$

where q is a constant and called the *charge*. A curve γ satisfying the Lorentz equation is referred to as a *magnetic trajectory* or *magnetic geodesic* with charge q.

One can see that the Lorentz equation defines a Hamiltonian system on the tangent bundle of (M, g, F) [1]. The study of magnetic trajectories in arbitrary Riemannian manifolds (as Hamiltonian systems) was developed mostly in the early 1990s, even though related pioneer works were published much earlier (see Arnold's articles [1,2]). We can refer to Arnold's problems concerning charges in magnetic fields on Riemannian manifolds of arbitrary dimension, commented by Ginzburg in [3] and references therein.

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In previous works [4–11], magnetic trajectories in the model spaces of three-dimensional geometries in the sense of Thurston have been studied extensively.

On even dimensional manifold M, we can consider the following two situations:

(1) M has an almost Hermitian structure (J, g). In this case, we need to assume that the fundamental 2-form $\Omega = g(\cdot, J)$ is *closed*. Then, (M, J, g) is called an *almost Kähler manifold*. Note that the fundamental 2-form Ω of an almost Kähler manifold is a *symplectic form*, i.e., a non-degenerate closed 2-form. Thus, (M, Ω) is a *symplectic manifold*.

(2) M has a closed 2-form Ω. In this case, we need Riemannian metric g. If Ω is *non-degenerate*, then we obtain an almost complex structure J on M so that J is the Lorentz force associated with the structure $(-\Omega, g)$. Hence, (M, J, g) is an almost Kähler manifold.

Kähler manifolds are characterized as integrable almost Kähler manifolds. Almost Kähler manifolds with non-integrable almost complex structure are called *strictly almost Kähler manifolds*.

On a symplectic manifold (M, Ω, J) equipped with a compatible almost complex structure J, or equivalently an almost Kähler manifold (M, J, g), one can take the fundamental 2-form Ω as a magnetic field and call it a *Kähler magnetic field*.

From the Hamiltonian viewpoint, homogeneous Kähler manifolds would be nice configuration spaces for magnetic trajectories. A *C-space* is a *compact* simply connected complex homogeneous space. A C-space that admits a Kähler metric with transitive holomorphic isometry group is called a *Kähler C-space*. Itoh [12] investigated the curvature properties of Kähler C-spaces with $b_2 = 1$, where b_2 is the second Betti number. On the other hand, let G be a compact simple Lie group with an abelian subgroup T. Denote by $C(T)$ the centralizer of T. Then, the coset manifold $G/C(T)$ is called a *generalized flag manifold*. It is known that generalized flag manifolds admit Kähler C-structures. Conversely, every Kähler C-space is represented as generalized flag manifold. In addition, every coadjoint orbit of a compact semi-simple Lie group is a Kähler C-space with respect to *Kirillov-Kostant-Souriau symplectic form* [13, 8.70] and compatible Killing metric. Conversely, every Kähler C-space is isomorphic to a coadjoint orbit of its connected group of isometries endowed with its canonical complex structure [13, 8.89]. For more information on generalized flag manifolds, we refer to [13].

Efimov [14] studied magnetic trajectories in homogeneous symplectic manifolds. More precisely, he proved the integrability of Kähler magnetic trajectories on simply connected homogenous symplectic manifold G/H with compact semi-simple G. Moreover, he proved the (non-commutative) integrability of magnetic trajectories on coadjoint orbits of compact semi-simple Lie groups (see also [15]).

Adachi initiated the study on Kähler magnetic trajectories in non-flat complex space forms (*cf.* [16]). Kalinin studied H-planar flows as an important class of Hamiltonian flows on Kähler manifolds [17]. He proved that the Lorentz equation on complex space forms can be reduced to one ordinary differential equation of second order, by virtue of H-projective mappings. Adachi et al. gave an representation formula for Kähler magnetic trajectories on Hermitian symmetric spaces [18]. In addition, Ikawa studied Kähler magnetic trajectories in generalized flag manifolds with two isotropy summands satisfying certain assumption [19]. Arvanitoyeorgos and Chrysikos [20] classified generalized flag manifolds considered in [19].

As far as the authors know, studies on Kähler magnetic fields on *strictly almost Kähler manifold* are very few. The lowest dimension of almost Kähler manifolds is 2. In two dimensions, almost Hermitian structures are automatically integrable. In four dimensions, there exist closed strictly almost Kähler manifolds (Kodaira-Thurston manifolds [21], Gompf's example [22], Yamato's example [23]) For more information on almost Kähler geometry, we refer to a survey [24].

We start our investigation with homogeneous examples. In two dimensions, Hermitian symmetric spaces are complex space forms (actually real space forms): complex plane $\mathbb{C}$, complex projective line $\mathbb{C}P_1$, and complex hyperbolic line $\mathbb{C}H_1$. Kähler magnetic trajectories in these spaces are folklore.

In four dimensions, as we have mentioned before, Kähler magnetic trajectories in Hermitian symmetric spaces are well investigated. We look for homogeneous almost Kähler 4-manifolds different from Hermitian symmetric spaces.

Filipkiewicz classified the four-dimensional maximal model geometries [25] (see also [26,27]). Wall gave the list of all four-dimensional geometries that admit compatible complex structure [28, Theorem 2.1]:

Complex space form	Hermitian symmetric	Kähler	LCK
$\mathbb{C}P^2$, $\mathbb{C}H^2$, $\mathbb{E}^4$	$\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$ $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	F^4	$\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$ Sol_0^4, Sol_1^4

The model space of F^4-*geometry* is the homogeneous Riemannian 4-space SA(2)/SO(2), where SA(2) is the orientation-preserving equiaffine transformations of the *equiaffine plane*, i.e., $\mathbb{R}^2$ equipped with a parallel area element. The F^4-geometry is the only four-dimensional geometry that admits finite-volume quotients but *no compact quotients*. The model space F^4 admits both Kähler structure and strictly almost Kähler structure. The pair of those fundamental 2-forms constitutes the so-called *symplectic pair*.

On the other hand, from another viewpoint, we may suggest to look for almost Kähler 4-manifolds, which are naturally reductive or generalized symmetric. The class of naturally reductive homogenous spaces is one of the generalizations of Riemannian symmetric spaces. The classification of naturally reductive 4-spaces due to Kowalski and Vanhecke [29] implies that naturally reductive almost Kähler 4-manifolds are Hermitian symmetric.

According to the classification of all simply connected and irreducible generalized Riemannian symmetric spaces of low dimension due to Kowalski, there exists only four-dimensional generalized Riemannian symmetric space. This space is realized as the homogeneous space SA(2)/SO(2) equipped with homogeneous almost Kähler structure. One can see that Kowalski's generalized Riemannian symmetric space is isometric to the space F^4 [30]. Fino [31, Corollary 3.1] gave certain curvature characterization of F^4.

As is well known, geodesics of a Riemannian symmetric space (more generally, naturally reductive homogeneous space) are *homogeneous*, i.e., orbits of 1-parameter subgroups of the isometry group. However, geodesics of F^4 are not homogeneous, in general. On this reason, determination of homogeneous geodesics in F^4 is a fundamental and important task.

These observations motivate us to study Kähler magnetic trajectories with respect to the symplectic pair and homogeneous geodesics in the model space F^4. However, comparing F^4 with other model spaces, the Lorentz equation of Kähler magnetic field in F^4 is very complicated. We anticipate that this project will extend over a considerable duration. As an initial phase for this project, this article is dedicated to delineating foundational information, especially homogeneous geometry of the model space F^4 and deducing the systems of Kähler magnetic trajectories with respect to the symplectic pair (Ω_+, Ω_-).

This article is organized as follows. We start with recalling the notion of symplectic pair and self-duality in Section 2. Section 3 is devoted to describing the four-dimensional simply connected Riemannian 3-symmetric space $\hat{M}^4_\lambda$ due to Kowalski. In particular, we describe the invariant almost complex structures $J_\pm$ of $\hat{M}^4_\lambda$ discovered by Kowalski. From Section 4, the homogeneous geometry of the model space F^4 will be started. More precisely, we discuss the coset space representation SA(2)/SO(2) for the 3-symmetric space $\hat{M}^4_\lambda$.

In Section 5, we recall the definition of the model space F^4 from the classification due to Filipkiewicz [25] and Wall [27]. It should be remarked that the model space F^4 is isometric to the 3-symmetric space $\hat{M}^4_\lambda$ (see [30]). In particular, we show that the model space F^4 is identified with certain solvable Lie group $\mathbb{R}^2_+ \ltimes \mathbb{R}^2$. We give the Levi-Civita connection, Riemannian curvature, Ricci operator, sectional curvatures, and scalar curvature of the model space F^4. Furthermore, the symplectic pair of two Kähler forms is introduced. Some typical submanifolds of F^4 are explored.

From Section 6, we start our investigation on geodesics and magnetic trajectories in F^4. In Section 6, the general properties of magnetic trajectories in an almost Kähler 4-manifold are studied. In Sections 7 and 8, Kähler magnetic trajectories with respect to the symplectic pair of Kähler forms are investigated. In the final section, we study homogeneous geodesics and homogeneous magnetic curves in F^4. Although the fact that

Kowalski et al. [32] gave a classification of homogeneous geodesics in F^4, here we give a new classification based on different expressions.

2 Preliminaries on almost Kähler geometry

2.1 Self-duality

Let (M, g) be an oriented Riemannian 4-manifold. Take a positively oriented orthonormal coframe field $\{\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4\}$. Then, the *Hodge star operator* $*$ acting on the space $A^2(M)$ of two-forms on M is described as

$$*(\vartheta^1 \wedge \vartheta^2) = \vartheta^3 \wedge \vartheta^4, \quad *(\vartheta^1 \wedge \vartheta^3) = \vartheta^4 \wedge \vartheta^2, \quad *(\vartheta^1 \wedge \vartheta^4) = \vartheta^2 \wedge \vartheta^3.$$

Since $*$ has eigenvalues ± 1, we have the decomposition

$$A^2(M) = A^2_+(M) \oplus A^2_-(M),$$

where

$$A^2_+(M) = \{\omega \in A^2(M) \mid *\omega = \omega\}, \quad A^2_-(M) = \{\omega \in A^2(M) \mid *\omega = -\omega\}.$$

A two-form ω is said to be *self-dual* (resp. *anti self-dual*) if $*\omega = \omega$ (resp. $*\omega = -\omega$). The space $A^2_+(M)$ is locally spanned by

$$\{\vartheta^1 \wedge \vartheta^2 + \vartheta^3 \wedge \vartheta^4, \vartheta^1 \wedge \vartheta^3 + \vartheta^4 \wedge \vartheta^2, \vartheta^1 \wedge \vartheta^4 + \vartheta^2 \wedge \vartheta^3\}.$$

On the other hand, $A^2_-(M)$ is locally spanned by

$$\{\vartheta^1 \wedge \vartheta^2 - \vartheta^3 \wedge \vartheta^4, \vartheta^1 \wedge \vartheta^3 - \vartheta^4 \wedge \vartheta^2, \vartheta^1 \wedge \vartheta^4 - \vartheta^2 \wedge \vartheta^3\}.$$

2.2 Symplectic pair

Let M be a $2n$-dimensional symplectic manifold, i.e., a $2n$-manifold equipped with a non-degenerate closed 2-form Ω called a *symplectic form*.

Definition 2.1. Let (M, Ω) be a symplectic manifold. An almost complex structure J is said to be *compatible* to Ω if

- Ω is *J-invariant*, i.e., $\Omega(JX, JY) = \Omega(X, Y)$,
- J is *Ω-tamed*, i.e., $\Omega(JX, X) > 0$, for any $X \neq 0$.

Such an almost complex structure J defines a Riemannian metric g by

$$g(X, Y) = \Omega(JX, Y).$$

One can see that g is J-invariant. Thus, (M, J, g) is an almost Kähler manifold, i.e., an almost Hermitian manifold with closed fundamental 2-form $\Omega = g(\cdot, J)$. On an almost Kähler manifold, its fundamental 2-form is also called the *Kähler form*.

Instead of a compatible almost complex structure J, we may equip a symplectic manifold (M, Ω) with a Riemannian metric g. Then, an almost complex structure J compatible to Ω is introduced by $g(\cdot, J) = \Omega$. Note that J is the Lorentz force associated with the magnetic field $-\Omega$.

An almost Kähler manifold is said to be a *Kähler manifold* if its almost complex structure is integrable. In other words, Kähler manifolds can be defined as Hermitian manifolds with closed fundamental 2-form.

Remark 1. (Critical metrics) Let (M, Ω) be a compact symplectic manifold. Denote by $\mathcal{AK}(M, \Omega)$ the Fréchet space of all compatible almost complex structures on (M, Ω). A compatible almost complex structure $J \in \mathcal{AK}(M, \Omega)$ is a critical point of the total scalar curvature, i.e., Einstein-Hilbert functional if and only if its Ricci tensor field is J-invariant [33].

Remark 2. On a 4-manifold M, a pair (Ω_+, Ω_-) of non-trivial symplectic forms is said to be a *symplectic pair* if

$$\Omega_+ \wedge \Omega_- = 0, \quad \Omega_+ \wedge \Omega_+ = -\Omega_- \wedge \Omega_-.$$

The kernels of $\Omega_+ \pm \Omega_-$ define complementary foliations with minimal leaves. Conversely, any symplectic pair on a 4-manifold is given by a pair of two-dimensional oriented complementary minimal foliations [34].

