Research Article

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Quot schemes and Fourier-Mukai transformation

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Abstract: We consider several related examples of Fourier-Mukai transformations involving the quot scheme. A method of showing conservativity of these Fourier-Mukai transformations is described.

Keywords: quot scheme, Hilbert scheme, Fourier-Mukai transformation, symmetric product

MSC 2020: 14C05, 14L30

1 Introduction

Fourier-Mukai transformations arise in numerous contexts in algebraic geometry [2,4]. Over time, it has emerged to be an immensely useful concept. Here, we investigate Fourier-Mukai transformations in a particular context, namely, in the set-up of quot schemes. We show that the conservativity of the Fourier-Mukai transformation holds in the following cases:

(1) Let M be an irreducible smooth projective variety over an algebraically closed field k such that $\dim M \geq 2$. Denote by $\mathrm{Hilb}^d(M)$ the Hilbert scheme parametrizing the zero-dimensional subschemes of M of length d. Let P_M (respectively, P_H) be the projection to M (respectively, $\mathrm{Hilb}^d(M)$) of the tautological subscheme $\mathcal{S} \subset M \times \mathrm{Hilb}^d(M)$. Let E and F be two vector bundles on M such that the vector bundles $P_{H*}P_M^*E$ and $P_{H*}P_M^*F$ on $\mathrm{Hilb}^d(M)$ are isomorphic. Then, we show that E and F are isomorphic (see Proposition 2.1).

(2) Let Q_M (respectively, Q_S) be the projection to M (respectively, $\mathrm{Sym}^d(M)$) of the tautological subscheme $\mathbb{S} \subset M \times \mathrm{Sym}^d(M)$. Let E and F be two vector bundles on M such that the vector bundles $Q_{S*}Q_M^*E$ and $Q_{S*}Q_M^*F$ on $\mathrm{Sym}^d(M)$ are isomorphic. Then, we show that E and F are isomorphic (see Lemma 2.2).

(3) Let C be an irreducible smooth projective curve defined over k. Fix a vector bundle E over C of rank at least two. Let $Q^d(E)$ denote the quot scheme parametrizing the torsion quotients of E of degree d. There is a tautological quotient $\Phi_C^*E \longrightarrow \mathbf{Q}$ over $C \times Q^d(E)$, where $\Phi_C : C \times Q^d(E) \longrightarrow C$ is the natural projection. Let V and W be vector bundles on C such that the vector bundles $\Phi_{Q*}((\Phi_C^*V) \otimes \mathbf{Q})$ and $\Phi_{Q*}((\Phi_C^*W) \otimes \mathbf{Q})$ on $Q^d(E)$ are isomorphic, where $\Phi_Q : C \times Q^d(E) \longrightarrow Q^d(E)$ is the natural projection. Then, we show that E and F are isomorphic (see Proposition 2.3).

We also prove a similar result in the context of vector bundles on curves equipped with a group action (see Section 3).

A key method in our proofs is the Atiyah's Krull-Schmidt theorem for vector bundles.

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2 A Fourier-Mukai transformation

2.1 Vector bundles on Hilbert schemes

Let k be an algebraically closed field. Let M be an irreducible smooth projective variety over k such that $\dim M \geq 2$. For any integer $d \geq 2$, let $\mathrm{Hilb}^d(M)$ denote the Hilbert scheme parametrizing the zero-dimensional subschemes of M of length d. We have the natural projections:

$$M \overset{p_M}{\leftarrow} M \times \mathrm{Hilb}^d(M) \overset{p_H}{\rightarrow} \mathrm{Hilb}^d(M).$$

There is a tautological subscheme

$$\mathcal{S} \subset M \times \mathrm{Hilb}^d(M),$$

such that for any $z \in \mathrm{Hilb}^d(M)$, the preimage $p_H^{-1}(z)$ is the subscheme $z \subset M$. The restriction of p_M (respectively, p_H) to $\mathcal{S}$ will be denoted by P_M (respectively, P_H).

For any vector bundle E on M, we have the direct image $P_{H*}P_M^*E$ on $\mathcal{S}$. We note that $P_{H*}P_M^*E$ is locally free because P_H is a finite morphism and P_M^*E is locally free. Let

$$\widetilde{E} \coloneqq P_{H*}P_M^*E \tag{2.1}$$

be this vector bundle; its rank is $d \cdot \mathrm{rank}(E)$. It is known that two vector bundles E and F on M are isomorphic if $\widetilde{E}$ and $\widetilde{F}$ are isomorphic [3,5]. We will give a very simple proof of it.

