Research Article

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Geometry of analytic continuation on complex manifolds – history, survey, and report

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Abstract: Beginning with the state of art around 1953, solutions of the Levi problem on complex manifolds will be recalled at first up to Takayama's result in 1998. Then, the activity of extending the results by the L^2 method in these decades will be reported. The method is by exploiting the finite dimensionality of certain $L^2\bar{\partial}$-cohomology groups to prove that a Hermitian holomorphic line bundle L over a complex manifold M is bimeromorphically equivalent to an ample bundle when it is restricted to a bounded locally pseudoconvex domain $\Omega \Subset M$ under the positivity of $L|_{\partial\Omega}$ and the regularity of $\partial\Omega$.

Keywords: plurisubharmonic function, locally pseudoconvex, bundle convexity, xxxxx-equation, L^2 estimates, curvature form, Bergman kernel

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In Memory of Jean-Pierre Demailly.

1 Introduction

In the theory of several complex variables, questions of extending analytic objects keeping their analyticity arise in many situations. The maximal domain of definition for such extensions has geometric properties similar to the convexity. At this point, basic questions arise characterizing the realm of several complex variables.

The study of domains of holomorphy was the starting point of general theory of several complex variables. Recall that a domain Ω over $\mathbb{C}^n$ is said to be a **domain of holomorphy** if $\Omega \in \pi_0(\mathcal{O}_{\mathbb{C}^n})$[1], i.e., if Ω is biholomorphically equivalent to a connected component of the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ of $\mathbb{C}^n$, which is, by definition, the sheaf $\pi : \mathcal{O}(=\mathcal{O}_{\mathbb{C}^n}) \to \mathbb{C}^n$, of the germs of holomorphic functions. That $\mathbb{C}^2\backslash\{(0,0)\}$ is not a domain of holomorphy was first explicitly stated in the international congress of mathematics talk of Hurwitz in 1897. There exist various geometric characterizations of domains of holomorphy. A successful example is the following equivalences that originated in the works of Hartogs, Levi, Cartan, and Thullen and finalized by Oka in 1953 as follows:

$$\begin{array}{llllll} \mathcal{O} & \supset & \Omega & \in \pi_0(\mathcal{O}) & \Leftrightarrow & \Omega \text{ is holomorphically convex} \\ \downarrow & \swarrow & & \Uparrow\Downarrow & & \\ \mathbb{C}^n & & & -\log\delta_\Omega \in \mathrm{PSH}(\Omega) & & \end{array}$$

1 $\pi_0(X)$ denotes the set of connected components of X.

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Here $\delta_\Omega(x) := \sup\{r; \exists \iota : \mathbb{B}(\pi(x), r) \overset{\text{holomorphic}}{\longrightarrow} \Omega \ \text{ s.t. } \pi \circ \iota = Id\}$,
$\mathbb{B}(w, r) := \{z \in \mathbb{C}^n;\ \|z - w\| < r\}$ and PSH = plurisubharmonic.

The equivalence between $-\log\delta_\Omega \in \mathrm{PSH}(\Omega)$ and $\Omega \in \pi_0(\mathcal{O})$ is the solution of the Levi problem[2] on $\mathbb{C}^n$. Recall that a complex manifold M is said to be **holomorphically convex** if

$$\forall \gamma \in M^{\mathbb{N}} \text{ s.t. } \gamma(\mathbb{N}) \not\Subset M, \quad \exists f \in \mathcal{O}(M) \text{ s.t. } f(\gamma(\mathbb{N})) \not\Subset \mathbb{C},$$

or equivalently,

$$\forall K \Subset M, \quad \{x;\ |f(x)| \le \sup_K |f|, \quad \text{for all } f \in \mathcal{O}(M)\} \Subset M.$$

A striking result in the early stage of several complex variables (SCV) was that a finitely sheeted domain over $\mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy (cf. [12]). We shall call a domain $\Omega \overset{\pi}{\to} M$ **locally pseudoconvex** if every point $x \in M$ has a neighborhood U such that $\pi^{-1}(U)$ is holomorphically convex[3]. It is known from the works of Fujita [25] and Takeuchi [72] that one can replace $\mathbb{C}^n$ by $\mathbb{CP}^n$ in Oka's theorem by interpreting δ as "the distance to the boundary of Ω" with respect to the Fubini-Study metric or replacing $\mathbb{B}(\pi(x), r)$ by the corresponding geodesic ball. The result was further generalized by Ueda [73] for the domains over Grassmannian manifolds.

Note that $\mathbb{C}^n$ is the unique simply connected complete Kähler manifold whose sectional curvature is zero (cf. [68]) and $\mathbb{CP}^n$ is characterized as a compact Kähler manifold whose biholomorphic sectional curvature is positive (cf. [45] and [69]). For any noncompact locally pseudoconvex domain $\pi : \Omega \to \mathbb{CP}^n$, the curvature property can be used to show that, for any fixed C^∞ real-valued function ψ on $\mathbb{CP}^n$, $-\log\delta_\Omega + \varepsilon\pi^*\psi \in \mathrm{PSH}(\Omega)$ for sufficiently small $\varepsilon > 0$. Recall that a holomorphically convex manifold M is called a **Stein manifold** if $\mathcal{O}(M) \twoheadrightarrow \mathbb{C}^{\{x,y\}}$[4], i.e., the natural restriction map $\mathcal{O}(M) \to \mathbb{C}^{\{x,y\}}$ is surjective, for all $x, y \in M$.