In [35], the authors considered almost Hermitian 4-manifold (M, J, g) such that $\Omega = g(\cdot, J)$ is anti self-dual [resp. self-dual]. Another almost complex structure J' is said to be *opposite* if $\Omega' = g(\cdot, J')$ is self-dual [resp. anti self-dual]. By definition, (Ω, Ω') is a symplectic pair if and only if $\mathrm{d}\Omega = \mathrm{d}\Omega' = 0$.

Remark 3. Let $\mathbb{H}^3$ be the hyperbolic 3-space. Then, its Riemannian product $\mathbb{H}^3 \times \mathbb{R}$ with the real line is one of the model spaces of four-dimensional geometry. Note that $\mathbb{H}^3 \times \mathbb{R}$ admits a solvable Lie group structure. However, this model space does not admit complex structure invariant under the isometry group $\mathrm{SO}^+(3,1) \times \mathbb{R}$. On the other hand, $\mathbb{H}^3 \times \mathbb{R}$ admits an non-integrable almost complex structure J invariant under left translations by the solvable Lie group $\mathbb{H}^3 \times \mathbb{R}$. The resulting left-invariant almost Hermitian structure is almost Kähler. In our previous work, we studied Kähler magnetic trajectories in $\mathbb{H}^3 \times \mathbb{R}$ equipped with this almost Kähler structure [36].

3 Generalized Riemannian symmetric 4-space

3.1 Riemannian structure

It is known that the underlying homogeneous Riemannian space of F^4 is a four-dimensional Riemannian 3-symmetric space. Here, we recall the explicit model of four-dimensional Riemannian 3-symmetric space due to Kowalski [37].

Let $\hat{M}^4_\lambda = (\hat{M}^4, \hat{g}_\lambda)$ be a Riemannian 4-manifold defined as the Cartesian 4-space $\mathbb{R}^4(x_1, x_2, x_3, x_4)$ with metric

$$\begin{aligned}\hat{g}_\lambda = {} & (-x_1 + \sqrt{x_1^2 + x_2^2 + 1})\mathrm{d}x_3^2 + (x_1 + \sqrt{x_1^2 + x_2^2 + 1})\mathrm{d}x_4^2 - 2x_2\mathrm{d}x_3\mathrm{d}x_4 \\ & + \lambda\left\{\frac{(1 + x_2^2)\mathrm{d}x_1^2 + (1 + x_1^2)\mathrm{d}x_2^2 - 2x_1x_2\mathrm{d}x_1\mathrm{d}x_2}{1 + x_1^2 + x_2^2}\right\},\end{aligned}$$

where λ is a positive constant. Kowalski showed that $\hat{M}^4_\lambda$ with symmetry

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) & 0 & 0 \\ \sin(4\pi/3) & \cos(4\pi/3) & 0 & 0 \\ 0 & 0 & \cos(2\pi/3) & -\sin(2\pi/3) \\ 0 & 0 & \sin(4\pi/3) & \cos(4\pi/3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

of order 3 at the origin is the only four-dimensional irreducible non-symmetric generalized Riemannian symmetric space [37].

We can take the following global orthonormal frame [38]:

$$\hat{e}_1 := \frac{1}{\sqrt{\lambda(x_1^2+x_2^2)}}\left(-x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}\right),\quad \hat{e}_2 := \sqrt{\frac{x_1^2+x_2^2+1}{\lambda(x_1^2+x_2^2)}}\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right),$$
$$\hat{e}_3 := \frac{x_1+\sqrt{x_1^2+x_2^2}}{\sqrt{2}\,x_2\mu_-}\frac{\partial}{\partial x_3}+\frac{1}{\sqrt{2}\,\mu_-}\frac{\partial}{\partial x_4},\quad \hat{e}_4 := \frac{x_1-\sqrt{x_1^2+x_2^2}}{\sqrt{2}\,x_2\mu_+}\frac{\partial}{\partial x_3}+\frac{1}{\sqrt{2}\,\mu_+}\frac{\partial}{\partial x_4},$$

where

$$\mu_\pm := \sqrt{-\frac{\sqrt{x_1^2+x_2^2}}{\pm x_2^2+(x_1-\sqrt{x_1^2+x_2^2+1})(\pm x_1+\sqrt{x_1^2+x_2^2})}}.$$

The dual coframe field $\{\hat{\vartheta}^1,\hat{\vartheta}^2,\hat{\vartheta}^3,\hat{\vartheta}^4\}$ of $\{\hat{e}_1,\hat{e}_1,\hat{e}_3,\hat{e}_4\}$ is given by

$$\hat{\vartheta}^1 = \frac{\sqrt{\lambda(x_1^2+x_2^2)}}{x_1^2+x_2^2}(-x_2\mathrm{d}x_1+x_1\mathrm{d}x_2),$$
$$\hat{\vartheta}^2 = \frac{\sqrt{\lambda(x_1^2+x_2^2)}}{(x_1^2+x_2^2)\sqrt{x_1^2+x_2^2+1}}(x_1\mathrm{d}x_1+x_2\mathrm{d}x_2),$$
$$\hat{\vartheta}^3 = \frac{\mu_-}{\sqrt{2}\sqrt{x_1^2+x_2^2}}(x_2\mathrm{d}x_3+(-x_1+\sqrt{x_1^2+x_2^2})\mathrm{d}x_4),$$
$$\hat{\vartheta}^4 = \frac{\mu_+}{\sqrt{2}\sqrt{x_1^2+x_2^2}}(-x_2\mathrm{d}x_3+(x_1+\sqrt{x_1^2+x_2^2})\mathrm{d}x_4).$$

We have

$$\hat{\vartheta}^1\wedge\hat{\vartheta}^2 = -\frac{\lambda}{\sqrt{x_1^2+x_2^2+1}}\mathrm{d}x_1\wedge\mathrm{d}x_2,\quad \hat{\vartheta}^3\wedge\hat{\vartheta}^4 = \mathrm{d}x_3\wedge\mathrm{d}x_4.$$

Calvaruso, Leo, and Van der Veken studied the curvature property and submanifold geometry of the four-dimensional semi-Riemannian 3-symmetric spaces in [39,40]. Here, we quote the following result.

Theorem 3.1. [40] *There are no parallel hypersurfaces in $\hat{M}^4_\lambda$. In particular, there are no totally geodesic hypersurfaces in $\hat{M}^4_\lambda$.*

This result is in a sharp contrast to Riemannian space forms.

3.2 Almost complex structures

According to [37] and [41, pp. 87–88], every invariant almost Hermitian structure on the 3-symmetric space $\hat{M}^4_\lambda$ is almost Kähler. We have two symplectic forms (see [38, pp. 53–54]):

$$\hat{\Omega}_\pm = \hat{\vartheta}^1\wedge\hat{\vartheta}^2 \pm \hat{\vartheta}^3\wedge\hat{\vartheta}^4 = -\frac{\lambda}{\sqrt{x_1^2+x_2^2+1}}\mathrm{d}x_1\wedge\mathrm{d}x_2 \pm \mathrm{d}x_3\wedge\mathrm{d}x_4.$$

Then $\hat{\Omega}_-$ gives a Kähler structure and anti self-dual 2-form. On the other hand, $\hat{\Omega}_+$ is non-Kähler. Note that $\hat{\Omega}_+$ is self-dual (see [38, pp. 53–54]).

The almost complex structures $\hat{J}_\pm$ defined by

$$\hat{g}(X,\hat{J}_\pm Y) = \hat{\Omega}_\pm(X,Y)$$

are expressed as [37, Example III.53]:

$$\hat{J}_+\hat{e}_1 = -\hat{e}_2, \quad \hat{J}_+\hat{e}_2 = \hat{e}_1, \quad \hat{J}_+\hat{e}_3 = -\hat{e}_4, \quad \hat{J}_+\hat{e}_4 = \hat{e}_3,$$
$$\hat{J}_-\hat{e}_1 = -\hat{e}_2, \quad \hat{J}_-\hat{e}_2 = \hat{e}_1, \quad \hat{J}_-\hat{e}_3 = \hat{e}_4, \quad \hat{J}_-\hat{e}_4 = -\hat{e}_3.$$

Note that in [38], the convention

$$\hat{\Omega}_\pm(X,Y) = \hat{g}(\hat{J}_\pm X, Y)$$

is used.

According to [38, p. 54], the Ricci operator has components:

$$-\frac{3}{2\lambda}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This formula implies that the Ricci tensor field $\hat{\rho}$ of $(\hat{M}, \hat{J}_\pm, \hat{g}_\lambda)$ is $\hat{J}_\pm$- *invariant*, i.e.,

$$\hat{\rho}(\hat{J}_\pm X, \hat{J}_\pm Y) = \hat{\rho}(X,Y).$$

Thus, the metric on the four-dimensional Riemannian 3-symmetric space is a critical metric in the sense of [33]. Moreover, $(\hat{\Omega}_+, \hat{\Omega}_-)$ is a symplectic pair.

Apostolov et al. [30] obtained the following *rigidity theorem*.

Theorem 3.2. *Let (M, g, J, J') be a Riemannian 4-manifold equipped with two orthogonal almost complex structures. If (g, J) is a strictly almost Kähler structure and (g, J') is a Kähler structure opposite to (g, J), then (M, g, J, J') is locally isometric to the Riemannian 3-symmetric 4-space $\hat{M}^4_\lambda$.*

4 Homogeneous space representation

In this section, we give a coset space representation of the four-dimensional Riemannian 3-symmetric space $\hat{M}^4_\lambda$.

4.1 Homogeneous manifold SA(2)/SO(2)

Let us denote by SA(2) the Lie group of all orientation-preserving equiaffine transformations of the *equiaffine plane* $\mathbb{R}^2 = (\mathbb{R}^2(x,y), dx \wedge dy)$. The Lie group SA(2) is explicitly given by

$$\mathrm{SA}(2) = \left\{ \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c, d, u, v \in \mathbb{R}, \quad ad - bc = 1 \right\}.$$

The Lie algebra $\mathfrak{sa}(2)$ of SA(2) is given by

$$\mathfrak{sa}(2) = \left\{ \begin{pmatrix} \alpha & \beta & \xi \\ \gamma & -\alpha & \eta \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma, \xi, \eta \in \mathbb{R} \right\}.$$

One can confirm that $[\mathfrak{sa}(2), \mathfrak{sa}(2)] = \mathfrak{sa}(2)$. Hence, $\mathfrak{sa}(2)$ is not solvable. Take a closed subgroup

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, 0 \le \theta < 2\pi \right\}$$

of SA(2). Then, the homogeneous manifold $M = \mathrm{SA}(2)/\mathrm{SO}(2)$ is four-dimensional and admits a structure of Riemannian 3-symmetric space.

Theorem 4.1. [37, p. 136, Theorem VI.3] *Any proper, simply connected and irreducible generalized Riemannian symmetric space* (M, g) *of dimension* 4 *is of order* 3 *and isomorphic to* $\hat{M}^4_\lambda$. *The underlying homogeneous manifold of* M *is* SA(2)/SO(2).

The reductive decomposition of the Lie algebra $\mathfrak{sa}(2)$ of SA(2) corresponding to $M = \mathrm{SA}(2)/\mathrm{SO}(2)$ is $\mathfrak{sa}(2) = \mathfrak{so}(2) + \mathfrak{m}_\lambda$ (see [37, p. 139]). Here, the Lie subspace $\mathfrak{m}_\lambda$ is spanned by the basis

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isotropy algebra $\mathfrak{so}(2)$ is spanned by

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations are

$$\begin{gathered} [X_1, X_2] = X_1, \quad [X_1, Y_1] = 0, \quad [X_1, Y_2] = -Y_1, \quad [X_1, B] = -Y_1, \\ [X_2, Y_1] = Y_1, \quad [X_2, Y_2] = 2B, \quad [X_2, B] = 2Y_2, \quad [Y_1, Y_2] = -X_1, \\ [Y_1, B] = X_1, \quad [Y_2, B] = -2X_2. \end{gathered} \tag{4.1}$$

For any vector $X \in \mathfrak{sa}(2)$, we denote by $X^\sharp$ the infinitesimal equiaffine transformation on $\mathbb{R}^2(x, y)$ induced by X, i.e.,

$$X^\sharp_{(x,y)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\exp_{\mathfrak{sa}(2)}(tX) \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

Then, we have

$$[X^\sharp, Y^\sharp] = -[X, Y]^\sharp.$$

The infinitesimal equiaffine transformations induced from X_1, Y_1, X_2, Y_2, and B are given by

$$X_1^\sharp = \frac{\partial}{\partial x}, \quad Y_1^\sharp = \frac{\partial}{\partial y}, \quad X_2^\sharp = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y_2^\sharp = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad B^\sharp = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

respectively (see [37, p. 139]). Note that our $X_1^\sharp$, $Y_1^\sharp$, $X_2^\sharp$, $Y_2^\sharp$, and $B^\sharp$ are denoted by X_1, Y_1, X_2, Y_2, and B in [37], respectively.