Proposition 2.1. *Let E and F be two vector bundles on M such that the corresponding vector bundles $\widetilde{E}$ and $\widetilde{F}$ on* $\mathrm{Hilb}^d(M)$ *are isomorphic (see* (2.1)). *Then, E and F are isomorphic.*

Proof. Since $\mathrm{rank}(\widetilde{E})$ and $\mathrm{rank}(\widetilde{F})$ are $d \cdot \mathrm{rank}(E)$ and $d \cdot \mathrm{rank}(F)$, respectively, it follows that $\mathrm{rank}(E) = \mathrm{rank}(F)$. Let $\mathrm{rank}(E) = r = \mathrm{rank}(F)$.

Fix a zero-dimensional subscheme $Z^0 \subset M$ of length $d - 1$. Let $Z^0_{\mathrm{red}} = \{x_1, \ldots, x_b\} \subset M$ be the reduced subscheme for Z^0. The complement $M \backslash Z^0_{\mathrm{red}} = M \backslash \{x_1, \ldots, x_b\}$ will be denoted by M^0. Let

$$\iota : M^0 \longrightarrow M \tag{2.2}$$

be the inclusion map. We have a morphism

$$\varphi : M^0 \longrightarrow \mathrm{Hilb}^d(M)$$

that sends any $x \in M^0$ to $Z_0 \cup \{x\}$. The pullback $\varphi^*\widetilde{E}$ (respectively, $\varphi^*\widetilde{F}$) is isomorphic to $\iota^*E \oplus V_0$ (respectively, $\iota^*F \oplus V_0$), where V_0 is a trivial vector bundle on M^0 of rank $(d-1)r$ and ι is the map in (2.2). The vector bundles $\varphi^*\widetilde{E}$ and $\varphi^*\widetilde{F}$ are isomorphic because $\widetilde{E}$ and $\widetilde{F}$ are isomorphic. So

$$\iota^*E \oplus V_0 = \iota^*F \oplus V_0. \tag{2.3}$$

There are no nonconstant functions on M^0 (recall that $\dim M \geq 2$). Hence, using [1, p. 315, Theorem 2(i)], from (2.3), it follows that $\iota^*E = \iota^*F$ (see [6] for vast generalizations of [1]). Hence, we have

$$\iota_*\iota^*E = \iota_*\iota^*F.$$

But $\iota_*\iota^*E$ (respectively, $\iota_*\iota^*F$) is E (respectively, F). This completes the proof. □

The line of arguments in Proposition 2.1 works in some other contexts. We will describe two such instances.

2.2 Vector bundles on symmetric product

As mentioned previously, M is an irreducible smooth projective variety of dimension at least two. For any integer $d \geq 2$, let $\mathrm{Sym}^d(M)$ denote the quotient of M^d under the action of the symmetric group S_d that permutes the factors of the Cartesian product. We recall that $\mathrm{Sym}^d(M)$ is a normal projective variety. There is a tautological subscheme

$$\mathbb{S} \subset M \times \mathrm{Sym}^d(M) \tag{2.4}$$

parametrizing all $(z, y) \in M \times \mathrm{Sym}^d(M)$ such that $z \in y$. Let

$$Q_M : \mathbb{S} \longrightarrow M \quad \text{and} \quad Q_S : \mathbb{S} \longrightarrow \mathrm{Sym}^d(M)$$

be the natural projections. For any vector bundle E on M, the direct image

$$\widehat{E} \coloneqq Q_{S*}Q_M^*E$$

on $\mathrm{Sym}^d(M)$ is locally free because Q_S is a finite morphism and Q_M^*E is locally free.

Lemma 2.2. *Let E and F be two vector bundles on M such that the corresponding vector bundles $\widehat{E}$ and $\widehat{F}$ on $\mathrm{Sym}^d(M)$ are isomorphic. Then, E and F are isomorphic.*

Proof. Fix any $z_0 = \{x_1, \ldots, x_{d-1}\} \in \mathrm{Sym}^{d-1}(M)$ (repetitions are allowed). Let

$$\iota : M^0 \coloneqq M \backslash z_0 \hookrightarrow M$$

be the inclusion map. We have a morphism

$$\phi : M \backslash z_0 \longrightarrow \mathrm{Sym}^d(M), \quad x \longmapsto \{x, \ z\}.$$