According to what J.-P. Serre told S. Hitotumatu, it had not yet been proven in 1954 that locally pseudoconvex domains over Stein manifolds are Stein (cf. [38])[5]. As is well known, this question became an exercise because of the following characterization of Steinness due to Grauert [27] (see also [35, pp. 280–284]).

Theorem 1. *For a connected complex manifold M, the following are equivalent*:
(1) *M is a Stein manifold.*
(2) *M admits a strictly plurisubharmonic (PSH) exhaustion function.*

Since every locally pseudoconvex domain Ω over $\mathbb{CP}^n$ can be shown to have strictly *PSH* exhaustion functions using the aforementioned property of $-\log\delta_\Omega$, one obtains Oka's theorem and its generalizations by Fujita and Takeuchi from Theorem 1. Roughly speaking, the curvature positivity implies strict pseudoconvexity and the latter yields holomorphic functions.

Grauert's method of showing "(2) ⇒ (1)" in Theorem 1 is basically a generalization of Oka's method in the sense that it is by solving Cousin's problem. However, it is not so constructive as Oka's since it is based on the finite dimensionality of sheaf cohomology groups rather than their vanishing. Nevertheless, a great advantage of Grauert's method is that it is available to produce holomorphic functions under weaker assumptions as in the following.

Theorem 2. *For a complex manifold M, the following are equivalent*:
(1') *M is holomorphically convex, and $\{(x, y) \in M \times M : \mathcal{O}(M) \twoheadrightarrow \mathbb{C}^{\{x,y\}}\}^c$ is a compact set.*
(2') *M admits a PSH exhaustion function, which is strictly PSH outside a compact set.*

M is called **strongly pseudoconvex** if it satisfies (1') or (2').

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2 Oka called it the Hartogs inverse problem.
3 Cartan [11] called such a domain "*partout pseudo-convexe*."
4 A^B denotes the set of maps from B to A.
5 Actually, it was not known even for the universal covering.

Strongly pseudoconvex manifolds arise naturally as neighborhoods of compact analytic subsets that are holomorphically contractible to points. A decisive result in this context is that strongly pseudoconvex manifolds are nothing but the nonsingular models of Stein spaces with finitely many singular points (cf. Grauert [28], Hironaka [36], and Artin [2]).

On the other hand, the picture of O_M changes quite a lot for other complex manifolds M. For instance, there exist a complex torus T and a domain $\Omega \subset T$ with $\partial\Omega \neq \phi$ and $\Omega \notin \pi_0(O_T)$ such that $-\log\delta_\Omega \in$ PSH with respect to a flat metric (cf. [48]).

This change enlarges the range of questions on the geometry of analytic continuation on complex manifolds. Grauert [29] suggested one direction by introducing the notion of bundle convexity. The idea is to find a geometric condition for a holomorphic vector bundle $\pi : E \to M$ so that the sheaf of germs of its holomorphic sections has properties similar to $O_{\mathbb{C}^n}$ and $O_{\mathbb{CP}^n}$.

Definition 1. M is said to be ***E*-convex in the sense of Grauert** if

$$\forall K \Subset M \ \exists \hat{K} \Subset E \ \text{ s.t. } \forall x \in M \backslash \pi(\hat{K}) \text{ and } \forall v \in E_x,$$

$$\exists\, s \in H^{0,0}(M, E) \ \text{ s.t. } s(K) \subset \hat{K} \ \text{ and } \ s(x) = v.$$

Definition 2. M is called ***E*-convex** if

$$\forall X \subset E \ \text{ s.t. } \pi|_X \text{ is proper and } \forall \gamma \in M^{\mathbb{N}} \text{ s.t. } \gamma(\mathbb{N}) \not\Subset M,$$

$$\exists s \in H^{0,0}(M, E) \ \text{ s.t. } \#(s(\gamma(\mathbb{N}))\backslash X) = \infty.$$

Note that

$$M \ is \ E\text{-convex} \Rightarrow M \ \text{ is } \ E \oplus F\text{-convex } f\text{or all } F$$

so that Grauert's E-convexity is more restrictive than the mere E-convexity. To find a reasonable class of (M, E) for which M is E-convex or so in Grauert's sense, basic things to be studied are consequences of the curvature properties of M and E. In order to describe a reasonable statement in terms of curvature properties, we shall fix a Hermitian metric along the fibers of E (a fiber metric of E in short) and restrict ourselves to the following weaker convexity notion.

Definition 3. Given a fiber metric h of E, M is called (E, h)-**convex** if

$$\forall \gamma \in M^{\mathbb{N}} \ \text{ s.t. } \gamma(\mathbb{N}) \not\Subset M, \exists s \in H^{0,0}(M, E) \ \text{ s.t. } \ |s(\gamma(\mathbb{N}))|_h \not\Subset \mathbb{R}.$$

Let $L \to M$ be a holomorphic line bundle and let $K_M \to M$ be the canonical line bundle of M. L is said to be positive (denoted $L > 0$) if it admits a fiber metric whose curvature form is everywhere positive. If the dual bundle L^* is positive, L is said to be negative (denoted $L < 0$). Kodaira's embedding theorem and Theorem 2 can be unified into the following.

Theorem 3. (See [46] for instance.) *A strongly pseudoconvex manifold of dimension n with a positive line bundle can be embedded into* $\mathbb{CP}^{2n+1}$.