4.2 Invariant almost Kähler structures

Kowalski's metric $\hat{g}_\lambda$ on SA(2)/SO(2) determined by the condition that $\{\hat{E}_1 := X_1, \hat{E}_2 := Y_1, \hat{E}_3 := X_2/\lambda, \hat{E}_4 := Y_2/\lambda\}$ is orthonormal with respect to it. Note that the Levi-Civita connection $\hat{\nabla}$ is computed in [39,40]. It should be remarked that this $\{\hat{E}_1, \hat{E}_2, \hat{E}_3, \hat{E}_4\}$ does not correspond to $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ in the previous section. In fact, the present one is left-invariant, but $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ is not.

The non-integrable almost complex structure $\hat{J}_+$ is

$$\hat{J}_+\hat{E}_1 = -\hat{E}_2, \quad \hat{J}_+\hat{E}_2 = \hat{E}_1, \quad \hat{J}_+\hat{E}_3 = -\hat{E}_4, \quad \hat{J}_+\hat{E}_4 = \hat{E}_3.$$

On the other hand, the complex structure $\hat{J}_-$ is

$$\hat{J}_-\hat{E}_1 = -\hat{E}_2, \quad \hat{J}_-\hat{E}_2 = \hat{E}_1, \quad \hat{J}_-\hat{E}_3 = \hat{E}_4, \quad \hat{J}_-\hat{E}_4 = -\hat{E}_3.$$

Unfortunately, magnetic equation for $\hat{\Omega}_\pm$ with respect to the coordinates (x_1, x_2, x_3, x_4) is complicated. So we introduce another model of $\hat{M}^4_\lambda$ in the next section.

5 Model space F^4

5.1 Riemannian structure of F^4

Among the list of four-dimensional Thurston geometries, there exists a geometry that has no compact models. The model space of this geometry is denoted by F^4. According to [27] (see also [26]), the model space $\mathrm{F}^4_c = (\mathrm{F}^4, g_c)$ of F^4-geometry is

$$\mathbb{H}^2(-4c^2) \times \mathbb{R}^2 = \{(x, y, u, v) \in \mathbb{R}^4 \mid y > 0\},$$

equipped with a homogeneous Riemannian metric

$$g_c = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4c^2y^2} + \frac{(\mathrm{d}u - x\mathrm{d}v)^2}{y} + y\mathrm{d}v^2. \tag{5.1}$$

On F^4_c, the equiaffine transformation group SA(2) acts isometrically and transitively via the action:

$$\begin{pmatrix} a_{11} & a_{12} & \xi \\ a_{21} & a_{22} & \eta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x + \sqrt{-1}y \\ u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{a_{11}(x + \sqrt{-1}y) + a_{12}}{a_{21}(x + \sqrt{-1}y) + a_{22}} \\ a_{11}u + a_{12}v + \xi \\ a_{21}u + a_{22}v + \eta \end{pmatrix}.$$

The isotropy subgroup of SA(2) at the origin $(0, 1, 0, 0)$ is SO(2). Hence, F^4_c is identified with SA(2)/SO(2).

On the other hand, according to [41, pp. 87–88], every invariant almost Hermitian structure on M is almost Kähler. Thus, we can study magnetic trajectories with respect to the Kähler structure and strictly almost Kähler structure.

Remark 4. Wall [27, p. 123] gave the following expression for the metric of F^4 (see also [42]):

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + \frac{1}{y}\left(\mathrm{d}u^2 + \mathrm{d}v^2 - \frac{2v}{y}(\mathrm{d}x\mathrm{d}u + \mathrm{d}y\mathrm{d}v) + \frac{v^2(\mathrm{d}x^2 + \mathrm{d}y^2)}{y^2}\right).$$

5.2 Solvable Lie group model

Let us recall the *polar decomposition* of $\mathrm{SL}_2\mathbb{R}$. The special linear group $\mathrm{SL}_2\mathbb{R}$ has the decomposition:

$$\mathrm{SL}_2\mathbb{R} = \mathcal{S} \cdot \mathrm{SO}(2),$$

where $\mathcal{S}$ is a solvable Lie group defined by

$$\mathcal{S} = \left\{ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \middle| (x, y) \in \mathbb{R}^2, \quad y > 0 \right\}.$$

The solvable Lie group $\mathcal{S}$ is identified with the upper-half plane

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | y > 0\}.$$

The Lie algebra $\mathfrak{s}$ of $\mathcal{S}$ is given by

$$\mathfrak{s} = \left\{ \begin{pmatrix} s_{11} & s_{12} \\ 0 & -s_{11} \end{pmatrix} \middle| s_{11}, s_{12} \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{sl}_2\mathbb{R}$ has the decomposition $\mathfrak{sl}_2\mathbb{R} = \mathfrak{so}(2) \oplus \mathfrak{s}$:

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \begin{pmatrix} 0 & -x_{21} \\ x_{21} & 0 \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} + x_{21} \\ 0 & -x_{11} \end{pmatrix}.$$

5.3 Semi-direct product model

Every homogeneous Riemannian 4-space is either locally symmetric or locally isometric to a Lie group with a left-invariant metric [43,44]. On the other hand, four-dimensional Lie groups that admit left-invariant symplectic form are solvable [45]. In this subsection, we give an explicit solvable Lie group model of F^4_c. Note that Fino proved that four-dimensional Lie groups equipped with left-invariant strictly almost Kähler structure with J-invariant Ricci tensor field are solvable [31].

Since the isotropy subgroup of SA(2) at the origin $(0, 1, 0, 0) \in \mathrm{F}^4_c$ is SO(2), the semi-direct product group

$$\mathcal{S} \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} & u \\ 0 & 1/\sqrt{y} & v \\ 0 & 0 & 1 \end{pmatrix} \middle| (x, y, u, v) \in \mathbb{R}^4, \quad y > 0 \right\} \subset \mathrm{SA}(2)$$

acts simply transitively on F^4_c. Indeed,

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} & u \\ 0 & 1/\sqrt{y} & v \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{-1} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x + \sqrt{-1}y \\ u \\ v \end{pmatrix}.$$

Hence, the model space $\mathrm{F}^4_c = \mathrm{SA}(2)/\mathrm{SO}(2)$ is identified with the semi-direct product

$$\mathbb{R}^2_+ \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} & u \\ 0 & 1/\sqrt{y} & v \\ 0 & 0 & 1 \end{pmatrix} \middle| (x, y, u, v) \in \mathbb{R}^4, \quad y > 0 \right\}.$$

The group multiplication is given explicitly by

$$(x, y, u, v)*(x', y', u', v') = \left(x + yx', yy', u + \sqrt{y}u' + \frac{x}{\sqrt{y}}v', v + \frac{v'}{\sqrt{y}} \right).$$

The inverse element of (x, y, u, v) is given by

$$(x, y, u, v)^{-1} = (-x/y, 1/y, (-u + xv)/\sqrt{y}, -\sqrt{y}v).$$

Let us consider the inclusion map $\iota : \mathbb{R}^2_+ \times \mathbb{R}^2 \to \mathrm{SA}(2)$ defined by

$$\iota(x,y,u,v) = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} & u \\ 0 & 1/\sqrt{y} & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$\iota^{-1}\iota_*\frac{\partial}{\partial x} = \frac{1}{y}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \iota^{-1}\iota_*\frac{\partial}{\partial y} = \frac{1}{y}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\iota^{-1}\iota_*\frac{\partial}{\partial u} = \frac{1}{\sqrt{y}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \iota^{-1}\iota_*\frac{\partial}{\partial v} = \begin{pmatrix} 0 & 0 & -x/\sqrt{y} \\ 0 & 0 & \sqrt{y} \\ 0 & 0 & 0 \end{pmatrix}.$$

These formulas suggest us to take the following basis of the Lie algebra $\mathfrak{sa}(2)$:

$$\bar{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\bar{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\bar{e}_1 = \frac{1}{2}(Y_2 - B), \quad \bar{e}_2 = -\frac{1}{2}X_2, \quad \bar{e}_3 = X_1, \quad \bar{e}_4 = Y_1, \quad \bar{e}_5 = B.$$

The commutation relations are ([31, Remark 2.2], [46, Remark 3.4])

$$[\bar{e}_1, \bar{e}_2] = -\bar{e}_1, \quad [\bar{e}_1, \bar{e}_3] = 0, \quad [\bar{e}_1, \bar{e}_4] = \bar{e}_3,$$
$$[\bar{e}_2, \bar{e}_3] = \frac{1}{2}\bar{e}_3, \quad [\bar{e}_2, \bar{e}_4] = -\frac{1}{2}\bar{e}_4, \quad [\bar{e}_3, \bar{e}_4] = 0.$$

5.4 Exponential map

Let us describe the tangent space $\mathfrak{f}$ of the homogeneous space $\mathrm{F}^4_c = \mathrm{SA}(2)/\mathrm{SO}(2)$ at the origin. The Lie algebra $\mathfrak{sa}(2)$ is decomposed as

$$\mathfrak{sa}(2) = \mathfrak{so}(2) \oplus \mathfrak{f},$$

where $\mathfrak{so}(2)$ is the Lie algebra of SO(2):

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -t_5 & 0 \\ t_5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t_5 \in \mathbb{R} \right\}$$

as we saw before. The tangent space $\mathfrak{f}$ of F^4_c at the origin is given by

$$\mathfrak{f} = \left\{ \begin{pmatrix} t_2/2 & t_1 & t_3 \\ 0 & -t_2/2 & t_4 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; t_1, t_2, t_3, t_4 \in \mathbb{R} \right\}.$$

The tangent space $\mathfrak{f}$ is the Lie algebra of $\mathcal{S} \ltimes \mathbb{R}^2$ and spanned by $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$. The Lie algebra $\mathfrak{sa}(2)$ is spanned by $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5\}$. The exponential map $\exp_{\mathfrak{f}}\colon \mathfrak{f} \to \mathcal{S} \ltimes \mathbb{R}^2$ is given explicitly by

$$\exp_{\mathfrak{f}}\begin{pmatrix} t_2/2 & t_1 & t_3 \\ 0 & -t_2/2 & t_4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{t_2/2} & 2t_1\sinh(t_2/2)/t_2 & 4(t_2t_3e^{t_2/4}\sinh(t_2/4) + t_1t_4(\cosh(t_2/2) - 1))/t_2^2 \\ 0 & e^{-t_2/2} & -4t_4e^{-t_2/4}\sinh(t_2/4)/t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It should be remarked that the decomposition $\mathfrak{sa}(2) = \mathfrak{so}(2) \oplus \mathfrak{f}$ is *not* reductive. Indeed, for any $T = t^1\bar{e}_1 + t^2\bar{e}_2 + t^3\bar{e}_3 + t^4\bar{e}_4 \in \mathfrak{f}$, we have

$$[\bar{e}_5, T] = 2t^2\bar{e}_1 - 2t^1\bar{e}_2 - t^4\bar{e}_3 + t^3\bar{e}_4 + t^2\bar{e}_5.$$

Thus, $[\mathfrak{so}(2), \mathfrak{f}] \not\subset \mathfrak{f}$.

5.5 Left-invariant metric

We denote by the left-translated vector field of $\bar{e}_i$ by the same latter. We obtain

$$\bar{e}_1 = y\frac{\partial}{\partial x}, \quad \bar{e}_2 = y\frac{\partial}{\partial y}, \quad \bar{e}_3 = \sqrt{y}\frac{\partial}{\partial u}, \quad \bar{e}_4 = \frac{x}{\sqrt{y}}\frac{\partial}{\partial u} + \frac{1}{\sqrt{y}}\frac{\partial}{\partial v}.$$

The dual 1-forms are

$$\bar{\vartheta}^1 = \frac{1}{y}\mathrm{d}x, \quad \bar{\vartheta}^2 = \frac{1}{y}\mathrm{d}y, \quad \bar{\vartheta}^3 = \frac{1}{\sqrt{y}}\mathrm{d}u - \frac{x}{\sqrt{y}}\mathrm{d}v, \quad \bar{\vartheta}^4 = \sqrt{y}\,\mathrm{d}v.$$

Here, we set

$$e_1 := 2c\bar{e}_1 = 2cy\partial_x = \begin{pmatrix} 0 & 2c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 := 2c\bar{e}_2 = 2cy\partial_y = \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 := \bar{e}_3 = \sqrt{y}\,\partial_u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 := \bar{e}_4 = \frac{x}{\sqrt{y}}\partial_u + \frac{1}{\sqrt{y}}\partial_v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad c > 0,$$

and equip a left-invariant metric g_c by the condition $\{e_1, e_2, e_3, e_4\}$ is orthonormal with respect to it. Then, g_c is expressed as

$$g_c = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4c^2y^2} + \frac{1}{y}\{(\mathrm{d}u - x\mathrm{d}v)^2 + y^2\mathrm{d}v^2\}.$$

This Riemannian metric coincides with (5.1). In this way, we obtain a solvable Lie group model of F^4_c.

Remark 5. In the literature, two normalizations $c = 1/2$ [46] or $c = 1$ [47] are used.
- Kiyota and Tsukada showed that the Singer invariant of $\mathrm{F}^4_{\frac{1}{2}}$ is 1 [46, Remark 3.4].
- Maier [47] showed that F^4_c does not admit half-conformally flat invariant metrics.