First, note that $\phi^*\widehat{E} = \phi^*\widehat{F}$ because $\widehat{E} = \widehat{F}$. Evidently, we have $\phi^*\widehat{E} = (\iota^*E) \oplus \mathcal{O}_{M^0}^{\oplus(d-1)\cdot \mathrm{rank}(E)}$ and $\phi^*\widehat{F} = (\iota^*F) \oplus \mathcal{O}_{M^0}^{\oplus(d-1)\cdot \mathrm{rank}(F)}$. Now, the argument in the proof of Proposition 2.1 goes through without any changes. □

2.3 Vector bundles on quot scheme

Let C be an irreducible smooth projective curve defined over k. Fix a vector bundle E over C of rank at least two. Fix an integer $d \geq 1$. Let $Q^d(E)$ denote the quot scheme parametrizing the torsion quotients of E of degree d. Let

$$\Phi_C : C \times Q^d(E) \longrightarrow C \quad \text{and} \quad \Phi_Q : C \times Q^d(E) \longrightarrow Q^d(E) \tag{2.5}$$

be the natural projections. There is a tautological quotient

$$\Phi_C^*E \longrightarrow \mathbf{Q} \tag{2.6}$$

over $C \times Q^d(E)$ whose restriction to any $C \times \{Q\}$, where $Q \in Q^d(E)$, is the quotient of E represented by Q.

Given a vector bundle V on C, we have the direct image

$$F(V) \coloneqq \Phi_{Q*}((\Phi_C^*V) \otimes \mathbf{Q}) \longrightarrow Q^d(E), \tag{2.7}$$

where Φ_Q and Φ_C are the projections in (2.5), and $\mathbf{Q}$ is the quotient in (2.6); this $F(V)$ is a vector bundle because the support of $\mathbf{Q}$ is finite over $Q^d(E)$.

Proposition 2.3. *Let V and W be vector bundles on C such that the corresponding vector bundles $F(V)$ and $F(W)$ are isomorphic* (*see* (2.7)). *Then, V and W are isomorphic.*

Proof. Since $\mathrm{rank}(F(V))$ and $\mathrm{rank}(F(W))$ are $d \cdot \mathrm{rank}(V)$ and $d \cdot \mathrm{rank}(W)$ respectively, from the given condition that $F(V)$ and $F(W)$ are isomorphic, we conclude that $\mathrm{rank}(V) = \mathrm{rank}(W)$. Let r denote $\mathrm{rank}(V) = \mathrm{rank}(W)$.

Let

$$\beta : \mathbb{P}(E) \longrightarrow X \tag{2.8}$$

be the projective bundle parametrizing the hyperplanes in the fibers of E. So $\mathbb{P}(E) = Q^1(E)$. For any $z \in \beta^{-1}(x) \subset \mathbb{P}(E)$, if $H(z) \subset E_x$ is the corresponding hyperplane, then the element of $Q^1(E)$ for z represents the quotient sheaf $E \longrightarrow E_x/H(z)$ of E. For any $z \in \mathbb{P}(E)$, the quotient sheaf map from E to the torsion quotient $E_x/H(z)$ of E of degree 1 corresponding to z will be denoted by $\mathbf{z}$.

Fix $d-1$ distinct points $x_1, \ldots, x_{d-1}$ of X. Fix points $y_i \in \beta^{-1}(x_i)$, $1 \le i \le d-1$, where β is the projection in (2.8). The complement $\mathbb{P}(E)\setminus\{y_1, \ldots, y_{d-1}\}$ will be denoted by $\mathcal{P}$. Let

$$\iota : \mathcal{P} \hookrightarrow \mathbb{P}(E) \tag{2.9}$$

be the inclusion map.

Note that the subset $\{y_1, \ldots, y_{d-1}\}$ defines a point of $Q^{d-1}(E)$ representing the quotient $\oplus_{j=1}^{d-1}\mathbf{y}_j$ of E; this point of $Q^{d-1}(E)$ will be denoted by $\mathbf{y}$. We have a morphism

$$\Psi : \mathcal{P} \longrightarrow Q^d(E), \quad z \longmapsto \mathbf{z} \oplus \mathbf{y};$$

recall that $\mathbb{P}(E) = Q^1(E)$ and both $\mathbf{y}$ and $\mathbf{z}$ are the quotients of E.