M is called a **weakly pseudoconvex manifold (= weakly 1-complete manifold)** if it admits a C^∞ plurisubharmonic exhaustion function. It is known that every complex Lie group is weakly pseudoconvex (cf. [42]). Fixing any C^∞ plurisubharmonic exhaustion function $\varphi : M \to \mathbb{R}$, we put $M_c = \{x;\ \varphi(x) < c\}$ for any $c \in \mathbb{R}$. By the use of sufficiently rapidly increasing convex functions on $[-\infty, c)$, the following can be deduced without difficulty, based on a standard L^2 method in [50]. (For the main result of [50], see Proposition 2 in §2.)

Proposition 1. *If a weakly pseudoconvex manifold M admits a positive line bundle* $L \to M$, *then the restriction map*

$$\Gamma(M, \mathcal{O}(K_M \otimes L)) \to \Gamma(M_c, \mathcal{O}(K_M \otimes L))$$

has a dense image and $\forall c \in \mathbb{R}\ \exists \mu_0 \in \mathbb{N}$ *s. t.* $\forall \mu \geq \mu_0$ *and* $\forall\ \gamma \in M_c^{\mathbb{N}}$ *s.t.* $\gamma(\mathbb{N}) \notin M_c, \exists \Sigma \subset \mathbb{N}$ *with* $\gamma(\Sigma) \notin M_c$ *s.t.,*

$$\Gamma(M_c, \mathcal{O}(K_M \otimes L^\mu)) \twoheadrightarrow \Gamma(\Sigma, \mathcal{O}(K_M \otimes L^\mu)).$$

Therefore, if M is connected and noncompact in the situation of Proposition 1, one has

$$\dim \Gamma(M, \mathcal{O}(K_M \otimes L^\mu)) = \infty, \text{ for } \mu \gg 1.$$

This observation enables us to construct singular fiber metrics (see §2) on L to conclude the following.

Theorem 4. *If a connected weakly pseudoconvex manifold M admits a positive line bundle L, then* $\exists \mu_0 \in \mathbb{N}$ s.t. $\forall \mu \geq \mu_0$ *and* $\forall \gamma \in M^{\mathbb{N}}$ *s.t.* $\gamma(\mathbb{N}) \notin M, \exists\ \Sigma \subset \mathbb{N}$ *with* $\gamma(\Sigma) \notin M$ *s.t.,*

$$\Gamma(M, \mathcal{O}(K_M \otimes L^\mu)) \twoheadrightarrow \Gamma(\Sigma, \mathcal{O}(K_M \otimes L^\mu)).$$

Corollary 1. *For any positive line bundle L over a weakly pseudoconvex manifold M, M is $K_M \otimes L^\mu$-convex for sufficiently large μ.*

If $K_M < 0$, by applying Theorem 4 to produce singular fiber metrics of K_M^* with positive curvature current and with enough singularities along Σ, one has the following.

Theorem 5. (cf. [70]) *A weakly pseudoconvex manifold is holomorphically convex if $K_M < 0$.*

Theorem 5 is essentially a small addendum of Theorem 4. A somewhat bigger one is the following.

Theorem 6. (cf. [71]) *Weakly pseudoconvex manifolds of dimension n with positive line bundles are holomorphically embeddable into $\mathbb{CP}^{2n+1}$.*

So, in view of Theorems 1–3, we are left with the following questions since 1998.

Q1. Does Theorem 4 remain true under the weaker assumption that $L|_{M\setminus U} > 0$ for some $U \Subset M$?

Q2. What about Theorem 6?

As for Theorem 5, one may ask the following.

Q3. Does Theorem 5 remain true if $K_M < 0$ is replaced by $K_M|_{M\setminus U} < 0$ for some $U \Subset M$?

Recently, it turned out that the answer to Q3 is affirmative. Namely, the following holds true.

Theorem 7. *A weakly pseudoconvex manifold is holomorphically convex if K_M is negative outside a compact set.*

It turned out that the answers to Q1 and Q2 are affirmative (cf. [60]). The next section will be devoted to showing an outline of the proof of Theorem 7.

2 Levi problem on weakly pseudoconvex manifolds

The proof of Theorem 7 is a combination of Takayama's proof of Theorem 5 and the following.

Proposition 2. [50] *Let M be a weakly pseudoconvex manifold of dimension n with a C^∞ plurisubharmonic exhaustion function φ and let $M_c := \{x \in M;\ \varphi(x) < c\}$ for $c \in \mathbb{R}$. Then, for any holomorphic line bundle $L \to M$ and $c \in \mathbb{R}$ satisfying $L|_{M\setminus M_c} > 0$,*

$$\dim H^{n,q}(M, L) < \infty \tag{1}$$

and

$$H^{n,q}(M,L) \cong H^{n,q}(M_c,L) \tag{2}$$

hold for $q \geq 1$ *with respect to the natural restriction homomorphisms, and*

$$H^{n,0}(M_c,L) = \overline{H^{n,0}(M,L)|_{M_c}}. \tag{3}$$

Here, $H^{p,q}(M,L)$ *denotes the* L*-valued* $\bar\partial$ *cohomology group of type* (p,q) *and* $\overline{H^{n,0}(M,L)|_{M_c}}$ *denotes the closure with respect to the topology of locally uniform convergence.*

Recall that $H^{n,q}(M,E)$ is canonically isomorphic to $H^{0,q}(M,K_M\otimes E)$ for any holomorphic vector bundle $E\to M$. We note that (3) is an extension of Runge's approximation theorem of Oka-Weil type (see [23]). (2) can be regarded as its extension to higher cohomology groups (cf. [1]). Because of (1), (2) is a consequence of the extension of (3) to the cohomology of higher degrees, but it can also be understood as a unique continuation of the cohomology classes (cf. [31, Part V, Commentary]).