The dual coframe field of $\{e_1, e_2, e_3, e_4\}$ is given by

$$\vartheta^1 = \frac{1}{2cy}\mathrm{d}x, \quad \vartheta^2 = \frac{1}{2cy}\mathrm{d}y, \quad \vartheta^3 = \frac{1}{\sqrt{y}}\mathrm{d}u - \frac{x}{\sqrt{y}}\mathrm{d}v, \quad \vartheta^4 = \sqrt{y}\,\mathrm{d}v.$$

5.6 Levi-Civita connection

The exterior derivatives of the coframe field $\{\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4\}$ are given by

$$\begin{aligned}
d\vartheta^1 &= \frac{1}{2cy^2} dx \wedge dy = (2c\vartheta^1) \wedge \vartheta^2, \quad d\vartheta^2 = 0,\\
d\vartheta^3 &= -\frac{1}{2y\sqrt{y}} dy \wedge du - \frac{1}{\sqrt{y}} dx \wedge dv + \frac{x}{2y\sqrt{y}} dy \wedge dv = c\vartheta^3 \wedge \vartheta^2 - 2c\vartheta^1 \wedge \vartheta^4,\\
d\vartheta^4 &= \frac{1}{2\sqrt{y}} dy \wedge dv = (-c\vartheta^4) \wedge \vartheta^2.
\end{aligned}$$

From the *first structure equations*:

$$d\vartheta^i + \sum_{j=1}^{4} \omega^i_j \wedge \vartheta^j = 0,$$

we obtain the following table of *connection forms*:

$$\omega^1_2 = -2c\vartheta^1, \quad \omega^1_3 = c\vartheta^4, \quad \omega^1_4 = c\vartheta^3, \quad \omega^2_3 = c\vartheta^3, \quad \omega^2_4 = -c\vartheta^4, \quad \omega^3_4 = c\vartheta^1.$$

In covariant derivative fashion, the Levi-Civita connection ∇ is described as

$$\begin{aligned}
&\nabla_{e_1} e_1 = 2ce_2, && \nabla_{e_1} e_2 = -2ce_1, && \nabla_{e_1} e_3 = -ce_4, && \nabla_{e_1} e_4 = ce_3,\\
&\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = 0, && \nabla_{e_2} e_4 = 0,\\
&\nabla_{e_3} e_1 = -ce_4, && \nabla_{e_3} e_2 = -ce_3, && \nabla_{e_3} e_3 = ce_2, && \nabla_{e_3} e_4 = ce_1,\\
&\nabla_{e_4} e_1 = -ce_3, && \nabla_{e_4} e_2 = ce_4, && \nabla_{e_4} e_3 = ce_1, && \nabla_{e_4} e_4 = -ce_2.
\end{aligned}$$

Hence, the non-vanishing commutators are given by

$$[e_1, e_2] = -2ce_1, \quad [e_1, e_4] = 2ce_3, \quad [e_2, e_3] = ce_3, \quad [e_2, e_4] = -ce_4.$$

If we determine the *curvature 2-forms*:

$$\Omega^i_j = d\omega^i_j + \sum_{k=1}^{4} \omega^i_k \wedge \omega^k_j = \sum_{k<l} R^i_{jkl} \vartheta^k \wedge \vartheta^l,$$

then the significant components of Riemann curvature tensor are given explicitly by

$$\begin{aligned}
&R^1_{212} = -4c^2, && R^1_{234} = 2c^2, && R^1_{313} = -c^2, && R^1_{324} = c^2,\\
&R^1_{423} = -c^2, && R^1_{414} = -c^2, && R^2_{323} = -c^2, && R^2_{314} = -c^2,\\
&R^2_{424} = -c^2, && R^2_{413} = c^2, && R^3_{412} = 2c^2, && R^3_{434} = 2c^2.
\end{aligned}$$

The Ricci operator is given by

$$\begin{pmatrix} -6c^2 & 0 & 0 & 0\\ 0 & -6c^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ of a tangent plane $e_i \wedge e_j$ spanned by e_i and e_j are given by

$$K_{12} = -4c^2, \quad K_{13} = K_{14} = K_{23} = K_{24} = -c^2, \quad K_{34} = 2c^2.$$

The scalar curvature is $-12c^2$.

5.7 Tensor U

For later use we give here the tensor $\mathsf{U}_{\mathfrak{f}}:\mathfrak{f}\times\mathfrak{f}\to\mathfrak{f}$ defined by

$$2\langle \mathsf{U}_{\mathfrak{f}}(X,Y),Z\rangle = -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle,\quad X,Y,Z\in\mathfrak{f}.$$

The non-vanishing terms of the tensor U are given by

$$\mathsf{U}_{\mathfrak{f}}(e_1,e_1)=2ce_2,\quad \mathsf{U}_{\mathfrak{f}}(e_1,e_2)=-ce_1,\ \ \mathsf{U}_{\mathfrak{f}}(e_1,e_3)=-ce_4,\ \ \mathsf{U}_{\mathfrak{f}}(e_2,e_3)=-\frac{1}{2}ce_3,$$
$$\mathsf{U}_{\mathfrak{f}}(e_2,e_4)=\frac{1}{2}ce_4,\ \ \mathsf{U}_{\mathfrak{f}}(e_3,e_3)=ce_2,\quad \mathsf{U}_{\mathfrak{f}}(e_3,e_4)=ce_1,\quad \mathsf{U}_{\mathfrak{f}}(e_4,e_4)=-ce_2.$$

Thus, for any vector $X=X^1e_1+X^2e_2+X^3e_3+X^4e_4\in\mathfrak{f}$,

$$\mathsf{U}_{\mathfrak{f}}(X,X)=2c(X^3X^4-X^1X^2)e_1+c(2(X^1)^2+(X^3)^2-(X^4)^2)e_2-cX^2X^3e_3+c(X^2X^4-2X^1X^3)e_4.$$

This formula implies the following result.

Proposition 5.1. *For a tangent vector $X=X^1e_1+X^2e_2+X^3e_3+X^4e_4\in\mathfrak{f}$, the curve*

$$\gamma(s)=\exp_{\mathfrak{f}}(sX):\mathbb{R}\to\mathrm{F}^4_c=\mathbb{R}^2_+\ltimes\mathbb{R}^2$$

is a geodesic starting at the origin of the solvable Lie group model of F^4_c if and only if X has the form

$$X=X^1(e_1\pm\sqrt{2}\,e_4)\quad \textit{or}\quad X=X^2e_2.$$

Proof. From $\mathsf{U}_{\mathfrak{f}}(X,X)=0$, we obtain the following system:

$$X^3X^4-X^1X^2=0,\quad 2(X^1)^2+(X^3)^2-(X^4)^2=0,\quad X^2X^3=0,\quad X^2X^4-2X^1X^3=0.$$

From the third equation, we have $X^2=0$ or $X^3=0$. In the first case, when $X^2=0$, we obtain $X^3=0$ and then finally, $X=X^1(e_1\pm\sqrt{2}\,e_4)$. In the second case, when $X^3=0$, for $X^2=0$, we obtain already mentioned solution, and for $X^4=0$, we obtain $X=X^2e_2$. □

5.8 Symplectic pair of F^4_c

On F^4_c, we introduce a two-parameter family

$$\Omega_{c_1,c_2}=c_1\vartheta^1\wedge\vartheta^2+c_2\vartheta^3\wedge\vartheta^4=\frac{c_1}{4c^2}\mathrm{d}x\wedge\mathrm{d}y+c_2\mathrm{d}u\wedge\mathrm{d}v$$

of left-invariant symplectic forms (c.f. [48]). Then, we associate an endomorphism fields J_{c_1,c_2} with Ω_{c_1,c_2} by

$$g_c(X,J_{c_1,c_2}Y)=\Omega_{c_1,c_2}(X,Y).$$

Then, we have

$$J_{c_1,c_2}e_1=-c_1e_2,\quad J_{c_1,c_2}e_2=c_1e_1,\quad J_{c_1,c_2}e_3=-c_2e_4,\quad J_{c_1,c_2}e_4=c_2e_3.$$

Then, J_{c_1,c_2} is a g_c-orthogonal almost complex structure when and only when

$$(c_1,c_2)=(1,1),\quad (1,-1),\quad (-1,1),\quad \text{or}\quad (-1,-1).$$

The only left-invariant g_c-orthogonal complex structures are $J_{-1,-1}$ and $J_{-1,1}$ [48, Theorem 3.2].

Hereafter, we set

$$J_+ \coloneqq J_{-1,-1}\quad\text{and}\quad J_- \coloneqq J_{-1,1}.$$

Note that J_- coincides with the complex structure introduced by Wall[1] [28, p. 273].

The corresponding Kähler forms $\Omega_\pm$ are

$$\Omega_+ = -\vartheta^1 \wedge \vartheta^2 - \vartheta^3 \wedge \vartheta^4 = -\frac{1}{4c^2}\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}u \wedge \mathrm{d}v \in A_+^2(\mathrm{F}_c^4),$$
$$\Omega_- = -\vartheta^1 \wedge \vartheta^2 + \vartheta^3 \wedge \vartheta^4 = -\frac{1}{4c^2}\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}u \wedge \mathrm{d}v \in A_-^2(\mathrm{F}_c^4).$$

Set $\varepsilon = 1$ for Ω_+ and $\varepsilon = -1$ for Ω_-, then the covariant derivatives ∇J_+ and ∇J_- are given by the following relations:

$$\begin{aligned}
&(\nabla_{e_1}J_\pm)e_1 = (\nabla_{e_1}J_\pm)e_2 = (\nabla_{e_1}J_\pm)e_3 = (\nabla_{e_1}J_\pm)e_4 = 0,\\
&(\nabla_{e_2}J_\pm)e_1 = (\nabla_{e_2}J_\pm)e_2 = (\nabla_{e_2}J_\pm)e_3 = (\nabla_{e_2}J_\pm)e_4 = 0,\\
&(\nabla_{e_3}J_\pm)e_1 = -c(1+\varepsilon)e_3, \quad (\nabla_{e_3}J_\pm)e_2 = c(1+\varepsilon)e_4,\\
&(\nabla_{e_3}J_\pm)e_3 = c(1+\varepsilon)e_1, \quad (\nabla_{e_3}J_\pm)e_4 = -c(1+\varepsilon)e_2,\\
&(\nabla_{e_4}J_\pm)e_1 = c(1+\varepsilon)e_4, \quad (\nabla_{e_4}J_\pm)e_2 = c(1+\varepsilon)e_3,\\
&(\nabla_{e_4}J_\pm)e_3 = -c(1+\varepsilon)e_2, \quad (\nabla_{e_4}J_\pm)e_4 = -c(1+\varepsilon)e_1.
\end{aligned}$$

In particular, J_- is parallel.

On the other hand, the covariant derivative ∇J_+ is given by

$$\begin{aligned}
&(\nabla_{e_1}J_+)Y = (\nabla_{e_2}J_+)Y = 0,\\
&(\nabla_{e_3}J_+)Y = 2c(Y^3e_1 - Y^4e_2 - Y^1e_3 + Y^2e_4),\\
&(\nabla_{e_4}J_+)Y = -2c(Y^4e_1 + Y^3e_2 - Y^2e_3 - Y^1e_4),
\end{aligned} \tag{5.2}$$

for any *left-invariant* vector field $Y = Y^1e_1 + Y^2e_2 + Y^3e_3 + Y^4e_4$.

5.9 Kowalski's 3-symmetric space

Apostolov et al. showed that $\mathrm{F}^4_{\frac{1}{2}} = (\mathrm{F}^4, g_{\frac{1}{2}})$ is isometric to the Kowalski's 3-symmetric space $\hat{M}_1^4$ under the isometry [30]:

$$x_1 = \frac{x^2 + y^2 - 1}{2y}, \quad x_2 = -\frac{x}{y}, \quad x_3 = u, \quad x_4 = -v.$$

The inverse isometry is

$$x = \frac{-x_2(x_1 + \sqrt{x_1^2 + x_2^2 + 1})}{1 + x_2^2}, \quad y = \frac{x_1 + \sqrt{x_1^2 + x_2^2 + 1}}{1 + x_2^2} > 0, \quad u = x_3, \quad v = -x_4.$$

The symplectic form $\hat{\Omega}_\pm$ of $\hat{M}_1^4$ is pull backed as

$$\hat{\Omega}_\pm = -\frac{1}{y^2}\mathrm{d}x \wedge \mathrm{d}y \mp \mathrm{d}u \wedge \mathrm{d}v = -(\vartheta^1 \wedge \vartheta^2 \pm \vartheta^3 \wedge \vartheta^4) = \Omega_\pm \in A_\pm^2(\mathrm{F}^4_{\frac{1}{2}}).$$

5.10 Some typical submanifolds

For later use, we give some typical submanifolds in F_c^4. First of all, we exhibit leaves of complementary foliations associated with the symplectic pair (Ω_+, Ω_-).