Now, the vector bundle $\Psi^*F(V)$ (respectively, $\Psi^*F(W)$) is isomorphic to $(\iota^*((\beta^*V) \otimes \mathcal{O}_{\mathbb{P}(E)}(1))) \oplus A$ (respectively, $(\iota^*((\beta^*W) \otimes \mathcal{O}_{\mathbb{P}(E)}(1))) \oplus A$), where ι and β are the maps in (2.9) and (2.8), respectively, and A is a trivial vector bundle on $\mathcal{P}$ of rank $r(d-1)$; the tautological line bundle on $\mathbb{P}(E)$ is denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Since V and W are isomorphic, we conclude that $(\iota^*((\beta^*V) \otimes \mathcal{O}_{\mathbb{P}(E)}(1))) \oplus A$ and $(\iota^*((\beta^*V) \otimes \mathcal{O}_{\mathbb{P}(E)}(1))) \oplus A$ are isomorphic. As there are no nonconstant functions on $\mathcal{P}$, it follows that $(\iota^*\beta^*V) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ and $(\iota^*\beta^*W) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ are isomorphic. This implies that $\iota^*\beta^*V$ and $\iota^*\beta^*W$ are isomorphic.

The direct image $\iota_*\iota^*\beta^*V$ (respectively, $\iota_*\iota^*\beta^*W$) is β^*V (respectively, β^*W). Hence, we conclude that β^*V and β^*W are isomorphic. So $\beta_*\beta^*V = V$ is isomorphic to $\beta_*\beta^*W = W$. □

3 Action of group on a curve

Let C be an irreducible smooth projective curve, and let Γ be a finite group acting faithfully on C. Consider the quotient curve

$$f : C \longrightarrow Y \coloneqq C/\Gamma. \tag{3.1}$$

For any vector bundle V on Y, the pullback f^*V is a Γ-equivariant vector bundle on C.

The order of the group Γ is denoted by d. We have a morphism

$$\rho : Y \longrightarrow \mathrm{Sym}^d(C) \tag{3.2}$$

that sends any $y \in Y$ to the element of $\mathrm{Sym}^d(C)$ given by the scheme-theoretic inverse image $f^{-1}(y)$, where f is the map in (3.1). To describe ρ explicitly, let $\{z_1, z_2, \ldots, z_n\}$ be the reduced inverse image $f^{-1}(y)_{\mathrm{red}}$. Then,

$$\rho(y) = \sum_{i=1}^{n} b_i z_i,$$

where b_i is the order of the isotropy subgroup $\Gamma_{z_i} \subset \Gamma$ of z_i for the action of Γ on C. Note that ρ is an embedding.

The action of Γ on C produces an action of Γ on $\mathrm{Sym}^d(C)$. The action of any $\gamma \in \Gamma$ sends any $(x_1, \ldots, x_d) \in \mathrm{Sym}^d(C)$ to $(\gamma(x_1), \ldots, \gamma(x_d))$. We have

$$\rho(Y) \subset \mathrm{Sym}^d(C)^\Gamma. \tag{3.3}$$

We note that $\mathrm{Sym}^d(C)$ is an irreducible smooth projective variety of dimension d. As in (2.4),

$$\mathbb{S} \subset C \times \mathrm{Sym}^d(C) \tag{3.4}$$

is the tautological subscheme parametrizing all $(c, x) \in C \times \mathrm{Sym}^d(C)$ such that $c \in x$. Let

$$Q_C : \mathbb{S} \longrightarrow C \quad \text{and} \quad Q_S : \mathbb{S} \longrightarrow \mathrm{Sym}^d(C) \tag{3.5}$$

be the natural projections. For any vector bundle E on C of rank r, the direct image

$$\widehat{E} := Q_{S*}Q_C^*E \tag{3.6}$$

is a vector bundle on $\mathrm{Sym}^d(C)$ of rank dr.

We will describe an alternative construction of the vector bundle $\widehat{E}$ in (3.6). For $1 \leq i \leq d$, let

$$p_i : C^d \longrightarrow C$$

be the projection to the i-th factor. Let

$$P : C^d \longrightarrow \mathrm{Sym}^d(C) \tag{3.7}$$

be the quotient map for the action of the symmetric group S_d that permutes the factors of C^d. The action of S_d on C^d lifts to the vector bundle

$$E^{[d]} := \bigoplus_{i=1}^{d} p_i^*E \longrightarrow C^d.$$

The action of S_d on $E^{[d]}$ produces an action of S_d on $P_*E^{[d]}$, where P is the projection in (3.7). The vector bundle $\widehat{E}$ in (3.6) coincides with the S_d-invariant part

$$(P_*E^{[d]})^{S_d} \subset P_*E^{[d]}.$$

The actions of Γ on C and $\mathrm{Sym}^d(C)$ (see (3.3)) together produce a diagonal action of Γ on $C \times \mathrm{Sym}^d(C)$. This action of Γ on $C \times \mathrm{Sym}^d(C)$ preserves the subscheme $\mathbb{S}$ in (3.4). For this action of Γ on $\mathbb{S}$, the projections Q_C and Q_S in (3.5) are evidently Γ-equivariant.