In order to apply an argument in the proof of Theorem 5, one needs to produce a singular fiber metric of K_M^* in such a way that the L^2 method is available to find enough holomorphic functions to conclude that M is holomorphically convex.

Here, by a singular fiber metric of a holomorphic line bundle L, we shall mean a system of measurable functions of the form $\{h_\alpha e^{-\psi_\alpha}\}_\alpha$ associated with a trivializing open covering $\{U_\alpha\}$ of M for L in such a way that $h_\alpha\in C^\infty(U_\alpha)$, $h_\alpha>0$, $\psi_\alpha\in \mathrm{PSH}(U_\alpha)\cap L^1(U_\alpha)$, and $h_\alpha e^{-\psi_\alpha}=h_\beta e^{-\psi_\beta}\,|\eta_{\beta\alpha}|^2$ are satisfied on $U_\alpha\cap U_\beta$ for a system of transition functions $\{\eta_{\alpha\beta}\}$ of L.

Definition 4. For a singular fiber metric h of a holomorphic line bundle $L\to M$ and for $x\in M$, we put

$$c_x(h) \coloneqq \sup\{p;\ \exists\psi\in\mathcal{L}^1_{\mathrm{loc},x}\ \text{s.t.}\ he^{\psi}\ \text{is}\ C^\infty\ \text{around}\ x\quad\text{and}\quad e^{-p\psi}\in\mathcal{L}^1_{\mathrm{loc},x}\}\in(0,\infty],$$

where $\mathcal{L}^1_{\mathrm{loc}}$ denotes the sheaf of germs of locally integrable functions.

Proposition 2 is applied to show the following.

Proposition 3. *In the situation of Proposition 2, for any* $d>c$ *and for any* $\gamma\in(M_d\setminus M_c)^{\mathbb{N}}$ *such that* $\varphi(\gamma(k))$ *is strictly increasingly convergent to* d*, one can find an analytic set* A *in* M_d *containing* $\gamma(\mathbb{N})$ *with compact components and a singular fiber metric* h *of* L *with strictly positive curvature current on* $M_d\setminus M_c$*, such that* h *is* C^∞ *on* $M_d\setminus A$ *and* $c_x(h)\leq 1$ *for all* $x\in\gamma(\mathbb{N})$*.*

For any singular fiber metric h of L, we denote by $L^{p,q}_{(2),\mathrm{loc}}(M,L,h)$ the set of measurable L-valued (p,q) forms u on M such that $|u|^2_{h,x}\in\mathcal{L}^1_{\mathrm{loc},x}$ for all $x\in M$ and set

$$H^{p,q}_{(2),\mathrm{loc}}(M,L,h) \coloneqq \frac{\mathrm{Ker}(\bar\partial : L^{p,q}_{(2),\mathrm{loc}}(M,L,h)\to L^{p,q+1}_{(2),\mathrm{loc}}(M,L,h))}{\mathrm{Im}(\bar\partial : L^{p,q-1}_{(2),\mathrm{loc}}(M,L,h)\to L^{p,q}_{(2),\mathrm{loc}}(M,L,h))}.$$

By applying Proposition 3, the proof of Proposition 2 can be generalized without difficulty to show the following.

Proposition 4. *Let* M *and* φ *be as in Proposition 2 and let* L *be a holomorphic line bundle with a singular fiber metric* h*, which is* C^∞ *on* M_c *and with strictly positive curvature current on* $M\setminus M_c$*. Then,*

$$\dim H^{n,q}_{(2),\mathrm{loc}}(M,L,h)<\infty$$

and

$$H^{n,q}_{(2),\mathrm{loc}}(M,L,h)\cong H^{n,q}_{(2),\mathrm{loc}}(M_c,L,h)$$

hold for $q \geq 1$ *with respect to the natural restriction homomorphisms, and*

$$H^{n,0}_{(2),\mathrm{loc}}(M_c, L, h) = \overline{H^{n,0}_{(2),\mathrm{loc}}(M, L, h)|_{M_c}}.$$

One can deduce the following from Proposition 4.

Corollary of Proposition 4. (An interpolation theorem) *In the situation of Proposition 2, one can find a fundamental neighborhood system* $\{U_\mu\}$ of A such that

$$H^{0,0}(M_d, K_M \otimes L) \twoheadrightarrow \mathrm{Im}(H^{0,0}(U_\mu, K_M \otimes L) \xrightarrow{\rho} H^{0,0}(\gamma(\mathbb{N}), K_M \otimes L)),$$

where ρ *denotes the restriction map.*

Hence, by letting $L = K_M^*$ in particular, one has

$$\mathcal{O}(M_d) \twoheadrightarrow \mathbb{C}^\Sigma,$$

for any $d > c$ and for any discrete set $\Sigma \subset M_d \backslash M_c$ such that $\varphi|_\Sigma$ is injective and $\sup_\Sigma \varphi = d$. Moreover, the preimages of the elements of $\mathbb{C}^\Sigma$ contain those functions that can be chosen arbitrarily small on $M_{c'}$ for any fixed $c' < c$. Hence, by a limiting argument, one has also

$$\mathcal{O}(M) \twoheadrightarrow \mathbb{C}^\Sigma,$$

for any discrete set $\Sigma \subset M \backslash M_c$ such that $\varphi|_\Sigma$ is injective and $\sup_\Sigma \varphi = \infty$.