1 He used the basis $\{\bar{e}_3, \bar{e}_4, 2\bar{e}_2, 2\bar{e}_1\}$.

Example 5.1. (Totally geodesic hyperbolic plane) For any constants u_0 and v_0, we consider a surface

$$M(1,2;\, u_0, v_0) = \{(x, y, u_0, v_0) \in \mathrm{F}^4_c\}.$$

The surface $M(1,2;\, u_0, v_0)$ is a leaf of the kernel foliation of $\Omega_+ + \Omega_-$; equivalently, it is an integral surface of the holomorphic distribution spanned by e_1 and e_2. One can see that $M(1,2;\, u_0, v_0)$ is isometric to hyperbolic plane $\mathbb{H}^2$ with a metric

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4c^2y^2}.$$

One can see that $M(1,2;\, u_0, v_0)$ is $J_\pm$-invariant and totally geodesic.

Example 5.2. (Minimal invariant Euclidean plane) The surface

$$M(3,4;\, x_0, y_0) = \{(x_0, y_0, u, v) \in \mathrm{F}^4_c\}$$

is a leaf of kernel foliation of $\Omega_+ - \Omega_-$. In other words, it is an integral surface of the holomorphic distribution spanned by e_3 and e_4. It is flat, minimal, and $J_\pm$-invariant. Note that it is non-totally geodesic. One can see that $M(3,4;\, x_0, y_0)$ is isometric to the Euclidean plane.

Example 5.3. (Heisenberg group) For any constant y_0, we consider a hypersurface

$$M(1,3,4;\, y_0) = \{(x, y_0, u, v) \in \mathrm{F}^4_c\}.$$

It should be remarked that $M(1,3,4;\, 1)$ is a nilpotent subgroup of F^4_c and isomorphic to the Heisenberg group. The induced metric of $M(1,3,4;\, y_0)$ is

$$g = \frac{\mathrm{d}x^2}{4c^2y_0^2} + \frac{1}{y_0}\{(\mathrm{d}u - x\mathrm{d}v)^2 + y_0^2\mathrm{d}v^2\}.$$

In particular, the induced metric of the Heisenberg group $M(1,3,4;\, 1)$ is

$$g = \frac{\mathrm{d}x^2}{4c^2} + \mathrm{d}v^2 + (\mathrm{d}u - x\mathrm{d}v)^2.$$

Thus, the Heisenberg group $M(1,3,4;\, 1)$ with $c = 1/2$ is isometric to the model space Nil_3. Hence, $M(1,3,4;\, y_0)$ is isometric to the Heisenberg group. Hereafter, we call $M(1,3,4;\, y_0)$ a *Heisenberg hypersurface* of F^4_c.

We can take a unit normal vector field $\nu = e_2$ for $M(1,3,4;\, y_0)$. Then, the shape operator derived from ν is given by

$$\begin{pmatrix} 2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix}.$$

Hence, $M(1,3,4;\, y_0)$ has constant mean curvature $2c/3$. The sectional curvatures K_M of $M(1,3,4;\, y_0)$ are given by

$$K_M(e_1 \wedge e_3) = K_M(e_3 \wedge e_4) = c^2, \quad K_M(e_1 \wedge e_4) = -3c^2.$$

We can introduce an almost contact structure $(\varphi^\pm, \xi, \eta)$ by

$$\xi = -J_\pm \nu = e_1, \quad \eta = \vartheta^1 = \frac{1}{2cy}\mathrm{d}x,$$

$$\varphi^\pm e_1 = 0, \quad \varphi^\pm e_3 = \pm e_4, \quad \varphi^\pm e_4 = \mp e_3.$$

One can see that the fundamental 2-forms $\Phi^\pm = g(\cdot, \varphi^\pm)$ are given by $\Phi^\pm = \pm \mathrm{d}u \wedge \mathrm{d}v$. Hence, $\Phi^\pm$ are magnetic fields on $M(1,3,4;\, y_0)$. Hence, $(M(1,3,4;\, y_0), \varphi^\pm, \xi, \eta)$ are strictly almost cosymplectic 3-manifolds.

Note that the nilradical of the model space Sol_1^4 is the Heisenberg group. The almost contact metric structure induced from the GCK-structure of Sol_1^4 is also strictly almost cosymplectic [49]. On the other hand, in the complex hyperbolic plane $\mathbb{C}H^2(-4)$, the Heisenberg group is embedded as the horosphere. The induced almost contact Riemannian structure is Sasakian.

As we mentioned earlier, F_c^4 does not admit any totally geodesic hypersurfaces. More strongly, the non-existence of parallel hypersurfaces in F_c^4 was proved in [40].

Problem 5.1. Classify totally umbilical hypersurfaces in F_c^4.

Totally, umbilical hypersurfaces in Sol_0^4 were classified in [50].

6 Magnetic trajectories in an almost Kähler manifold

6.1 Frenet curves

Definition 6.1. If γ is a curve in a Riemannian manifold $M = (M, g)$, parametrized by arc length s, we say that γ is a *Frenet curve of osculating order* r when there exist orthonormal vector fields $E_1, E_2, \ldots, E_r$ along γ such that

$$\begin{aligned} &\dot\gamma = E_1, \quad \nabla_{\dot\gamma}E_1 = \kappa_1 E_2, \quad \nabla_{\dot\gamma}E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \ \ldots, \\ &\nabla_{\dot\gamma}E_{r-1} = -\kappa_{r-2}E_{r-2} + \kappa_{r-1}E_r, \quad \nabla_{\dot\gamma}E_r = -\kappa_{r-1}E_{r-1}, \end{aligned} \tag{6.1}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are the positive C^∞ functions of s. The function κ_j is called the jth *curvature* of γ.

A *geodesic* is regarded as a Frenet curve of osculating order 1. A *circle* is defined as a Frenet curve of osculating order 2 with *constant* κ_1. A *helix* of order r is a Frenet curve of osculating order r, such that all the curvatures κ_1, $\kappa_2, \ldots, \kappa_{r-1}$ are constants.

For Frenet curves in almost Kähler manifolds, we recall the following notion:

Definition 6.2. Let $\gamma(s)$ be a Frenet curve of osculating order $r > 0$ in an almost Kähler manifold (M, J, g). The *complex torsions* τ_{ij} $(1 \leq i < j \leq r)$ are smooth functions along γ defined by $\tau_{ij} = g(E_i, JE_j)$ [51]. A helix of order r in (M, J, g) is said to be a *holomorphic helix* of order r if all complex torsions are constant. In particular, holomorphic helices of order 2 are called *holomorphic circles*.

6.2 Kähler magnetic trajectory equation

Hereafter, we assume that $M = (M, J, g)$ is an almost Kähler 4-manifold. Then, $-\Omega = g(\cdot, J)$ is a magnetic field with Lorentz force J on M and called the *Kähler magnetic field.*

Definition 6.3. A curve $\gamma(t)$ in an almost Kähler manifold (M, J, g) is said to be a *Kähler magnetic trajectory* with *strength* q if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma = qJ\dot\gamma, \tag{6.2}$$

for some constant q.

One can see that every Kähler magnetic trajectory has constant speed. Thus, hereafter, we parameterize Kähler magnetic trajectory by arc length parameter s. In addition, if necessarily, by the affine parameter change $s \longmapsto -s$, we may assume that $q > 0$.

Remark 6. The Kähler magnetic equation (6.2) is valid on general almost Hermitian manifolds. On an arbitrary almost Hermitian manifold (M, g), regular curves γ satisfying (6.2) are called *J-trajectories*. J-trajectories in *non-Kähler* locally conformal Kähler manifolds were investigated in [49,52–55].

Now, let $\gamma(s)$ be a unit speed Kähler magnetic trajectory of charge $q > 0$ in an almost Kähler 4-manifold $M = (M, J, g)$. First, we observe that the first curvature κ_1 is constant $|q|$ by comparing the magnetic curve equation and the Frenet formula (6.1). The Frenet formula implies that the first normal vector field E_2 is given by $E_2 = J\dot{\gamma}$ and $\kappa_1 = \varepsilon q > 0$.

Next, the second curvature κ_2 is determined by the equation $\nabla_{\dot{\gamma}}E_2 = -\kappa_1 E_1 + \kappa_2 E_3$. The covariant derivative $\nabla_{\dot{\gamma}}E_2$ is computed as

$$\nabla_{\dot{\gamma}}E_2 = \nabla_{\dot{\gamma}}(JE_1) = (\nabla_{\dot{\gamma}}J)\dot{\gamma} + J(\nabla_{\dot{\gamma}}\dot{\gamma}) = (\nabla_{\dot{\gamma}}J)\dot{\gamma} - q\dot{\gamma} = (\nabla_{\dot{\gamma}}J)\dot{\gamma} - \kappa_1 E_1.$$

Hence, we obtain

$$\kappa_2 E_3 = (\nabla_{\dot{\gamma}}J)\dot{\gamma}. \tag{6.3}$$

This formula implies that a Kähler magnetic curve is a Riemannian circle if and only if J is parallel along the magnetic curve. In particular, when the ambient manifold is Kähler, every Kähler magnetic curve is a Riemannian circle.

Remark 7. If a Frenet curve γ in an almost Kähler manifold (M, J, g) is Kähler magnetic, then

$$\tau_{12} = g(E_1, JE_2) = -1.$$

If M is a Kähler manifold, every Kähler magnetic curve is a holomorphic circle.

Let us assume that (M, J, g) be an almost Kähler 4-manifold and γ a unit speed Kähler magnetic curve with charge q, then its complex torsions are

$$\tau_{12} = -1, \quad \tau_{13} = \tau_{14} = \tau_{23} = \tau_{24} = 0.$$

The complex torsion τ_{34} satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau_{34} = g(E_3, (\nabla_{\dot{\gamma}}J)E_4).$$

7 Kähler magnetic curves with respect to Ω_-

In this section, we study Kähler magnetic curves in $(\mathrm{F}^4_c, J_-, g_c)$. As we mentioned before, Kähler magnetic curves in $(\mathrm{F}^4_c, J_-, g_c)$ are holomorphic circles.

7.1 Magnetic equations

Let $\gamma(s) = (x(s), y(s), u(s), v(s))$ be an arc length parameterized curve in $\mathrm{F}^4_c = \mathbb{R}^2_+ \ltimes \mathbb{R}^2$. Then, its unit tangent vector field is expressed as

$$\dot{\gamma}(s) = \dot{x}(s)\frac{\partial}{\partial x} + \dot{y}(s)\frac{\partial}{\partial y} + \dot{u}(s)\frac{\partial}{\partial u} + \dot{v}(s)\frac{\partial}{\partial v}.$$

Having in mind,

$$e_1 = 2cy\ \partial_x, \quad e_2 = 2cy\ \partial_y, \quad e_3 = \sqrt{y}\ \partial_u, \quad e_4 = \frac{x}{\sqrt{y}}\ \partial_u + \frac{1}{\sqrt{y}}\ \partial_v,$$

we obtain

$$\dot{\gamma} = \frac{\dot{x}}{2cy}e_1 + \frac{\dot{y}}{2cy}e_2 + \frac{(\dot{u} - x\dot{v})}{\sqrt{y}}e_3 + \sqrt{y}\,\dot{v}e_4.$$

Hence,

$$\begin{aligned}\nabla_{\dot{\gamma}}\dot{\gamma} = &\left(\frac{\ddot{x}}{2cy} - \frac{\dot{x}\dot{y}}{cy^2} + 2c\dot{v}(\dot{u} - x\dot{v})\right)e_1 + \left(\frac{\ddot{y}}{2cy} + \frac{\dot{x}^2 - \dot{y}^2}{2cy^2} + \frac{c}{y}((\dot{u} - x\dot{v})^2 - (y\dot{v})^2)\right)e_2 \\ &+ \left(\frac{-y\dot{x}\dot{v} - \dot{y}(\dot{u} - x\dot{v})}{y\sqrt{y}} + \frac{\ddot{u} - x\ddot{v}}{\sqrt{y}}\right)e_3 + \left(\frac{\dot{y}\dot{v}}{\sqrt{y}} + \sqrt{y}\,\ddot{v} - \frac{\dot{x}(\dot{u} - x\dot{v})}{y\sqrt{y}}\right)e_4.\end{aligned} \tag{7.1}$$

Introducing substitutions

$$X(s) = \frac{\dot{x}(s)}{2cy(s)}, \quad Y(s) = \frac{\dot{y}(s)}{2cy(s)}, \quad U(s) = \frac{\dot{u}(s) - x(s)\dot{v}(s)}{\sqrt{y(s)}}, \quad V(s) = \sqrt{y(s)}\,\dot{v}(s), \tag{7.2}$$

we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = (\dot{X} - 2cXY + 2cUV)e_1 + (\dot{Y} + 2cX^2 + c(U^2 - V^2))e_2 + +(\dot{U} - cYU)e_3 + (\dot{V} - 2cXU + cYV)e_4.$$

Then, the arc length condition is given by

$$X^2 + Y^2 + U^2 + V^2 = 1. \tag{7.3}$$

On the other hand, using

$$J_- e_1 = e_2, \quad J_- e_2 = -e_1, \quad J_- e_3 = -e_4, \quad J_- e_4 = e_3,$$

we have

$$J_- \dot{\gamma} = -Ye_1 + Xe_2 + Ve_3 - Ue_4.$$

Hence, the magnetic curves are given as the solutions of the following system:

$$\begin{aligned} \dot{X} - 2cXY + 2cUV &= -qY, \\ \dot{Y} + 2cX^2 + c(U^2 - V^2) &= +qX, \\ \dot{U} - cYU &= +qV, \\ \dot{V} - 2cXU + cYV &= -qU. \end{aligned} \tag{7.4}$$

Since $(\mathrm{F}^4_c, J_-, g_c)$ is Kähler, the Kähler magnetic curve $\gamma(s)$ has constant curvatures $\kappa_1^- = q$ and $\kappa_2^- = 0$. Take a Frenet frame field $E_1^- = \dot{\gamma}$ and $E_2^- = \varepsilon J_- \dot{\gamma}$. Then, the first curvature $\kappa_1^- = \varepsilon q$ is computed as

$$\begin{aligned} q = \varepsilon\ \kappa_1^- &= \varepsilon\ g(\nabla_{\dot{\gamma}}\dot{\gamma}, E_2) = g(\nabla_{\dot{\gamma}}\dot{\gamma}, J_- \dot{\gamma}) \\ &= -Y(\dot{X} - 2cXY + 2cUV) + X(\dot{Y} + 2cX^2 + c(U^2 - V^2)) + V(\dot{U} - cYU) - U(\dot{V} - 2cXU + cYV) \\ &= (X\dot{Y} - \dot{X}Y) + (\dot{U}V - U\dot{V}) + 2cX(X^2 + Y^2 + U^2) + cX(U^2 - V^2) - 4cYUV. \end{aligned}$$

Hence, using (7.3), we have the conservation law:

$$(X\dot{Y} - \dot{X}Y) + (\dot{U}V - U\dot{V}) + cX(2 + U^2 - 3V^2) - 4cYUV = q.$$

Every Kähler magnetic curve $\gamma(s)$ is of order 2, so along γ, we obtain a J_--invariant plane field $\operatorname{span}\{\dot{\gamma}(s), J_- \dot{\gamma}(s)\}$ along γ. This fact suggests us to study Kähler magnetic curves lying in J_--invariant surfaces.