Now, let E be a Γ-equivariant vector bundle on C. Since the projections Q_C and Q_S in (3.5) are Γ-equivariant, the vector bundle $\widehat{E}$ in (3.6) is also Γ-equivariant. From (3.3), it now follows that the vector bundle

$$\rho^*\widehat{E} \longrightarrow Y \tag{3.8}$$

is equipped with an action of Γ over the trivial action of Γ on Y.

Proposition 3.1. *Let E and F be vector bundles on Y such that the corresponding Γ-equivariant vector bundles $\rho^*\widehat{f^*E}$ and $\rho^*\widehat{f^*F}$ on Y are isomorphic. Then, E and F are isomorphic.*

Proof. The vector bundle f^*E has a natural action of Γ because it is pulled back from C/Γ. The action of Γ on f^*E produces an action of Γ on f_*f^*E over the trivial action of Γ on Y. Similarly, Γ acts on f_*f^*F.

Consider $Q_S^{-1}(\rho(Y)) \subset \mathbb{S}$, where Q_S and ρ are the maps in (3.5) and (3.2), respectively. Let

$$Q_C' := Q_C\,|_{Q_S^{-1}(\rho(Y))} : Q_S^{-1}(\rho(Y)) \longrightarrow C$$

be the restriction of the map Q_C in (3.5). It is straightforward to check that this map Q_C' is an isomorphism. So we have the commutative diagram

$$\begin{tikzcd} C \arrow[d, "f"'] & Q_S^{-1}(\rho(Y)) \arrow[l, "Q_C'"'] \arrow[d, "Q_S"] \\ Y \arrow[r, "\rho"] & \rho(Y) \end{tikzcd} \tag{3.9}$$

where the horizontal maps are isomorphisms. Moreover, all the maps in (3.9) are Γ-equivariant with Γ acting trivially on Y and $\rho(Y)$. Therefore, from (3.9), we conclude that there are isomorphisms

$$f_* f^* E \xrightarrow{\sim} \rho^* \widehat{f^* E} \quad \text{and} \quad f_* f^* F \xrightarrow{\sim} \rho^* \widehat{f^* F} \tag{3.10}$$

as Γ-equivariant vector bundles.

Since the Γ-equivariant vector bundles $\rho^* \widehat{f^* E}$ and $\rho^* \widehat{f^* F}$ are isomorphic, from (3.10), it follows that

$$f_* f^* E \xrightarrow{\sim} f_* f^* F \tag{3.11}$$

as Γ-equivariant vector bundles

Next, we will show that

$$(f_* f^* E)^\Gamma = E \quad \text{and} \quad (f_* f^* F)^\Gamma = F. \tag{3.12}$$

To prove (3.12), first note that the action of Γ on C produces an action of Γ on $f_* \mathcal{O}_C$. The projection formula gives that

$$f_* f^* E \xrightarrow{\sim} E \otimes (f_* \mathcal{O}_C).$$

The action of Γ on $f_* \mathcal{O}_C$ and the trivial action of Γ on E together produce an action of Γ on $E \otimes (f_* \mathcal{O}_C)$. The aforementioned isomorphism between $f_* f^* E$ and $E \otimes (f_* \mathcal{O}_C)$ is evidently Γ-equivariant. Since $(f_* \mathcal{O}_C)^\Gamma = \mathcal{O}_Y$, we conclude that (3.12) holds.

Finally, the proposition follows from (3.11) and (3.12). □

4 Alternative constructions

Let C be a smooth projective curve over k and E a vector bundle on C. Unlike in Section 2.3, E can be a line bundle; we no longer assume $\operatorname{rank}(E)$ to be at least two. As mentioned earlier, $\mathcal{Q}^d(E)$ denotes the quot scheme that parametrizes the torsion quotients of E of degree d. Let

$$\gamma : \mathcal{Q}^d(E) \longrightarrow \operatorname{Sym}^d(C) \tag{4.1}$$

be the natural Chow morphism.