Hence, M is holomorphically convex.

In short, on weakly 1-complete domains, line bundles that are positive near the boundary have sufficiently many holomorphic sections over compact sets if sufficiently high tensor power is taken, so that it is possible to construct singular fiber metrics by using them in order to solve an interpolation problem. On locally pseudoconvex bounded domains in complex manifolds, the L^2 method works to solve similar interpolation problems. Some of the specific outcomes will be reviewed in the following.

3 Pseudoconvexity and the Bergman kernel

Every holomorphic map

$$\left\{(z, w) \in \mathbb{D}^2;\ |z| > \frac{1}{2} \text{ or } |w| < \frac{1}{2}\right\} \to \Omega \in \pi_0(\mathcal{O}) \quad (\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\})$$

is extendable to a holomorphic map $\mathbb{D}^2 \to \Omega$ (Hartogs). If $\Omega \subset \mathbb{C}^n$ and $\partial\Omega$ is a C^2-smooth real hypersurface with a defining function ρ, $\partial\bar{\partial}\rho|_{\mathrm{Ker}\,\partial\rho}$ is called the **Levi form** of $\partial\Omega$. Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary has a defining function ρ whose Levi form is everywhere semipositive on $\partial\Omega$ (Levi).

In 1933, Bergman observed the following in some special cases:

$$\Omega \underset{hol.cvx}{\Subset} \mathbb{C}^2 \text{ and } \partial\Omega \in C^2 \Rightarrow \delta_\Omega(z)^{-2} \lesssim B_\Omega(z, z) \lesssim \delta_\Omega(z)^{-3},$$

where, $B_\Omega(z, w)$ denotes the Bergman kernel function of Ω (cf. [5]).

In 1965, Hörmander [40] proved that, given a domain $\Omega \subset \mathbb{C}^n$,

$$\lim_{z \to z_0} B_\Omega(z, z)\delta_\Omega(z)^{n+1} \text{ exists and } > 0$$

if the range of the $\bar{\partial}$-operator $L^{0,0}_{(2)}(\Omega) \to L^{0,1}_{(2)}(\Omega)$ is closed and $\partial\Omega$ is strongly pseudoconvex at z_0. Here, $L^{p,q}_{(2)}(\Omega)$ denotes the space of $L^2(p, q)$-forms on Ω.

In 1974, Fefferman proved for strongly pseudoconvex domains Ω with C^∞-smooth boundary that

$$B_\Omega(z,z) = \varphi(z)\delta_\Omega(z)^{-n-1} + \psi(z)\log\delta_\Omega(z)$$

holds for some C^∞ functions φ and ψ on $\overline{\Omega}$. The following is an application.

Fefferman's theorem. Every biholomorphic map between two strongly pseudoconvex bounded domains Ω_1 and Ω_2 with C^∞-smooth boundary extends as a diffeomorphism from $\overline{\Omega_1}$ to $\overline{\Omega_2}$.

By an L^2 extension theorem in [62],

$\Omega \underset{\psi cvx}{\Subset} \mathbb{C}^n$ and $\partial\Omega \in Lip \Rightarrow \delta_\Omega(z)^{-2} \lesssim B_\Omega(z,z)$.[6]

Recently, Chen [13] proved that $\delta_\Omega(z)^{-2} \lesssim B_\Omega(z,z)$ also holds if $\partial\Omega$ is locally the graph of a continuous function.

A connection between the weighted Bergman kernels and pluripotential theory is shown in the following diagram:

$$\begin{array}{ccccccc} \{f_\mu\}_\mu & \overset{CONS}{\subset} & A^2_\varphi & \in & \text{RKHS's} & \ni & A^2_\varphi := \left\{ f \in \mathcal{O}(\Omega); \int_\Omega e^{-\varphi}|f|^2 < \infty \right\} \\ & & \downarrow & & \swarrow & & \uparrow \\ B_\varphi(z,w) := \sum f_\mu(z)\overline{f_\mu(w)} & \in & \text{Bergman kernels} & \supset & \{B_{m\varphi}\}_{m=1}^\infty & \overset{\mathscr{D}}{\mapsto} & \text{PSH} \ni \varphi, \end{array}$$

where RKHS := reproducing kernel Hilbert space, CONS := complete orthonormal system, and

$$\mathscr{D}(\{B_{m\varphi}\}) := \lim_{m\to\infty} \frac{1}{m}\log B_{m\varphi}(z,z).$$

Demailly's approximation theorem in [17] asserts that

$$\mathscr{D}(\{B_{m\varphi}\}) = \varphi \tag{4}$$

holds for any $\varphi \in \mathrm{PSH}(\Omega)$ if Ω is pseudoconvex.

Looking for even better approximations, Demailly [18] asked whether or not $|f|^2e^{-\varphi} \in L^1(\Omega)$ for $\varphi \in \mathrm{PSH}(\Omega)$ and $f \in \mathcal{O}(\Omega)$ implies $|f|^2e^{-p\varphi} \in L^1_{\mathrm{loc}}(\Omega')$ for $\Omega' \Subset \Omega$ and for $p > 1$ sufficiently close to 1. Recently, a sharp and effective affirmative answer was given by Guan [32].