We pay our attention to leaves of the kernel foliations of $\Omega_+ \pm \Omega_-$ exhibited in Examples 5.1 and 5.2. We know that leaves of $\operatorname{Ker}(\Omega_+ + \Omega_-)$ are totally geodesic, but leaves of $\operatorname{Ker}(\Omega_+ - \Omega_-)$ are not. The different behavior of these foliations under the Levi-Civita connection ∇ makes the behavior of magnetic curves significant different.

7.2 Magnetic curves in hyperbolic plane $M(1, 2;\ u_0, v_0)$

Next, we study magnetic curves satisfying $u = u_0$ and $v = v_0$. As we mentioned in Example 5.1, in this case, our submanifold is a hyperbolic plane, which is $J_\pm$-invariant. The magnetic curve equations (7.4) are reduced to

$$\dot{X} - 2cXY = -qY, \quad \dot{Y} + 2cX^2 = qX.$$

Multiplying the first equation by X and the second by Y, after addition and integration, and taking in account arc length condition, we obtain

$$X^2 + Y^2 = 1.$$

This equation we considered in detail in [49], so in the following proposition, we give only the results.

Proposition 7.1. *Kähler magnetic curves of* $(\mathrm{F}^4_c, J_-, g_c)$ *lying in a hyperbolic plane* $M(1, 2;\ u_0, v_0) \subset \mathrm{F}^4_c$ *are congruent to the one of the curves from the list*:

(1a) *a horizontal line*

$$x(s) = \pm q y_0 s + x_0, \quad y(s) = y_0 > 0, \quad \text{for } q = \mp 2c.$$

(1b) *an oblique half line* $x = \frac{\pm q}{\sqrt{4c^2 - q^2}}(y - y_0) + x_0$ *with the arc length parametrization*

$$x(s) = \frac{\mp q y_0}{\sqrt{4c^2 - q^2}}(e^{\pm\sqrt{4c^2-q^2}\,s} - 1) + x_0,$$

$$y(s) = y_0 e^{\pm\sqrt{4c^2-q^2}\,s}, \quad \text{for } |q| < 2c.$$

(2) *a Riemannian circle*

$$x(s) = a + r\sin\phi(s), \quad y(s) = \pm(qr + r\cos\phi(s))$$

for $|q| \geq 2c$ *and some non-constant function* $\phi(s)$ *satisfying* $\dot{\phi}(s) = -q - \cos\phi(s)$.

7.3 Magnetic curves in the Euclidean plane $M(3, 4;\ x_0, y_0)$

Assume that a unit speed Kähler magnetic curve $\gamma(s)$ lies in the Euclidean plane $M(3, 4;\ x_0, y_0)$. Then, the magnetic equations are reduced to

$$2cUV = 0, \quad U^2 - V^2 = 0, \quad \dot{U} = qV, \quad \dot{V} = -qU.$$

The only solution to this system is $U = V = 0$. This contradicts with the arc length condition $U^2 + V^2 = 1$.

Proposition 7.2. *There is no Kähler magnetic curve of* $(\mathrm{F}^4_c, J_-, g_c)$ *lying in the Euclidean plane* $M(3, 4;\ x_0, y_0)$.

7.4 Magnetic curves in the Heisenberg hypersurface $M(1, 3, 4;\ y_0)$

Let us study Kähler magnetic curves lying in the Heisenberg hypersurface $M(1, 3, 4;\ y_0)$. The magnetic curve equations are

$$\begin{cases} \dot{X} + 2cUV = 0, \\ 2cX^2 + c(U^2 - V^2) = qX, \\ \dot{U} = qV, \\ \dot{V} - 2cXU = -qU. \end{cases} \tag{7.5}$$

Let us determine the Kähler magnetic curves lying in $M(1,3,4;y_0)$ under the initial condition $\gamma(0) = (x_0, y_0, u_0, v_0)$.

Differentiating the second equation of (7.5), we obtain

$$4cX\dot{X} + 2cU\dot{U} - 2cV\dot{V} = q\dot{X}.$$

Substituting $\dot{X}, \dot{U}, \dot{V}$ from other equations of (7.5) into the previous one, we have

$$-6cUV(2cX + q) = 0.$$

– Case 1: Assume that $U = 0$, then we have

$$V = 0, \quad \dot{X} = 0, \quad X(2cX - q) = 0.$$

Hence, $X = 0$ or $X = \frac{q}{2c}$. From the arc length condition $X^2 + U^2 + V^2 = 1$, the only possibility is

$$X(s) = \pm 1.$$

Thus, $q = \pm 2c$ and $x(s) = \pm q y_0 s + x_0$. From (7.2), it follows $\dot{v} = 0$, and hence, by the initial condition, $v = v_0$. Analogously, $u = u_0$.

Thus, we obtain the geodesic

$$\gamma(s) = (\pm q y_0 s + x_0,\ y_0,\ u_0,\ v_0), \quad \text{for } q = \pm 2c.$$

– Case 2: Let us assume that $V = 0$, then from (7.5), we obtain

$$\dot{X} = 0, \quad 2cX^2 + cU^2 = qX, \quad \dot{U} = 0, \quad (2cX - q)U = 0.$$

Hence, from the last equation, $U = 0$ or $X = q/(2c)$. In the first case, the arc length condition implies $X = \pm 1$. In the second case, the second equation implies $U = 0$. So, both cases lead to the geodesic from Case 1.

– Case 3: $X = -q/(2c)$. Then, from the first equation, we have $UV = 0$. Both cases, $U = 0$ and $V = 0$, are already examined in the previous consideration.

Hence, we proved the following corollary.

Corollary 7.1. *The only unit speed Kähler magnetic curves of* $(\mathrm{F}^4_c, J_-, g_c)$ *lying in the Heisenberg hyperspace are geodesics in hyperbolic plane* $M(1,2;u_0,v_0)$, *which are obtained as the intersection* $M(1,3,4;y_0) \cap M(1,2;u_0,v_0)$ *parameterized by*

$$\gamma(s) = (\pm q y_0 s + x_0,\ y_0,\ u_0,\ v_0), \quad \textit{for } q = \pm 2c,$$

where $\gamma(0) = (x_0, y_0, u_0, v_0)$.

8 Kähler magnetic curves with respect to Ω_+

In this section, we deduce the equations for the Kähler magnetic curves in the strictly almost Kähler manifold $(\mathrm{F}^4_c, J_+, g_c)$.

Here, using

$$J_+ e_1 = e_2, \quad J_+ e_2 = -e_1, \quad J_+ e_3 = e_4, \quad J_+ e_4 = -e_3,$$

we have

$$J_+ \dot{\gamma} = -Ye_1 + Xe_2 - Ve_3 + Ue_4.$$

Hence, using (7.1), the magnetic curves are given as the solutions of the following system:

$$\begin{aligned}\dot{X} - 2cXY + 2cUV &= -qY,\\ \dot{Y} + 2cX^2 + c(U^2 - V^2) &= +qX,\\ \dot{U} - cYU &= -qV,\\ \dot{V} - 2cXU + cYV &= +qU.\end{aligned} \tag{8.1}$$

Note that the first and second equations are identical to the ones of (7.4).

Take a Frenet frame field $E_1^+ = E_1^- = \dot{\gamma}$ and $E_2^+ = J_+\dot{\gamma}$. Then, the first curvature $\kappa_1^+ = \varepsilon\, q$ is computed as

$$\begin{aligned}q = \kappa_1^+ &= \varepsilon\, g(\nabla_{\dot{\gamma}}\dot{\gamma}, E_2^+) = g(\nabla_{\dot{\gamma}}\dot{\gamma}, J_+\dot{\gamma})\\ &= -Y(\dot{X} - 2cXY + 2cUV) + X(\dot{Y} + 2cX^2 + c(U^2 - V^2)) - V(\dot{U} - cYU) + U(\dot{V} - 2cXU + cYV)\\ &= (X\dot{Y} - \dot{X}Y) + (U\dot{V} - \dot{U}V) + 2cX(X^2 + Y^2) - cX(U^2 + V^2).\end{aligned}$$

Hence, using (7.3), we have the conservation law:

$$(X\dot{Y} - \dot{X}Y) + (U\dot{V} - \dot{U}V) + 3cX(X^2 + Y^2) - cX = q.$$

Next, the almost complex structure J_+ is non-parallel; thus, magnetic curves are not of order 2, in general. Let us compute $\kappa_2^+ E_3^+$.

Using (5.2) and (6.3), we obtain

$$\begin{aligned}\kappa_2^+ E_3^+ &= 2cU(Ue_1 - Ve_2 - Xe_3 + Ye_4) - 2cV(Ve_1 + Ue_2 - Ye_3 - Xe_4)\\ &= 2c\{(U^2 - V^2)e_1 - 2UVe_2 + (YV - XU)e_3 + (XV + YU)e_4\}.\end{aligned}$$

Hence,

$$(\kappa_2^+)^2 = 4c^2(U^2 + V^2).$$

Thus, the solution of the equation $\kappa_2^+ = 0$ is $U = V = 0$. Hence, we have the following proposition.

Proposition 8.1. *The only Kähler magnetic curves of order 2 in* $(\mathrm{F}^4_c, J_+, g_c)$ *are curves lying in hyperbolic plane given in Proposition* 7.1.

Proof. Inserting $U = V = 0$ into (8.1), we obtain the system

$$\dot{X} - 2cXY = -qY, \quad \dot{Y} + 2cX^2 = qX.$$

The solutions of this system are given in Proposition 7.1. □

8.1 Magnetic curves in the Euclidean plane $M(3, 4;\, x_0, y_0)$

Assume that a unit speed Kähler magnetic curve $\gamma(s)$ lies in the Euclidean plane $M(3, 4;\, x_0, y_0)$, i.e., $X = Y = 0$. Then the magnetic equations (8.1) are reduced to

$$2cUV = 0, \quad U^2 - V^2 = 0, \quad \dot{U} = -qV, \quad \dot{V} = qU.$$

The only solution to this system is $U = V = 0$. This contradicts with the arc length condition $U^2 + V^2 = 1$. This fact together with Proposition 7.2 implies the following result.

Proposition 8.2. *There is no Kähler magnetic curves of* $(\mathrm{F}^4_c, J_\pm, g_c)$ *lying in the Euclidean plane* $M(3, 4;\, x_0, y_0)$.

8.2 Magnetic curves in the Heisenberg hypersurface $M(1,3,4;y_0)$

Let us determine the Kähler magnetic curves lying in $M(1,3,4;y_0)$ under the initial condition $\gamma(0)=(x_0,y_0,u_0,v_0)$.