For any vector bundle V on C, consider the vector bundle $F(V)$ on $\mathcal{Q}^d(E)$ constructed in (2.7). We will describe its direct image $\gamma_* F(V)$ on $\operatorname{Sym}^d(C)$, where γ is the map in (4.1).

For every $1 \le j \le d$, let $\varphi_j : C^d \longrightarrow C$ be the projection to the j-th factor. Take a vector bundle V on C. We have the vector bundle

$$\mathcal{V} := \bigoplus_{j=1}^{d} \varphi_j^*(V \otimes E) \longrightarrow C^d. \tag{4.2}$$

The symmetric group S_d acts on C^d by permuting the factors of the tensor product (see Section 2.2). The corresponding quotient is $\operatorname{Sym}^d(C)$. As in (3.7), let

$$P : C^d \longrightarrow C^d/S_d = \operatorname{Sym}^d(C) \tag{4.3}$$

be the quotient map. The action of S_d on C^d has a natural lift to an action of S_d on the vector bundle $\mathcal{V}$ in (4.2). This action of S_d on $\mathcal{V}$ produces an action of S_d on the direct image $P_* \mathcal{V}$, where P is the projection in (4.3).

Lemma 4.1. *The direct image $\gamma_* F(V)$ on $\operatorname{Sym}^d(C)$, where $F(V)$ and γ are as in* (2.7) *and* (4.1), *respectively, is naturally identified with the S_d-invariant part*

$$(P_* \mathcal{V})^{S_d} \subset P_* \mathcal{V}$$

for the aforementioned action of S_d on $P_* \mathcal{V}$.

Proof. There is a natural homomorphism

$$\varpi : F(V) \longrightarrow P_*\mathcal{V}.$$

It is straightforward to check that $\varpi(F(V)) \subset (P_*\mathcal{V})^{S_d} \subset P_*\mathcal{V}$ and that the resulting homomorphism $F(V) \longrightarrow (P_*\mathcal{V})^{S_d}$ is an isomorphism. □

Let

$$\Psi_C : C \times \mathrm{Sym}^d(C) \longrightarrow C \quad \text{and} \quad \Psi_S : C \times \mathrm{Sym}^d(C) \longrightarrow \mathrm{Sym}^d(C) \tag{4.4}$$

be the natural projections.

Consider $(\mathrm{Id}_C \times \gamma)_*\mathbf{Q}$ on $C \times \mathrm{Sym}^d(C)$, where $\mathbf{Q}$ is the sheaf in (2.6) and γ is the map in (4.1). Given a vector bundle V on C, we have the direct image

$$G(V) := \Psi_{S*}(\Psi_C^*V \otimes (\mathrm{Id}_C \times \gamma)_*\mathbf{Q}) \longrightarrow \mathrm{Sym}^d(C),$$

where Ψ_C and Ψ_S are projections in (4.4).

Proposition 4.2. *For any vector bundle V on C, there is a natural isomorphism*

$$\gamma_*F(V) \ \simeq \ G(V),$$

where $F(V)$ is constructed in (2.7).

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & C \times \mathcal{Q}^d(E) \arrow[r, "\Phi_Q"] \arrow[dl, "\Phi_C"'] \arrow[d, "(\mathrm{Id}_C \times \gamma)"'] & \mathcal{Q}^d(E) \arrow[d, "\gamma"'] \\
C & C \times \mathrm{Sym}^d(C) \arrow[l, "\Psi_C"] \arrow[r, "\Psi_S"'] & \mathrm{Sym}^d(C).
\end{tikzcd}$$

Now, using the aforementioned commutative diagram, we can obtain the required isomorphism as follows:

$$\begin{aligned}
\gamma_*F(V) &= \gamma_*\Phi_{Q*}((\Phi_C^*V) \otimes \ \mathbf{Q}) \\
&\simeq \Psi_{S*}(\mathrm{Id}\times\gamma)_*((\Phi_C^*V) \otimes \ \mathbf{Q}) \\
&= \Psi_{S*}(\mathrm{Id}\times\gamma)_*((\mathrm{Id}\times\gamma)^*(\Psi_C^*V) \otimes \ \mathbf{Q}) \\
&\simeq \Psi_{S*}((\Psi_C^*V) \otimes \ (\mathrm{Id}\times\gamma)_*\mathbf{Q}) \quad \text{(projection formula)} \\
&= G(V).
\end{aligned}$$

□

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