If Ω admits a divisor $A \subset \Omega$ such that $\Omega \setminus A$ is Stein, generalization of (4) holds for the families of weighted Bergman kernels, say B_{h^m}, for the spaces $H^{n,0}_{(2)}(\Omega, L, h^m)(m = 1, 2, \dots)$ for the line bundles $L \to M$ with singular fiber metrics h with positive curvature current. An important property of B_h is that

$$\int_\Omega B_h = \dim H^{n,0}(\Omega, L \otimes \mathcal{I}_h)$$

holds if $\Omega \Subset \Omega$ (=Ω is compact), where $\mathcal{I}_h$ denotes the multiplier ideal sheaf of h defined by

$$\mathcal{I}_h := \{f \in \mathcal{O};\ |f|^2e^{-\varphi_\alpha} \in \mathcal{L}^1_{\mathrm{loc}}\}\ \ (h = h_\alpha e^{-\varphi_\alpha}, h_\alpha \in C^\infty, \varphi_\alpha \in \mathrm{PSH}).$$

Therefore, it is tempting to ask for the behavior of $\log B_{h^m}(z,z)$ as $m \to \infty$ and as $z \to \partial\Omega$. In particular, the asymptotics of functions

$$\log B_{h^m}(z,z) - \log B_{h^m h_0}(z,z)$$

might be interesting for any fixed fiber metric h_0 of a fixed line bundle $L_0 \to M$.

4 Some geometry beyond holomorphic convexity

In Oka's theory, important existence theorems are tied together by an approximation theorem of Runge type, so that the existence of a *PSH* exhaustion function is crucial to let the limiting arguments run. Hence,

6 ψcvx=pseudoconvex.

Bergman, Oka → Grauert, Hörmander → Fefferman, et al.
$\approx$
restricted class of *PSH* exhaustions ⇒ sharper analytic results.

On the other hand, the principal idea of [29] can be understood as follows:

$$\Omega \underset{\text{str.}\psi\text{cvx}}{\Subset} M \overset{(Gr-4)}{\approx} \text{“}\Omega \underset{\text{loc.}\psi\text{cvx}}{\Subset} M\text{”+“}; E \to M \text{ and } E|_{\partial\Omega} > 0.\text{”}$$

In short, as far as the existence theorems are concerned, one should be able to replace the assumption of strict pseudoconvexity of Ω, in many cases, by the combination of the weak pseudoconvexity of Ω and the positivity of bundles along the boundary of Ω. Hence, if one wants to study the bundle-valued Bergman kernels on complex manifolds, one has to extend the application of the L^2 method **to the situation where the domain does not admit *PSH* exhaustion functions in canonical ways**. Such an extension of the objects seems to be indispensable because the following questions remain open for Kähler manifolds M.

Conjecture 1. $\Omega \underset{\text{loc.}\psi cvx}{\Subset} M \underset{?}{\Rightarrow}$ Ω is weakly pseudoconvex. (cf. [30]),

Conjecture 2. $M \Subset M$ and $\tilde{M} \underset{\text{covering}}{\to} M \underset{?}{\Rightarrow} \tilde{M}$ is weakly pseudoconvex. (cf. [67]).

Continuation of analytic objects on such Ω and $\tilde{M}$ will be accompanied with interesting questions. For instance, one may ask $O(X) \overset{?}{\twoheadrightarrow} O(X \setminus K)$ if $X = \Omega$ or $\tilde{M}$ as above when X is connected, $\dim X \geq 2$ and $K \Subset X$. The answer is no in general, but something not totally stupid can be said in some cases. For instance, if Ω is a smooth and locally pseudoconvex bounded domain in a Kähler manifold of dimension ≥2 whose complement is not locally pseudoconvex, then $\partial\Omega$ must be connected (cf. [52]).

We also note that $\Omega \underset{str.\psi cvx}{\Subset} M$ is an intrinsic property of Ω (cf. [47]), whereas $\Omega \underset{\text{loc.}\psi cvx}{\Subset} M$ is not, as one can see from the following examples.

(1) $\mathbb{CP}^2 \supset \Omega = \mathbb{CP}^2 \setminus \{p\} \cong O(1)_{\mathbb{CP}^1} \underset{\text{loc.}\psi cvx}{\subset} \mathbb{CP}^2$ blown-up at p.

(2) $\mathbb{CP}^n \times \mathbb{CP}^1 \supset (\mathbb{C}^n \setminus \{0\}) \times \{\zeta \in \mathbb{C}; 1 < |\zeta| < \exp(2\pi^2/\log 2)\}$ (not locally pseudoconvex if $n \geq 2$) $\cong \Omega \underset{\text{loc.}\psi cvx \text{ and } \partial\Omega \in C^\omega}{\subset} (\mathbb{C}^n \setminus \{0\}) \times \mathbb{CP}^1 / \langle (z, \zeta) \mapsto (2z, 2\zeta) \rangle$.

(See [51] for $n = 1$ and [20] for $n \geq 2$. See also [49].)

Therefore, one has to impose conditions more than the mere local convexity on $\partial\Omega$ to extend Grauert's theorem for $E|_{\partial\Omega} > 0$ along the idea (Gr-4).

Remark. As a supporting evidence of conjecture 1, one can mention the following.