The magnetic curve equations are

$$\begin{cases}\dot X + 2cUV = 0,\\ 2cX^2 + c(U^2 - V^2) = qX,\\ \dot U = -qV,\\ \dot V - 2cXU = qU.\end{cases} \tag{8.2}$$

Differentiating the second equation of (8.2), we obtain

$$4cX\dot X + 2cU\dot U - 2cV\dot V = q\dot X.$$

Substituting $\dot X, \dot U, \dot V$ from other equations of (8.2) into the previous one, we have

$$-2cUV(6cX+q)=0.$$

– Case 1: Assume that $U=0$, then we have

$$V=0,\quad \dot X=0,\quad X(2cX-q)=0.$$

Hence, $X=0$ or $X=\frac{q}{2c}$. From the arc length condition $X^2+U^2+V^2=1$, the only possibility is

$$X(s)=\pm 1.$$

Thus, $q=\pm 2c$ and $x(s)=\pm q y_0 s + x_0$. From (7.2), it follows $\dot v=0$, and hence, by the initial condition, $v=v_0$. Analogously, $u=u_0$. Thus, we obtain geodesic

$$\gamma(s)=(\pm q y_0\ s+x_0,\quad y_0,\quad u_0,\quad v_0),\quad \text{for } q=\pm 2c.$$

– Case 2: Let us assume that $V=0$, then from (8.2), we obtain

$$\dot X=0,\quad 2cX^2+cU^2=qX,\quad \dot U=0,\quad (2cX-q)U=0.$$

Hence, from the last equation, $U=0$ or $X=-q/(2c)$. In the first case, the arc length condition implies $X=\pm 1$. So, this case leads to the geodesic from Case 1. In the second case, the second equation implies $U^2=-q^2/c^2$, i.e., the contradiction.

– Case 3: $X=-q/(6c)$. Then, from the first equation, we have $UV=0$. The both cases $U=0$ and $V=0$ lead to the contradiction.

Combining these arguments and Corollary 7.1, we obtain

Corollary 8.1. *The only unit speed Kähler magnetic curves of* $(F^4_c, J_\pm, g_c)$ *lying in the Heisenberg hyperspace are geodesics in hyperbolic plane* $M(1,2;\,u_0,v_0)$, *which are obtained as the intersection* $M(1,3,4;\,y_0)\cap M(1,2;\,u_0,v_0)$ *parameterized by*

$$\gamma(s)=(\pm q y_0 s+x_0,\ y_0,\ u_0,\ v_0),\quad \textit{for } q=\pm 2c,$$

where $\gamma(0)=(x_0,y_0,u_0,v_0)$.

We will continue to investigate Kähler magnetic curves of (F^4_c, J_+, g_c) in a separate publication.

Problem 8.1. Determine Kähler magnetic curves of (F^4_c, J_+, g_c).

Problem 8.2. Determine minimal $J_\pm$-invariant surfaces as well as minimal surfaces that are totally real with respect to $J_\pm$ in F^4.

8.3 Geodesics equations

Assume that $\gamma(s)$ is a geodesic in F^4_c, then from (7.4) for $q = 0$, we obtain the following system:

$$\begin{aligned}\dot{X} - 2cXY + 2cUV &= 0,\\ \dot{Y} + 2cX^2 + c(U^2 - V^2) &= 0,\\ \dot{U} - cYU &= 0,\\ \dot{V} - 2cXU + cYV &= 0,\end{aligned}$$

for geodesics.

Solving this system in the general case presents a considerable challenge.

9 Homogeneous magnetic curves in F^4_c

Since $(\mathrm{F}^4_c, J_\pm, g_c)$ is a homogeneous almost Kähler manifold, classification of homogeneous Kähler magnetic curves is a crucial task. In this section, we reinvestigate homogeneous magnetic curves and homogeneous geodesics.

9.1 Homogeneous geodesics

9.1.1 Riemannian geodesic orbit spaces

Let $M = G/K$ be a homogeneous Riemannian space. A curve $\gamma(s)$ starting at the origin o is called *homogeneous* if it is an orbit of o under the action of some one-parameter subgroup of G. Namely, a homogeneous curve starting at o is represented as

$$\gamma(s) = \exp_{\mathfrak{g}}(sX)\cdot o,$$

for some vector $X \in \mathfrak{g}$.

It is known that every homogeneous Riemannian space admits at least one homogeneous geodesic starting at the origin.

Definition 9.1. A reductive homogeneous Riemannian space $M = G/K$ is called a *space with homogeneous geodesics* or a *Riemannian g.o. space* if every geodesic $\gamma(s)$ of M is an orbit of a one-parameter subgroup of the *largest* connected group of isometries.

Now, let us assume that $M = G/K$ is a *reductive* homogeneous Riemannian space with Lie subspace $\mathfrak{m}$. For any vector $X \in \mathfrak{g}$, we decompose it as

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} \in \mathfrak{k}, \quad X_{\mathfrak{m}} \in \mathfrak{m},$$

along the reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

Next, we introduce a tensor $\mathsf{U}_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$2\langle \mathsf{U}_{\mathfrak{m}}(X, Y), Z\rangle = -\langle X, [Y, Z]_{\mathfrak{m}}\rangle + \langle Y, [Z, X]_{\mathfrak{m}}\rangle, \quad X, Y, Z \in \mathfrak{m}.$$

A homogeneous Riemannian space is said to be *naturally reductive* if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ with vanishing $\mathsf{U}_{\mathfrak{m}}$. It is well known that a naturally reductive homogeneous space $M = G/K$ is a Riemannian g. o. space. The model space $\mathrm{F}^4_c = \mathrm{SA}(2)/\mathrm{SO}(2)$ is *not* a Riemannian g. o. space.

Let $M = G/K$ be a reductive homogeneous Riemannian space with Lie subspace $\mathfrak{m}$. Take vectors $X, Z \in \mathfrak{g}$ and set $\phi_t = \exp(tX)$ and $\psi_s = \exp(sZ)$. The *fundamental vector field* $X^\sharp$ derived from X is defined by

$$X_x^\sharp = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp_{\mathfrak{g}}(tX)\cdot x, \quad x \in M.$$

At any point $x \in M$, we have (see [29] or Appendix of this article):

$$Z^\sharp_{\phi_t(x)} = \phi_{*(\phi_t^{-1}\circ\psi_s\circ\phi_t)(x)}(Z - t[X,Z] + o(t^2))^\sharp_x. \tag{9.1}$$

From the Koszul formula, we have

$$2g(\nabla_{X^\sharp}X^\sharp, Z^\sharp) = 2X^\sharp g(X^\sharp, Z^\sharp) - Z^\sharp g(X^\sharp, X^\sharp) + 2g([Z^\sharp, X^\sharp], X^\sharp).$$

Since

$$X_x^\sharp g(X^\sharp, Z^\sharp) = g_x(X^\sharp, [X^\sharp, Z^\sharp]), \quad Z_x^\sharp\, g(X^\sharp, X^\sharp) = 2g_x(X^\sharp, [Z^\sharp, Z^\sharp]),$$

we deduce that

$$g_x(\nabla_{X^\sharp}X^\sharp, Z^\sharp) = -g_x(X^\sharp, [X,Z]^\sharp) = -\langle X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}\rangle = \langle [X,Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle.$$

This equation implies the following useful criterion (see [29, Proposition 2.1], [56, Theorem 5.2]).

Proposition 9.1. *Let $M = G/K$ be a reductive homogeneous Riemannian space equipped with a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Take a vector $X = X_{\mathfrak{k}} + X_{\mathfrak{m}} \in \mathfrak{g}$ such that $X_{\mathfrak{m}} \neq 0$. Then,*

$$\gamma(s) = \exp_{\mathfrak{g}}(sX)\cdot o$$

is a geodesic if and only if one of the following conditions is fulfilled:
(1) $[X_{\mathfrak{m}}, X_{\mathfrak{k}}] = \mathsf{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}})$.
(2) $\langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}], Z\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z]_{\mathfrak{m}}\rangle$, *for any* $Z \in \mathfrak{m}$.
(3) $\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0$, *for any* $Z \in \mathfrak{m}$.

In such a case, X is called a geodesic vector.

9.1.2 Reductive decomposition of F_c^4

Now, we apply the criterion (Proposition 9.1) for geodesics in F_c^4. However, the decomposition $\mathfrak{sa}(2) = \mathfrak{so}(2) + \mathfrak{f}$ is *not* reductive. Thus, here, we give a reductive decomposition for the homogeneous space F_c^4. Following Kiyota and Tsukada [46, p. 728], we take the following basis $\{E_1, E_2, E_3, E_4, E_5\}$ of $\mathfrak{sa}(2)$:

$$E_1 = \begin{pmatrix} 0 & c & 0\\ c & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = e_1 + c\bar{e}_5 = cY_2, \quad E_2 = e_2 = \begin{pmatrix} c & 0 & 0\\ 0 & -c & 0\\ 0 & 0 & 0\end{pmatrix} = -cX_2,$$
$$E_3 = e_3 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = X_1, \quad E_4 = e_4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} = Y_1,$$
$$E_5 = \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = B.$$

Denote by $\mathfrak{m}$ the linear subspace spanned by $\{E_1, E_2, E_3, E_4\}$. Then, $\mathfrak{m}$ is a linear subspace of $\mathfrak{sa}(2)$ complementary to $\mathfrak{so}(2) = \mathbb{R}E_5$. Indeed, every element of $\mathfrak{sa}(2)$ is decomposed as

$$\begin{pmatrix} \alpha & \beta & \xi\\ \gamma & -\alpha & \eta\\ 0 & 0 & 0\end{pmatrix} = \frac{\gamma+\beta}{2c}E_1 + \frac{\alpha}{c}E_2 + \xi E_3 + \eta E_4 + \frac{\gamma-\beta}{2}E_5.$$

The commutation relations are

$$[E_1, E_2] = 2c^2E_5, \quad [E_1, E_3] = cE_4, \quad [E_1, E_4] = cE_3,$$
$$[E_2, E_3] = cE_3, \quad [E_2, E_4] = -cE_4, \quad [E_3, E_4] = 0,$$

and

$$[E_5, E_1] = -2E_2, \quad [E_5, E_2] = 2E_1, \quad [E_5, E_3] = E_4, \quad [E_5, E_4] = -E_3. \tag{9.2}$$

Hence, the decomposition $\mathfrak{sa}(2) = \mathfrak{so}(2) + \mathfrak{m}$ is reductive. We identify the tangent space of F^4_c at the origin with $\mathfrak{m}$. Let us equip a Riemannian metric on SA(2)/SO(2) so that $\{E_1, E_2, E_3, E_4\}$ is orthonormal with respect to it. Then, the resulting homogeneous Riemannian space is isometric to the solvable Lie group model of F^4_c (see [46, Remark 3.3, 3.4]).

Remark 8. The basis $\{X_1, Y_1, X_2, Y_2, B\}$ used in [37, p. 139] is related to our $\{E_1, E_2, E_3, E_4, E_5\}$ by the correspondence:

$$X_1 \leftrightarrow E_3, \quad Y_1 \leftrightarrow E_4, \quad X_2 \leftrightarrow -\frac{1}{c}E_2, \quad Y_2 \leftrightarrow \frac{1}{c}E_1, \quad B \leftrightarrow E_5, \tag{9.3}$$

for $c = 1$.

The tensor $\mathsf{U}_\mathfrak{m}$ is computed as

$\mathsf{U}_\mathfrak{m}(\cdot,\cdot)$	E_1	E_2	E_3	E_4
E_1	$\mathbf{0}$	$\mathbf{0}$	$-\frac{c}{2}E_4$	$-\frac{c}{2}E_3$
E_2	$\mathbf{0}$	$\mathbf{0}$	$-\frac{c}{2}E_3$	$\frac{c}{2}E_4$
E_3	$-\frac{c}{2}E_4$	$-\frac{c}{2}E_3$	cE_2	cE_1
E_4	$-\frac{c}{2}E_3$	$\frac{c}{2}E_4$	cE_1	$-cE_2$

9.1.3 System of equations for homogeneous geodesics

Take a vector

$$X = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 + X^5E_5 \in \mathfrak{sa}(2) = \mathfrak{so}(2) + \mathfrak{m}, \quad (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \neq 0,$$

and denote its $\mathfrak{so}(2)$-part and $\mathfrak{m}$-part by $X_\mathfrak{k}$ and $X_\mathfrak{m}$, respectively.

If we compute the system of equations

$$\langle [X, E_k]_\mathfrak{m}, X_\mathfrak{m}\rangle = 0, \quad k = 1, 2, 3, 4,$$

we obtain

$$\begin{aligned}
[X, E_1] &= -2X^5E_2 - cX^4E_3 - cX^3E_4 - 2c^2X^2E_5,\\
[X, E_2] &= 2X^5E_1 - cX^3E_3 + cX^4E_4 + 2c^2X^1E_5,\\
[X, E_3] &= cX^2E_3 + (cX^1 + X^5)E_4,\\
[X, E_4] &= (cX^1 - X^5)E_3 - cX^2E_4.
\end{aligned}$$

From these, we deduce the system for homogeneous geodesics in F^4_c:

$$\begin{aligned}
&X^2X^5 + cX^3X^4 = 0,\\
&2X^1X^5 + c((X^4)^2 - (X^3)^2) = 0,\\
&X^4X^5 + c(X^1X^4 + X^2X^3) = 0,\\
&X^3X^5 + c(X^2X^4 - X^1X^3) = 0.
\end{aligned} \tag{9.4}$$

Remark 9. Using Proposition 9.1-(2), i.e., $\langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}], Z\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Z]_{\mathfrak{m}}\rangle$, we can again obtain System (9.4). Also, one can check that using Proposition 9.1-(1) and computing $[X_{\mathfrak{m}}, X_{\mathfrak{k}}] = \mathrm{U}_{\mathfrak{m}}(X_{\mathfrak{m}}, X_{\mathfrak{m}})$, we again obtain System (9.4).