Theorem 8. (Diederich-Ohsawa [21]) *For any compact Kähler manifold M and $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{Aut}\mathbb{D})$, $M \times_\rho \mathbb{D}$ ($\Subset_{\text{loc.}\psi cvx} M \times_\rho \mathbb{CP}^1$) is weakly pseudoconvex.*

As for conjecture 2, partial answers are given in [24] and [10].

5 From Riemann to Demailly

In [19], Demailly remarked as follows.

> It is remarkable that Bernhard Riemann already anticipated in [66] the use of L^2 estimates and the idea of minimizing energy, even though his terminology was very different from the one currently in use.

As is well known, Riemann's idea was realized, or rather justified, by Hilbert and Weyl and then further extended by Hodge and Kodaira. In particular, Kodaira characterized projective algebraic varieties as compact complex manifolds that admit positive line bundles, by establishing a cohomology vanishing theorem.

Demailly's thesis [16] is one of the generalizations of Kodaira's vanishing theorem. Demailly proved a vanishing theorem with L^2 estimates on complete Kähler manifolds under the semipositivity conditions on the curvature of the bundles. It was first observed by Grauert [26] that complete Kähler metrics live naturally on Stein manifolds as well as on quasi-projective manifolds. The reason why Demailly's L^2 vanishing theorem is effective in algebraic geometry is that L^2 holomorphic functions extend analytically across proper analytic subsets of the domains in $\mathbb{C}^n$ as in the case of Riemann's removable singularity theorem in one variable.

The method of Demailly is a natural extension of Skoda's variant of Andreotti-Vesentini-Hörmander's refinement of Oka-Kodaira's solution of the generalized Cousin problem. According to what I heard, Skoda, who was the adviser of Demailly, explored his method of solving a division problem with L^2 estimates after reading Oka's paper recommended by Lelong. Demailly's works have clearly shown that the method of L^2 estimates was a big breakthrough in SCV and complex geometry.

6 Bundle-convexity theorems

Here is a pseudo-chronologically ordered collection of works of the author related to Demailly's:

$$\begin{array}{llllll}
L^2 \text{ estimates for } \bar\partial & \underset{[22]}{\longrightarrow} \text{ Bergman metric} & & & & \\
\swarrow\text{[62]} & \searrow\text{[54]} & (\overset{[55]}{\longrightarrow}) & \overset{[57]}{\longrightarrow} \text{ bundle-convexity} & & \\
L^2 \text{ extension} & \underset{[53]}{\longrightarrow} & \text{Nishino's rigidity} & \searrow\text{[56]}: & \text{vanishing} & \\
 & & \text{of } \mathbb{C} \text{ in Stein families} & & \text{for } H^{p,q}_{\mathrm{alg}}(\Omega, L) &
\end{array}$$

Note that "L^2 extension" is closely related to "$\log B_{\Omega_t}(z,z) \in \mathrm{PSH}$ w.r.t. (t,z)," which was first discovered in [41,44] in special cases and established in [6] for Stein families $\{\Omega_t\}$. L^2 extension problems have been solved in this context from various viewpoints [4,7,8,19,32–34,63]. Another connection between the L^2 extension and the Bergman kernel is given by Demailly's aforementioned approximation theorem.

Let $E \to M$ be a holomorphic vector bundle equipped with a fiber metric h, and let $\Omega \subset M$ be a relatively compact open set. In this situation, the (E,h)-convexity of Ω can be expressed more concisely, i.e.,

$$\Omega \text{ is } \boldsymbol{E}\text{-convex} \quad \Leftrightarrow \quad \forall \gamma \in \Omega^{\mathbb{N}} \text{ s.t. } \gamma(\mathbb{N}) \not\Subset \Omega \ \exists s \in H^{0,0}(\Omega, E) \text{ s.t. } s(\gamma(\mathbb{N})) \not\Subset E.$$

By an abuse of language, we shall confuse E-convexity with (E,h)-convexity for the bounded domains. Nontrivial bundle-convexity theorems were first obtained by Pinney [64] and Asserda [3].

Theorem 9. (cf. [64]) *Assumption*: $M \Subset M$, $\partial\Omega \in C^2$, $\mathrm{rank} E = 1$, *and* $E > 0$.
Conclusion: Ω *is* E^μ*-convex for* $\mu \gg 1$.

Theorem 9 realizes the idea (Gr-4) under $\partial\Omega \in C^2$, but only partially because $E > 0$ seems obviously superfluous.

Theorem 10. (cf. [3]) *Assumption*: $M \Subset M$, $\mathrm{rank} E = 1$, $E > 0$, *and* $\Omega \underset{\mathrm{loc}.\psi cvx}{\Subset} M$.
Conclusion: Ω *is* E^μ*-convex for* $\mu \gg 1$.

Since Theorem 10 becomes false if one replaces $E > 0$ by $E|_{\partial\Omega} > 0$ (see Example 1 in §5), it is still necessary to impose some condition on $\partial\Omega$ to extend it in the direction of (Gr-4).

Theorems 9 and 10 have been extended in [58] as follows.

Theorem 11. (Bundle-convexity I) *Assumption*: $\Omega \underset{\mathrm{loc}.\psi cvx}{\Subset} M$, $\partial\Omega \in C^2$, $E \to M$, $\mathrm{rank} E = 1$, *and* $E|_{\partial\Omega} > 0$.
Conclusion: Ω *is* E^μ*-convex for* $\mu \gg 1$.