Remark 10. As we mentioned in Introduction, Kowalski et al. [32] gave a classification of homogeneous geodesics in F^4 based on the system different from (9.4). More precisely, we take a basis $\{E_1, E_2, E_3, E_4, E_5\}$ of $\mathfrak{sa}(2)$ satisfying (9.2). The commutation relations (9.2) coincide with (4.1) and therefore (up to signs) with commutation relations given in [37,48,57] via the correspondence (9.3).

On the other hand, in [32], the authors use the basis $\{X_1, X_2, Y_1, Y_2, B\}$ of $\mathfrak{sa}(2)$ satisfying (4.1) except $[X_1, Y_2]$. Instead of $[X_1, Y_2] = Y_1$, they used $[X_1, Y_2] = X_1$. This mistake propagated computational errors and produced incorrect results on homogeneous geodesics.

9.1.4 Determining the homogeneous geodesics in F_c^4

Next, we solve System (9.4). First, we consider special cases when one-solution component is zero. If we assume $X^5 = 0$, then directly from (9.4), we have

$$X = X^1E_1 + X^2E_2. \tag{9.5}$$

For $X^4 = 0$, we obtain $X = X^1E_1$, which is particular solution of (9.5) and two new solutions,

$$X = X^1(E_1 \pm \sqrt{2}E_3 + cE_5).$$

Assuming $X^3 = 0$, besides some already mentioned solutions, we obtain

$$X = X^1(E_1 \pm \sqrt{2}E_4 - cE_5).$$

If we assume $X^2 = 0$, we do not obtain a new solution. Finally, for $X^1 = 0$, we obtain eight new solutions:

$$X = X^5(\pm E_2 + E_3 \pm E_4 - cE_5) \quad \text{and} \quad X = X^5(\pm E_2 + E_3 \mp E_4 + cE_5).$$

Furthermore, we assume that all components are different from zero. First, we obtain $X^3 = X^4$, and hence, $X^1 = 0$. So, we have a contradiction.

Hence, we proved the following theorem.

Theorem 9.1. *For a tangent vector* $X = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 + X^5E_5 \in \mathfrak{sa}(2)$ *satisfying* $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \neq 0$, *the curve*

$$\gamma(s) = \exp_{\mathfrak{sa}(2)}(sX)\cdot o$$

is a geodesic starting at the origin o *of* $\mathrm{F}_c^4 = \mathrm{SA}(2)/\mathrm{SO}(2)$ *if and only if the geodesic vector* X *has one of the following forms*:

$$\begin{aligned}
X &= X^1E_1 + X^2E_2,\\
X &= X^1(E_1 \pm \sqrt{2}E_3 + cE_5),\\
X &= X^1(E_1 \pm \sqrt{2}E_4 - cE_5),\\
X &= X^5(\pm E_2 + E_3 \pm E_4 - cE_5),\\
X &= X^5(\pm E_2 + E_3 \mp E_4 + cE_5),
\end{aligned}$$

where $X^1, X^2, X^5 \in \mathbb{R}$.

Remark 11. In Proposition 5.1, we considered geodesics of the form $\exp_{\mathfrak{f}}(sX) : \mathbb{R} \to \mathrm{F}_c^4 = \mathbb{R}_+^2 \ltimes \mathbb{R}^2$, where

$$X = X^1(e_1 \pm \sqrt{2}e_4), \quad \text{or} \quad X = X^2e_2.$$

Since $E_1 = e_1 + cE_5$, $E_2 = e_2$, and $E_4 = e_4$, these vectors are rewritten as

$$X = X^1(E_1 \pm \sqrt{2}E_4 - cE_5), \quad X = X^2E_2.$$

Thus, these geodesics are included in the list of theorem mentioned earlier.

9.2 Homogeneous magnetic curves

Now, we magnetize the homogeneous geodesics by the Kähler magnetic fields $-\Omega_\pm$.

9.2.1 Systems of equations for homogeneous magnetic trajectories

For a vector $X = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 + X^5E_5 \in \mathfrak{so}(2) + \mathfrak{m}$ with $X_\mathfrak{m} = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 \neq 0$, the curves $\gamma(s) = \exp_{\mathfrak{sa}(2)}(sX)$ are Kähler magnetic curves starting at the identity with respect to the Kähler magnetic field $-\Omega_\pm$ if and only if

$$[X_\mathfrak{k}, X_\mathfrak{m}] + \mathsf{U}_\mathfrak{m}(X_\mathfrak{m}, X_\mathfrak{m}) = qJ_\pm X_\mathfrak{m}. \tag{9.6}$$

Since

$$J_+E_1 = E_2, \quad J_+E_2 = -E_1, \quad J_+E_3 = E_4, \quad J_+E_4 = -E_3,$$
$$J_-E_1 = E_2, \quad J_-E_2 = -E_1, \quad J_-E_3 = -E_4, \quad J_-E_4 = E_3,$$

we obtain the systems

$$\begin{aligned} &2X^2X^5 + 2cX^3X^4 = -qX^2, \\ &2X^1X^5 + c((X^4)^2 - (X^3)^2) = -qX^1, \\ &X^4X^5 + c(X^1X^4 + X^2X^3) = qX^4, \\ &X^3X^5 + c(X^2X^4 - X^1X^3) = qX^3, \end{aligned} \tag{9.7}$$

for Ω_+, and

$$\begin{aligned} &2X^2X^5 + 2cX^3X^4 = -qX^2, \\ &2X^1X^5 + c((X^4)^2 - (X^3)^2) = -qX^1, \\ &X^4X^5 + c(X^1X^4 + X^2X^3) = -qX^4, \\ &X^3X^5 + c(X^2X^4 - X^1X^3) = -qX^3, \end{aligned} \tag{9.8}$$

for Ω_-.

9.2.2 Determining the homogeneous magnetic curves with respect to $-\Omega_+$

We solve System (9.7). First, we consider special cases when one component of solution is zero.

If we assume $X^5 = 0$, then from the first equation of (9.7), we have

$$X^2 = -\frac{2c}{q}X^3X^4, \quad q \neq 0.$$

Substituting this relation in the third and the fourth equation of (9.7), and assuming $X^3 \neq 0$ and $X^4 \neq 0$, we have

$$(X^3)^2 = \frac{q}{2c}\left(X^1 - \frac{q}{c}\right) \quad \text{and} \quad (X^4)^2 = -\frac{q}{2c}\left(X^1 + \frac{q}{c}\right).$$

These two relations imply a contradiction. Also, for $X^3 = 0$ (or $X^4 = 0$), we again have a contradiction. Thus, $X^5 \neq 0$.

If we assume $X^5 \neq 0$ and $X^1 = 0$, then from the second equation of (9.7) follows $X^3 = \pm X^4$. Next, the fourth equation gives $X^4X^5 \pm X^2X^4 = qX^4$. Hence, for $X^4 = 0$ by further calculations, we have a solution $X = X^2E_2 - \frac{q}{2}E_5$ since $X_{\mathrm{m}} \neq 0$.

If $X^4 \neq 0$, then we obtain four new solutions

$$X = X^2E_2 \pm \sqrt{(X^2)^2 - \frac{3q}{2c}X^2}\,E_3 \pm \sqrt{(X^2)^2 - \frac{3q}{2c}X^2}\,E_4 + (q - cX^2)E_5,$$
$$X = X^2E_2 \pm \sqrt{(X^2)^2 + \frac{3q}{2c}X^2}\,E_3 \mp \sqrt{(X^2)^2 + \frac{3q}{2c}X^2}\,E_4 + (q + cX^2)E_5.$$

If we assume $X^5 \neq 0$ and $X^2 = 0$, then from the first equation of (9.7) follows $X^3X^4 = 0$. For $X^3 = 0$, we obtain two new solutions

$$X = X^1E_1 - \frac{q}{2}E_5 \quad \text{and} \quad X = X^1E_1 \pm \sqrt{2(X^2)^1 - \frac{3q}{c}X^1}\,E_4 + (q - cX^1)E_5.$$

For $X^4 = 0$, we obtain one new solution

$$X = X^1E_1 \pm \sqrt{2(X^1)^2 + \frac{3q}{c}X^1}\,E_3 + (q + cX^1)E_5.$$

If we assume $X^5 \neq 0$ and $X^3 = 0$ or $X^4 = 0$, (9.7) gives only already mentioned solutions. Also, if we assume that all components of X are different from zero, we obtain a contradiction.

Hence, we proved the following theorem.

Theorem 9.2. *For a tangent vector $X = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 + X^5E_5 \in \mathfrak{so}(2) + \mathfrak{m}$, satisfying $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \neq 0$, the curve*

$$\gamma(s) = \exp_{\mathfrak{so}(2)}(sX)$$

is a Kähler magnetic curves starting at the identity with respect to the Kähler magnetic field-Ω_+ if and only if the magnetic vector X has one of the following forms:

$$X = X^1E_1 - \frac{q}{2}E_5,$$
$$X = X^2E_2 - \frac{q}{2}E_5,$$
$$X = X^1E_1 \pm \sqrt{2(X^1)^2 + \frac{3q}{c}X^1}\,E_3 + (q + cX^1)E_5,$$
$$X = X^1E_1 \pm \sqrt{2(X^1)^2 - \frac{3q}{c}X^1}\,E_4 + (q - cX^1)E_5,$$
$$X = X^2E_2 \pm \sqrt{(X^2)^2 + \frac{3q}{2c}X^2}\,E_3 \mp \sqrt{(X^2)^2 + \frac{3q}{2c}X^2}\,E_4 + (q + cX^2)E_5,$$
$$X = X^2E_2 \pm \sqrt{(X^2)^2 - \frac{3q}{2c}X^2}\,E_3 \pm \sqrt{(X^2)^2 - \frac{3q}{2c}X^2}\,E_4 + (q - cX^2)E_5,$$

where $X^1, X^2, X^5 \in \mathbb{R}$.

9.2.3 Determining the homogeneous magnetic curves with respect to Ω

Analogous to the previous consideration, we solve System (9.8). We obtain the following theorem.

Theorem 9.3. *For a tangent vector* $X = X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 + X^5E_5 \in \mathfrak{so}(2) + \mathfrak{m}$, *satisfying* $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \neq 0$, *the curve*

$$\gamma(s) = \exp_{\mathfrak{sa}(2)}(sX)$$

is a Kähler magnetic curve starting at the identity with respect to the Kähler magnetic field-Ω_- *if and only if the magnetic vector X has one of the following forms*:

$$\begin{aligned}
X &= X^1E_1 - \frac{q}{2}E_5,\\
X &= X^2E_2 - \frac{q}{2}E_5,\\
X &= X^1E_1 \pm \sqrt{2(X^1)^2 + \frac{q}{c}X^1}\,E_3 - (q + cX^1)E_5,\\
X &= X^1E_1 \pm \sqrt{2(X^1)^2 - \frac{q}{c}X^1}\,E_4 - (q - cX^1)E_5,\\
X &= X^2E_2 \pm \sqrt{(X^2)^2 + \frac{q}{2c}X^2}\,E_3 \pm \sqrt{(X^2)^2 + \frac{q}{2c}X^2}\,E_4 - (q + cX^2)E_5,\\
X &= X^2E_2 \pm \sqrt{(X^2)^2 - \frac{q}{2c}X^2}\,E_3 \mp \sqrt{(X^2)^2 - \frac{q}{2c}X^2}\,E_4 - (q - cX^2)E_5,
\end{aligned}$$

where $X^1, X^2, X^5 \in \mathbb{R}$.

10 Conclusion

In this article, we describe the four-dimensional simply connected Riemannian 3-symmetric space $\hat{M}^4_\lambda$ due to Kowalski. We explain the homogeneous geometry of the model space F^4 and give the Levi-Civita connection, Riemannian curvature, Ricci operator, sectional curvatures, and scalar curvature of the model space F^4. Next, we introduce the symplectic pair of two Kähler forms and explore some typical submanifolds of F^4. Furthermore, we study geodesics and magnetic curves in F^4. We explore the general properties of magnetic curves in an almost Kähler 4-manifold and characterize Kähler magnetic curves with respect to the symplectic pair of Kähler forms. In Section 9, we study homogeneous geodesics and homogeneous magnetic curves in F^4.

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Appendix
Formula (9.1)

In this appendix, we give a proof of Formula (9.1) for reader's convenience.

Take vectors $X, Y \in \mathfrak{g}$ and set $\phi_t = \exp(tX)$ and $\psi_s = \exp(sZ)$. Then,

$$\begin{aligned}
Z^\sharp_{\phi_t(x)} &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \psi_s(\phi_t(x)) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \phi_t(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x) \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} (\phi_t^{-1} \circ \psi_s \circ \phi_t)(x) \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} \{\exp(tX)^{-1} \exp(sZ) \exp(tX)\}(x) \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} \mathrm{Ad}(\exp(tX)^{-1}) \exp(sZ)(x) \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \exp\{s\mathrm{ad}(e^{-tX})Z\} \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} (\mathrm{ad}(e^{-tX})Z)^\sharp_x \\
&= \phi_{*(\phi_t^{-1} \circ \psi_s \circ \phi_t)(x)} (Z - t[X, Z] + o(t^2))^\sharp_x.
\end{aligned}$$

Hence, we obtain (9.1).