Theorem 12. (Bundle-convexity II) *Assumption*: $\Omega \underset{\mathrm{loc}.\psi cvx}{\Subset} M$, $\partial\Omega = |D|$ *for some effective divisor* D *on* M *s.t.* $[D]|_{|D|} \geq 0$, $E \to M$, $\mathrm{rank} E = 1$, *and* $E|_{\partial\Omega} > 0$.
Conclusion: Ω *is* E^μ*-convex for* $\mu \gg 1$.

Consequently, one can extend Theorem 7:

Theorem 13. (cf. [56,12]) *In the situation of Theorems 11 or 12, assume that* $E = K_M^*$. *Then,* Ω *is holomorphically convex.*

See also [65], which concludes holomorphic convexity in a similar situation but under a stronger assumption. On the other hand, negative but interesting examples exist also in this direction (cf. [14]).

Remark. In view of Theorem 10, the E^μ-convexity of $M\backslash|D|$ does not imply that $[D]|_{|D|}$ is semipositive. It seems to be open whether or not $[D]|_{|D|}$ is nef if $M\backslash|D|$ is E^μ-convex in a "transcendental sense." It is in a good contrast with a recent result by Höring and Peternell [39] saying that $[D]|_{|D|}$ is pseudoeffective[7] if D is a smooth hypersurface of a compact Kähler manifold M and $M\backslash D$ is Stein. It is known from Ueda's theory [74] that the complement of a smooth curve C in a compact complex surface S is strongly pseudoconvex if $C \hookrightarrow S$ is of finite type, i.e., if $\deg[C]|_C = 0$ and the germ of the embedding $C \hookrightarrow S$ is not formally equivalent to that of the embedding $C \hookrightarrow [C]|_C$ as the zero section. In [55], it was shown in this situation that for any line bundle $L \to S$ with $L|_C > 0$

$$H^{2,0}(S\backslash C, L) = \overline{\cup_{\mu=1}^{\infty} H^{2,0}(S, L \otimes [C]^\mu)},$$

where $H^{2,0}(S, L \otimes [C]^\mu)$ is naturally identified with the space of meromorphic sections of $K_S \otimes L$ with poles of order at most μ (only) along C. There is a famous example of $C \hookrightarrow S$ by Serre, where C is an elliptic curve, $[C]|_C$ is trivial, and $S\backslash C \cong \mathbb{C}^* \times \mathbb{C}^*$. The point is that $S\backslash C$ is Stein, does not admit any plurisubharmonic exhaustion function φ such that $\varphi \leq \log\frac{1}{\delta_C}$, and does admit one of growth $\frac{1}{\delta_C}$, where δ_C denotes the distance to C. Recently Koike and Ueda [43] showed that certain affine bundles over compact Kähler manifolds have a property similar to $S\backslash C$ as in Serre's example and [74]. Many other interesting things seem to be left undiscussed in this direction.

A result on the kernel asymptotics for the case $\partial\Omega \in C^2$ is the following.

Theorem 14. (cf. [59]) *Let* Ω *be a bounded locally pseudoconvex domain with* C^2*-smooth boundary in a complex manifold* M *and let* $E \to M$ *be a holomorphic line bundle with a* C^∞ *fiber metric* h *whose curvature form is positive at every point of* $\partial\Omega$. *Then, for any* $\varepsilon > 0$ *one can find* $\nu_0 \in \mathbb{N}$ *such that*

$$\liminf_{z \to \partial\Omega} B_{\Omega, E^\nu}(z) \cdot \rho(z)^{2-\varepsilon} > 0$$

holds for any $\nu \geq \nu_0$. *Here,* B_{Ω,E^ν} *denotes the Bergman kernel for the* L^2E^ν*-valued holomorphic n-forms with respect to* h^ν.

Sketchy accounts of the proofs of Theorems 11 and 12 are given below.

Proof of bundle-convexity I. Finite dimensionality of the $L^2\bar\partial$-cohomology with respect to a complete metric on $\Omega\backslash\gamma(\mathbb{N})$ for a class of $\gamma : \mathbb{N} \to \Omega$ is applied. More precisely, for any $z_0 \in \partial\Omega$, one can find a sequence $\gamma \in \Omega^{\mathbb{N}}$ with

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7 For the definition, see [9] (Definition 7.1).

$\lim_{k\to\infty}\gamma(k) = z_0$, a complete metric g on $\Omega\backslash\gamma(\mathbb{N})$, $\psi : \Omega \to [-\infty, -1)$ with $\psi^{-1}(-\infty) = \gamma(\mathbb{N})$ and $-\partial\bar{\partial}\log(-\psi) \approx g$ near $\gamma(\mathbb{N})$ such that

$$\dim H^{n,1}_{(2)}(\Omega\backslash\gamma(\mathbb{N}), E^{\mu^2}, g, h^{\mu^2}e^{-\psi}(-\psi)\delta^{\mu}_{\Omega}) < \infty. \tag{5}$$

One can apply (5) to find desired sections by choosing ψ so that $e^{-\psi}$ is non-integrable around any point of $\gamma(\mathbb{N})$. □

Proof of bundle-convexity II. $\exists m \in \mathbb{N}$ s.t. $H^{0,q}(M, K_M \otimes E^m \otimes [D]^{\mu}) \twoheadrightarrow H^{0,q}(D, K_M \otimes E^m \otimes [D]^{\mu}|_D)$ for $\mu \gg 1$. □

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